ИТОГИ НАУКИ И ТЕХНИКИ.
Современная математика и ее приложения.
Тематические обзоры.
Том 213 (2022). С. 110–144
DOI: 10.36535/0233-6723-2022-213-110-144

УДК 512.7

ПОЛИНОМИАЛЬНЫЕ АВТОМОРФИЗМЫ, КВАНТОВАНИЕ И ЗАДАЧИ ВОКРУГ ГИПОТЕЗЫ ЯКОБИАНА. I. ВВЕДЕНИЕ

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Посвящается памяти Евгения Соломоновича Голода

Аннотация. Целью данного обзора является систематизация результатов, касающихся квантового подхода к некоторым классическим аспектам некоммутативных алгебр, особенно к гипотезе о якобиане. Работа начинается с квантования доказательства теоремы Бергмана о централизации, затем обсуждаются автоморфизмы автоморфизмов INd-схем и вопросы аппроксимации. Последняя глава посвящена связи между теоремами типа Бернсайда теории PI и гипотезой Якоби (подход Ягжева). В данном выпуске публикуется первая часть работы; продолжение будет опубликовано в следующих выпусках.

***Ключевые слова*:** автоморфизм, квантование, гипотеза о Якобиане.

POLYNOMIAL AUTOMORPHISMS, QUANTIZATION, AND JACOBIAN CONJECTURE RELATED PROBLEMS. I. INTRODUCTION

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Abstract. The purpose of this review is the collection and systematization of results concerning the quantization approach to the some classical aspects of non-commutative algebras, especially to the Jacobian conjecture. We start with quantization proof of Bergman centralizing theorem, then discourse authomorphisms of INd-schemes authomorphisms, then go to aproximation issues. Last chapter dedicated to relations between PI-theory Burnside type theorems and Jacobian Conjecture (Jagzev approach). This issue contains the first part of the work; continuation will be published in future issues.

***Keywords and phrases*:** automorphism, quantization, Jacobian conjecture.
***AMS Subject Classification*:** 14R10, 18G85

Работа выполнена при поддержке Российского научного фонда (проект № 17-11-01377). Работа Ф. Разавиния была также поддержана Фондом науки и технологий Португалии (национальный грант PD/BD/142959/2018).

ISSN 2782–4438

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PREFACE

The purpose of this review is the collection and systematization of results concerning the quantization approach to the Jacobian conjecture.

O.-H. Keller's Jacobian conjecture remains, as of the writing of this text, an open and apparently unassailable problem. Various possible approaches to the Jacobian conjecture have been explored, resulting in accumulation of a substantial bibliography, while the development of vast parts of modern algebra and algebraic geometry were in part stimulated by a search for an adequate framework in which the Jacobian conjecture could be investigated. This has engendered a situation of simultaneous existence of circumstantial evidence in favor and against the positivity of this conjecture.

One of the more established plausible approaches to the Jacobian problem concerns the study of infinite-dimensional algebraic semigroups of polynomial endomorphisms and groups of automorphisms of associative algebras, as well as mappings between them. The foundation for this approach was laid by I. R. Shafarevich. During the last several decades, the theory was developed and vastly enriched by the works of Anick, Artamonov, Asanuma, Bass, Bergman, Białynicki-Birula, Czerniakiewicz, Dicks, Dixmier, Kambayashi, Lewin, Makar-Limanov, Shestakov, Umirbaev, Wright, and many others. In particular, the results of Anick, Makar-Limanov, Shestakov, and Umirbaev established a connection between the Jacobian conjecture for the commutative polynomial algebra and its associative analogs on the one hand with combinatorial and geometric properties (stable tameness, approximation) of the spaces of polynomial automorphisms on the other.

More recently, the stable equivalence between the Jacobian conjecture and a conjecture of Dixmier on the endomorphisms of the Weyl algebra has been discovered by Kanel-Belov and Kontsevich and, independently, by Tsuchimoto. The cornerstone of this rather surprising feature is a certain mapping (sometimes referred to as the anti-quantization map) from the semigroup of Weyl algebra endomorphisms (a quantum object) to the semigroup of endomorphisms of the corresponding Poisson algebra (the appropriate classical object). In view of that, it seems reasonable to think there are insights to be gained by studying quantization of spaces of polynomial mappings and properties of the corresponding quantization morphisms.

In this direction, one of the larger milestones is given by a series of conjectures of Kontsevich concerning equivalences between polynomial symplectomorphisms, holonomic modules over algebras of differential operators, and automorphisms of such algebras. Another rather nontrivial side of the quantization program rests upon the interaction with universal algebra.

In this review, we present some of our progress regarding quantization, Kontsevich conjecture, as well as recall some of our recent results on the geometry of Ind-scheme automorphisms, approximation by tame automorphisms together with its symplectic version, and torus actions on free associative algebras. We also provide a review of the work of Kanel-Belov, Bokut, Rowen and Yu, which sought to connect the Jacobian problem with various problems in universal algebra, as conceived by the brilliant late mathematician A. V. Yagzhev.

We have benefitted greatly from extensive and fruitful discussions with E. Aljadeff, V. A. Artamonov, I. V. Arzhantsev, V. L. Dolnikov, A. E. Guterman, R. N. Karasev, I. V. Karzhemanov,

D. Kazhdan, V. O. Manturov, A. A. Mikhalev, S. Yu. Orevkov, E. B. Plotkin, B. I. Plotkin, A. M. Raigorodskii, E. Rips, A. L. Semenov, G. B. Shabat, G. I. Sharygin, N. A. Vavilov, E. B. Vinberg, U. Vishne, I. Yu. Zhdanovskii, and A. B. Zheglov. It is a pleasant task to express our utmost gratitude to our esteemed colleagues.

Chapter 1

INTRODUCTION

1.1. Quantization and Algebra Problems

This section provides an overview of the Jacobian conjecture together with motivation for the theory of Ind-schemes and quantization, as well as some necessary preliminaries on the proof of Bergman's centralizer theorem. Throughout this paper, all rings are assumed to be associative with multiplicative identity.

1.1.1. Free algebras. A free algebra is a noncommutative analog of a polynomial ring since its elements can be described as "polynomials" with noncommuting variables, while the free commutative algebra is the polynomial algebra. First, we give the definition of a free monoid, which is needed in our definition of free algebras (see [174]).

Definition 1.1.1. Let $X = \{x_i : i \in I\}$. A *word* is a string with elements in X. A *free associative monoid* X^* on a set X is the set of words in X, including the empty product to represent 1. The multiplication on X^* is given by the juxtaposition of words.

Next, we can naturally define the free associative algebra with respect to a generating set over a commutative ring.

Definition 1.1.2. Let C be a commutative ring with a multiplicative identity. A *free associative C-algebra* $C\langle X\rangle$ with respect to a generating set $X = \{x_i : i \in I\}$ is the free C-module with base X^*.

Remark 1.1.3. This C-module becomes a C-algebra by introducing a multiplication as follows: the product of two basis elements is the concatenation of the corresponding words and the product of two arbitrary C-module elements are thus uniquely determined. Note that $C\langle X\rangle := \bigoplus_{w\in X^*} Cw$ and the elements of $C\langle X\rangle$ are called *noncommutative polynomials* over C generated by X.

Similarly, we can also define the free associative algebra $k\langle X\rangle$ with respect to a generating set $X = \{x_i : i \in I\}$ over an arbitrary field k.

Remark 1.1.4. If C is an integral domain, then the product of leading monomials of two noncommutative polynomials f and g in $C\langle X\rangle$ is the leading monomial of fg. It follows that $C\langle X\rangle$ is also a domain (but still noncommutative).

We finally note that throughout this review, we will only discuss free associative k-algebras with respect to a generating set $X = \{x_1, \dots, x_s\}$ (for $s \geqslant 2$) over a field k instead of a commutative ring.

1.1.2. Matrix representations of algebras. Let A be a k-algebra and K be a field extension of k. In this work, we consider only finite-dimensional representations of A, i.e., the word "representation" means a "finite-dimensional representation."

Definition 1.1.5. An n-dimensional *matrix representation* over K is a k-homomorphism $\rho : A \to M_n(K)$ to the matrix algebra over K.

Remark 1.1.6. Two representations ρ' and ρ are said to be equivalent if they are conjugate, i.e., $\rho' = \tau\rho\tau^{-1}$ for some invertible matrix $\tau \in M_n(K)$.

The representation is said to be *irreducible* if the images of A generates the matrix algebra as a K-algebra, or if the map $A \otimes K \to M_n(K)$ is surjective. Usually, we study the case where $K = k$. With this assumption, we say that a representation $\rho : A \to M_n(k)$ is irreducible if and only if it is surjective.

1.1.3. Algebra of generic matrices. In order to use the concept of generic matrices, we first need to introduce the matrix representation of a free associative algebra (see [16]). We introduce matrix representations of any k-algebras in Sec. 1.1.2. A matrix representation of the free associative ring $k\langle X\rangle = k\langle x_1, \dots, x_s\rangle$ over k generated by a finite set $X = \{x_1, \dots, x_s\}$ of s indeterminates ($s \geqslant 2$) is given by assigning arbitrary matrices as images of the variables. In itself, this is not very interesting. However, when one asks for equivalence classes of irreducible representations, the other is directed to an interesting problem in invariant theory. We will discuss this topic below.

Definition 1.1.7. Let n be a positive integer and $\{x_{ij}^{(\nu)} \mid 1 \leqslant i, j \leqslant n,\ \nu \in \mathbb{N}\}$ be independent commuting indeterminates over k. Then

$$X_\nu := (x_{ij}^{(\nu)}) \in M_n(k[x_{ij}^{(\nu)}])$$

is called an $n \times n$ *generic matrix* over k, and the k-subalgebra of $M_n(k[x_{ij}^{(\nu)}])$ generated by X_ν is called the *algebra of generic matrices*; we denote it by $k\langle X_1, \dots, X_s\rangle$ or simply $k\{X\}$.

The algebra of generic matrices is a basic object in the study of the polynomial identities and invariants of $n \times n$ matrices.

There is a canonical homomorphism

$$\pi : k\langle x_1, \dots, x_s\rangle \to k\langle X_1, \dots, X_s\rangle \tag{1.1.1}$$

from the free associative ring on variables $x_1, \dots, x_s$ to this ring.

If $u_1, \dots, u_s$ are $n \times n$ matrices with entries in a commutative k-algebra R, then we can substitute u_j for X_j and hence obtain a homomorphism

$$k\langle X_1, \dots, X_s\rangle \to M_n(R).$$

The homomorphism π possesses the following important property: an element f of the free associative algebra lies in the kernel of the map π if and only if it vanishes identically on $M_n(R)$ for every commutative k-algebra R, and this is valid if and only if f vanishes identically on $M_n(k)$. In addition, (irreducible) matrix representations of a free ring A of dimension $\leqslant n$ correspond bijectively to (irreducible) matrix representations of the ring of generic matrices. This result is a core tool in our proof.

Let $u_1, \dots, u_N$ ($N = n^2$) be a basis for the matrix algebra $M_n(\bar{K})$ and $z_1, \dots, z_N$ be indeterminates. Then the entries of the matrix $Z = \sum z_j u_j$ are all algebraically independent. Moreover, the following Amitsur theorem is well known.

Theorem 1.1.8 (Amitsur). *The algebra $k\langle X_1, \dots, X_s\rangle$ of generic matrices is a domain.*

For the proof, see [16, Theorem V.10.4] or [233, Theorem 3.2].

1.1.4. The Amitsur–Levitzki theorem. For the free associative algebra $A = k\langle X\rangle$, the *commutator* of two elements in A is defined as $[x, y] = xy - yx$. The commutator has analogs for several variables, called *generalized commutators* of elements $x_1, \dots, x_n$ of A:

$$S_n(x_1, x_2, \dots, x_n) := \sum (-1)^\sigma x_{\sigma 1} \cdots x_{\sigma n} \tag{1.1.2}$$

(see [16]), where σ runs over the group of all permutations. It is clear that $S_2(x, y) = [x, y]$. Note that the generalized commutators are multilinear and alternating polynomials in the variables. Moreover, a general multilinear polynomial in n variables has the form $p(x_1, \dots, x_n) = \sum c_\sigma x_{\sigma 1} \cdots x_{\sigma n}$, where the coefficients c_σ are elements of k.

There is an important and powerful result (see [6]), which was first proved by A. S. Amitsur and J. Levitzki in 1950.

Theorem 1.1.9 (A. S. Amitsur and J. Levitzki). *Let R be a commutative ring and r be an integer.*

(i) *If $r \geqslant 2n$, then $S_r(a_1, \dots, a_r) = 0$ for every set $a_1, \dots, a_r$ of $n \times n$ matrices with entries in R.*
(ii) *Let $p(x_1, \dots, x_r)$ be a nonzero multilinear polynomial. If $r < 2n$, then there exist $n \times n$ matrices $a_1, \dots, a_r$ such that $p(a_1, \dots, a_r) \neq 0$. In particular, $S_r(x_1, \dots, x_r)$ is not identically zero.*

The identity $S_{2n} \equiv 0$ is called the *standard identity* of $n \times n$ matrices. Note that $S_2 \equiv 0$ is the commutative law, which holds for any 1×1 matrices but not for any $n \times n$ matrices for $n > 1$.

Remark 1.1.10. The Amitsur–Levitzki theorem is quite important (see [16]). Assume that study a representation $A \to M_n(k)$ of a k-algebra A. Let $I \subset A$ be the ideal generated by all substitutions of elements of A into S_{2n} and let $\tilde{A} = A/I$. The Amitsur–Levitzki theorem asserts that the relation $S_{2n} = 0$ is valid in $M_d(k)$ if $d \leqslant n$ whereas it is invalid for $d > n$. Killing I has the effect of keeping the representations of dimensions $\leqslant n$, and cutting out all irreducible representations of higher dimension.

The original proof of Theorem (1.1.9) obtained by Amitsur and Levitzki is a direct calculation, which is quite involved. Rosset proposed a short proof (see [173]) based on the exterior algebra of a vector space of dimension $2n$. This proof can be also found in [233, Theorem 1.7]. Then we obtain the following proposition.

Proposition 1.1.11. *Let $k\{X\}$ be the algebra of generic matrices.*

(a) *Every minimal polynomial of $A \in k\{X\}$ is irreducible. In particular, A is diagonalizable.*
(b) *Eigenvalues of $A \in k\{X\}$ are roots of irreducible minimal polynomial of A, and every eigenvalue the appears same amount of times.*
(c) *The characteristic polynomial of A is a power of the minimal polynomial of A.*

The following important open problem is well known.

Problem 1.1.12. For sufficiently large n, every nonscalar element in the algebra of generic matrices has a minimal polynomial, which always coincides with its characteristic polynomial.

This is an important open problem. For small n, the Galois group of an extension quotient field of the center of the algebra of generic matrices might not be symmetry. But it still unknown for sufficiently large n.

From Proposition 1.1.11(c), for a sufficiently large prime $n = p$, we can obtain the following assertion.

Corollary 1.1.13. *Let $k\{X\}$ be the algebra of generic matrices of a sufficiently large prime order $n := p$. Assume that A is a nonscalar element in $k\{X\}$. Then the minimal polynomial of A coincides with its characteristic polynomial.*

Proof. Let $m(A)$ and $c(A)$ be the minimal polynomial and the characteristic polynomial of A, respectively. Note that $\deg c(A) = n$ and $c(A) = (m(A))^k$. Since A is not scalar, $\deg m(A) > 1$. Since n is a prime, k divides n. Hence $k = 1$. ☐

Recall the following fact.

Proposition 1.1.14. *Two matrices with the same eigenvectors commute.*

Proof. Let $A, B \in M_{n\times n}(k)$ have the same n linearly independent eigenvectors. Since A and B are $n \times n$ matrices with n eigenvectors, they are diagonalizable and hence $A = Q^{-1}D_AQ$ and $B = P^{-1}D_BP$, where Q and P are matrices whose columns are eigenvectors of A and B associated with the eigenvalues listed in the diagonal matrices D_A and D_B, respectively. According to the assumption, A and B have the same eigenvectors and hence $P = Q =: S$. Therefore, $A = S^{-1}D_AS$ and $B = S^{-1}D_BS$ and hence

$$AB = S^{-1}D_ASS^{-1}D_BS = S^{-1}D_AD_BS;$$

similarly, we have $BA = S^{-1}D_BD_AS$. Since D_A and D_B are diagonal matrices, they commute and hence A and B also commute. ☐

Proposition 1.1.15. *If n is prime and A is a nonscalar element of the algebra of generic matrices, then all eigenvalues of A are pairwise different.*

Proposition 1.1.15 directly follows from Proposition 1.1.11 and Corollary 1.1.13.
Proposition 1.1.15 implies the following results.

Proposition 1.1.16. *Generic matrices commuting with A are simultaneously diagonalizable with A in the same eigenvectors basis as A.*

If A is a nonscalar matrix, then we have following assertion.

Proposition 1.1.17. *If A is a nonscalar element of the algebra of generic matrices $k\{X\}$, then each eigenvalue of A is transcendental over k.*

1.1.5. Deformation quantization.

1.1.5.1. Literature review. In general, the "quantization problem" can be stated as follows. Given a classical physical model (Hamiltonian system, Lagrange system on a Riemannian manifold, etc.), quantization amounts to replacing the observable functions with operators acting on a Hilbert space such that they satisfy some specific quantization conditions. In quantum mechanics, this quantization condition is called the *canonical commutation relation*, which is the fundamental relation between canonical conjugate quantities. For example, the commutation relation between different components of position and momentum can be expressed as $[\hat{P}_i, \hat{Q}_j] = i\hbar\delta_{ij}$, where i is the imaginary unit and δ_{ij} is the Kronecker delta. Hermann Weyl studied the Heisenberg uncertainty principle in quantum mechanics by considering the operator ring generated by P and Q. For any $2n$-dimensional linear space V, the Kronecker delta can be realized as a symplectic form ω such that $u \otimes v - v \otimes u = \omega(u, v)$ defines a Weyl algebra $W(V)$ over V. In this sense, classical mechanics corresponds to symmetric algebra, while the Weyl algebra is the "quantization" of symmetric algebra.

In the 1940s, J. E. Moyal (see [162]) conducted a more in-depth study of the Weyl quantization. Unlike Weyl, the object he was interested in was not operators, but the classical function space: Weyl ignores the Poisson structure of the classical function space. Instead of constructing a Hilbert space from a Poisson manifold and associating an algebra of operators to it, he was only concerned with the algebra. He used the star product and Moyal bracket to define a Poisson algebraic structure named Moyal algebra over the classical function space. Through the investigation of the Moyal algebra, F. Bayen et al. (see [32]) raised that the quantum algebra can be considered as a deformation of the classical algebra if we take $\hbar$ as the deformation parameter. In particular, they proved that for the classical Poisson algebraic structure on the symmetric algebra over $\mathbb{R}^{2n}$, the Moyal algebra is the only possible deformation in the sense of normative equivalence. That is, quantum mechanics is the only possible "deformation" of classical mechanics.

We use the Poisson bracket to "deform" the ordinary commutative product of observables in classical mechanics—elements of our function algebra—and obtain a noncommutative product suitable for quantum mechanics. In order to perform a deformation, we ask that the Moyal product is not only an asymptotic expansion, but also a real analytical expansion. There is no a priori guarantee for this. By the Darboux theorem, the local Poisson algebra structure on the symplectic manifold can always be deformed into the Moyal algebra. We only need to extend this local deformation to the whole manifold after equipping it with a flat symplectic connection. However, for a typical Poisson manifold, the situation is much more complicated.

In the mid 1970s, the existence of star-products for symplectic manifolds with trivial third cohomology group was proved, but this restriction turned out to be merely technical. In the early 1980s, the existence of star-products for wide classes of symplectic manifolds was proved, and finally it was shown that any symplectic manifold can be "quantized." Further generalizations were achieved with [98] where Fedosov proved that the results about the canonical star-product on an arbitrary symplectic manifold can be used to prove that all regular Poisson manifolds can be quantized. However, in physics we sometimes require manifolds that have a degenerate Poisson bracket and thus are not symplectic. Therefore, all the results mentioned above provided only a partial answer to the problem of quantization.

In 1993–1994, M. Kontsevich proposed the *Formality Conjecture* which would imply the desired result. If the Formality Conjecture could be proved, this would infer that any finite-dimensional Poisson manifold can be canonically quantized in the sense of deformation quantization. The Formality Conjecture is proved by Kontsevich in [128]. Kontsevich then derived an explicit quantization formula, which gives a formal definition of the Moyal product for any Poisson manifold. However, it is not clear whether it gives the only possible deformation quantization in the sense of canonical equivalence.

Another direction of developing researches in deformation quantization is strict deformation quantization, where the parameter is no longer a formal parameter, but a real one. In a way, the deformed algebras $A[\![\hbar]\!]$ are identified with the original algebra A.

1.1.5.2. Definitions and basic results.

Definition 1.1.18 (ring of formal power series)**.** Let R be a commutative ring with identity. $R[\![X]\!]$ is called the ring of formal power series in the variable X over R if and only if any element of $R[\![X]\!]$ has the form $\sum\limits_{i\in\mathbb{N}} a_i X^i$ and satisfies the following relations:

$$\sum_{i\in\mathbb{N}} a_i X^i + \sum_{i\in\mathbb{N}} b_i X^i = \sum_{i\in\mathbb{N}} (a_i + b_i) X^i, \tag{1.1.3}$$

$$\sum_{i\in\mathbb{N}} a_i X^i \times \sum_{i\in\mathbb{N}} b_i X^i = \sum_{n\in\mathbb{N}} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) X^n. \tag{1.1.4}$$

The above product (1.1.4) of coefficients is called the Cauchy product of the two sequences of coefficients; it is a kind of discrete convolutions. Note that the zero element and the multiplicative identity of the ring of formal power series are the same as in the ring R.

Remark 1.1.19. The series $A = \sum\limits_{n\in\mathbb{N}} a_n X^n \in R[\![X]\!]$ is invertible if and only if its constant coefficient a_0 is invertible in R. The inverse series of an invertible series A is $B = \sum\limits_{n\in\mathbb{N}} b_n X^n \in R[\![X]\!]$ with

$$b_0 = \frac{1}{a_0}, \quad b_n = -\frac{1}{a_0} \sum_{i=1}^{n} a_i b_{n-i}, \ n \geqslant 1.$$

An important example is the geometric series

$$(1 - X)^{-1} = \sum_{n=0}^{\infty} X^n.$$

If R is a field, then a series is invertible if and only if the constant term is nonzero.

Definition 1.1.20. A *Lie algebra* is a vector space with a skew-symmetric bilinear operation $(f, g) \to [f, g]$ satisfying the Jacobi identity

$$[[f, g], h] + [[g, h], f] + [[h, f], g] = 0.$$

Definition 1.1.21. A *Poisson algebra* is a vector space equipped with a structure of a commutative associative algebra $(f, g) \to fg$ and a Lie-algebra structure $(f, g) \to \{f, g\}$ satisfying the *Leibniz rule*

$$\{fg, h\} = f\{g, h\} + \{f, h\}g.$$

Definition 1.1.22. A *Poisson manifold* is a manifold M whose function space $C^\infty(M)$ is a Poisson algebra with the pointwise multiplication as the commutative product.

Let k be an arbitrary field and A be a unitary k-algebra. Denote by $k[\![\hbar]\!]$ the *ring of formal power series* in the indeterminate $\hbar$ and by $A[\![\hbar]\!]$ the $k[\![\hbar]\!]$-module of formal power series with coefficients in A.

Definition 1.1.23. A *formal deformation* or a *star product* of the algebra A is an associative, $\hbar$-adic continuous, $k[\![\hbar]\!]$ bilinear product

$$\star : A[\![\hbar]\!] \times A[\![\hbar]\!] \to A[\![\hbar]\!]$$

satisfying the following rule on A:

$$f \star g = \sum_{n=0}^{\infty} B_n(f, g)\hbar^n = fg + \sum_{n=1}^{\infty} B_n(f, g)\hbar^n \quad \forall f, g \in A, \tag{1.1.5}$$

where $B_n : A \times A \to A$ are bilinear operators.

Remark 1.1.24. We usually require that the bilinear operators B_n are bidifferential operators, i.e., bilinear maps that are differential operators with respect to each argument.

Remark 1.1.25. The formal deformation extends $k[\![\hbar]\!]$-linearity in $A[\![\hbar]\!]$ with respect to the bilinear product

$$\left(\sum_{k=0}^{\infty} f_k \hbar^k\right) \star \left(\sum_{m=0}^{\infty} g_m \hbar^m\right) = \sum_{n=0}^{\infty}\left(\sum_{m+k+r=n}^{\infty} B_r(f_k, g_m)\right)\hbar^n.$$

There is a natural gauge group acting on star-products. This group consists of automorphisms of $A[\![\hbar]\!]$ considered as an $k[\![\hbar]\!]$-module of the following form:

$$f \longmapsto f + \sum_{n=0}^{\infty} D_n(f)\hbar^n \qquad \forall f \in A \subset A[\![\hbar]\!],$$
$$\sum_{n=0}^{\infty} f_n \hbar^n \longmapsto \sum_{n=0}^{\infty} f_n \hbar^n + \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} D_m(f_m)\hbar^{n+m} \quad \forall \sum_{n=0}^{\infty} f_n\hbar^n \in A[\![\hbar]\!],$$

where $D_i : A \to A$ are differential linear operators.

Definition 1.1.26. The operators $D(\hbar)$ defined above are called *gauge transformations* in A. The set of all such operators is naturally a group.

If $D(\hbar) = 1 + \sum\limits_{m=1}^{\infty} D_m \hbar^m$ is such an automorphism, then it defines an equivalence and acts on the set of star products as follows:

$$\star \mapsto \star', \quad f(\hbar) \star' g(\hbar) := D(\hbar)(D(\hbar)^{-1}(f(\hbar)) \star D(\hbar)^{-1}(g(\hbar))) \quad \forall f(\hbar), g(\hbar) \in A[\![\hbar]\!]. \tag{1.1.6}$$

Each associative formal deformation $\star$ of the multiplication of A admits a unit element $1_\star$. Moreover, such an associative formal deformation $\star$ is always equivalent to another formal deformation $\star'$ with $1_{\star'} = 1_A$, where 1_A is the unit element of A. We are interested in star products up to gauge equivalence.

The following lemma gives a Poisson structure for an associative formal deformation of the multiplication of an associative and commutative k-algebra A.

Lemma 1.1.27 (see [124, Lemma 1.1])**.** *Let $\star$ be an associative formal deformation of the multiplication of an associative and commutative k-algebra A. For $f, g \in A$, we set*

$$\{f, g\} := B_1(f, g) - B_1(g, f).$$

Then the map $\{\cdot,\cdot\}$ is a Poisson bracket on A, i.e., a k-linear map such that the bracket is a Lie bracket satisfying the Leibniz rule. In addition, the bracket is dependent only on the equivalence class of $\star$.

Proof. For brevity, we denote by $[f, g]_\star$ the commutator of the star product $f \star g - g \star f$. Clearly, the map

$$(f, g) \mapsto \frac{1}{\hbar}[f, g]_\star \tag{1.1.7}$$

defines a Lie bracket on $A[\![\hbar]\!]$. The bracket $\{\cdot,\cdot\}$ is equal to the reduction of this Lie bracket modulo $\hbar$, i.e., it satisfies the relation

$$\frac{1}{\hbar}[f, g]_\star \equiv \{f, g\} \mod \hbar A[\![\hbar]\!]. \tag{1.1.8}$$

We write this formula as follows:

$$\{f, g\} := \left.\frac{[f, g]_\star}{\hbar}\right|_{\hbar=0} = B_1(f, g) - B_1(g, f). \tag{1.1.9}$$

Therefore, the bracket $\{\cdot,\cdot\}$ is still a Lie bracket, and it also satisfies the Leibniz rule since the Lie bracket defined in (1.1.7) satisfies the associativity rule of the star product.

Assume that $D(\hbar)$ is an automorphism, which yields equivalence of $\star$ and $\star'$. Then we have

$$B_1(f,g) + D_1(fg) = B_1'(f,g) + D_1(f)g + fD_1(g)$$

for all $f, g \in A$. Thus, the difference $B_1(f,g) - B_1'(f,g)$ is symmetric in f and g and does not contribute to $\{\cdot,\cdot\}$. □

One can also decompose the operator B_1 into the sum of the symmetric and anti-symmetric parts:

$$B_1 = B_1^+ + B_1^-, \quad B_1^+(f,g) = B_1^+(g,f), \quad B_1^-(f,g) = -B_1^-(g,f).$$

Then gauge automorphisms affect only the symmetric part of B_1, i.e., $B_1^- = (B_1')^-$. The symmetric part is annihilated by gauge automorphisms. In this notation, we may write

$$\{f,g\} = B_1(f,g) - B_1(g,f) = 2B_1^-(f,g).$$

Thus, gauge equivalence classes of star products modulo $\hbar^2 A[\![\hbar]\!]$ are classified by Poisson structures. However, it is not clear whether there exists a star product for a given Poisson structure. Moreover, we may ask whether there exists a preferred choice of an equivalence class of star products. As we mentioned above, M. Kontsevich showed (see [128]) that there is a canonical construction of an equivalence class of star products for any Poisson manifold.

1.1.5.3. Formal deformation quantization. In this section, we may assume that A is the algebra of smooth functions on a Poisson manifold M.

Definition 1.1.28. A *deformation quantization* of a Poisson manifold M is a star product on A such that $2B_1^- = \{\cdot,\cdot\}$.

We do not reproduce Kontsevich's proof here and do not deal with it below. His proof is based on the cohomology of the Hochschild complex. By the following theorem, there is a surjection from the equivalence classes of formal deformations of A onto Poisson brackets on A.

Theorem 1.1.29 (M. Kontsevich, [128])**.** *Let M be a smooth manifold and $A = C^\infty(M)$. Then there is a natural isomorphism between equivalence classes of deformations of the null Poisson structure on M and equivalence classes of smooth deformations of the associative algebra A.*

In particular, any Poisson bracket on M comes from a canonically defined (*modulo equivalence*) *star product.*

Moreover, Kontsevich constructed a section of map, and his construction is canonical up to equivalence for general manifolds. A later result shows that, in addition to the existence of a canonical way of quantization, we can define a universal infinite-dimensional manifold parametrizing quantizations.

The simplest example of a deformation quantization is the Moyal product for the Poisson structure on $\mathbb{R}^n$. This is the first known example of a nontrivial deformation of the Poisson bracket.

Example 1.1.30. Let $M = \mathbb{R}^n$. We consider a Poisson structure with constant coefficients

$$\alpha = \sum_{i,j} \alpha^{ij} \partial_i \wedge \partial_j, \quad \alpha^{ij} = -\alpha^{ji} \in \mathbb{R},$$

where $\partial_i = \partial/\partial x^i$ is the partial derivative with respect to the coordinate x^i, $i = 1, 2, \ldots, n$. In this we have

$$\{f,g\} = \sum_{i,j} \alpha^{ij} \partial_i(f) \partial_j(g).$$

The Moyal $\star$-product is then given by exponentiating this Poisson operator:

$$f \star g = e^{\hbar\alpha}(f,g) = fg + \hbar \sum_{i,j} \alpha^{ij} \partial_i(f)\partial_j(g) + \frac{h^2}{2} \sum_{i,j,k,l} \alpha^{ij}\alpha^{kl} \partial_i\partial_k(f)\partial_j\partial_l(g) + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \sum_{\substack{i_1,\ldots,i_n \\ j_1,\ldots,j_n}} \prod_{k=1}^{n} \alpha^{i_k j_k} \prod_{k=1}^{n} \partial_{i_k}(f) \prod_{k=1}^{n} \partial_{j_k}(g).$$

The Moyal product is a deformation of (M, α), but this formula is only valid when α has constant coefficients.

In particular, consider the following example.

Example 1.1.31. Let $M = \mathbb{R}^2$. Consider the Poisson bracket

$$\{f, g\} = \mu \circ \left(\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}\right)(f \otimes g) = \frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_2} - \frac{\partial f}{\partial x_2}\frac{\partial g}{\partial x_1},$$

where μ is the multiplication of functions on M. Then Kontsevich's construction yields the associative formal deformation given by

$$f \star g = \sum_{n=0}^{\infty} \frac{\partial^n f}{\partial x_1^n}\frac{\partial^n g}{\partial x_2^n}\frac{\hbar^n}{n!}.$$

The explicit construction of Kontsevich's formal quantization is based on some combinatoric tools such as quivers. We complete this section here since we do not need to construct an explicit formula of deformation quantization in our proof.

1.1.6. Algebraically closed skew field. The role of algebraically closed fields in commutative algebra is well known. There are some parallel generalizations of the concept of an algebraically closed skew field to noncommutative skew fields have proved useful for settling various questions in the ring theory. However, there are various definitions. The diversity of definitions of algebraically closed skew fields is based on different choices of some particular characteristic of a commutative algebraically closed field. The most natural generalization is in the sense of solvability of arbitrary equations which was brought in sight by Bokut (see [57–59]). In particular, in [59] Bokut raises a question whether algebraically closed skew fields exist or not. The positive answer to this question was given by L. Makar-Limanov (see [144]). His result is one of the fundamental contributions to the theory of noncommutative, algebraically closed skew fields. In [64], P. M. Cohn outlined a wide research program for skew fields that are algebraically closed in the various senses. Note that not every associative algebra can be embedded into an algebraically closed algebra, in the sense of solvability of arbitrary equations. For example, the "metro-equation" $ax - xa = 1$ (cf. [63]) is never solvable in any extension of the quaternionic skew field. In [126], P. S. Kolesnikov reproved the Makar-Limanov theorem on the existence of an algebraically closed skew field in the sense of solution for any generalized polynomial equation. He used a simpler argument for proving that the skew field constructed is algebraically closed.

1.1.6.1. Existence of algebraically closed skew field. We construct a noncommutative skew field A satisfying the following definition (cf. [126]).

Definition 1.1.32. A skew field A with center F is said to be *algebraically closed* if, for any $S(x) \in A * F[x] \setminus A$, there exists an element $a \in A$ such that $S(a) = 0$; here, $*$ stands for a free product.

It is easy to see that if A is a field, that is, $A = F$, then Definition 1.1.32 checks with the usual definition of an algebraically closed field.

Let F be an algebraically closed field of characteristic 0 and G be the commutative group generated by the elements

$$p_1^{\lambda_1},\ q_1^{\mu_1},\ p_2^{\lambda_2},\ q_2^{\mu_2},\ \ldots,$$

where $\lambda_i, \mu_i \in \mathbb{Q}$ and p_1 and q_i are symbols in some countable alphabet. The group is isomorphic to the direct sum of countably many additive groups $\mathbb{Q}$ of rational numbers. Then we introduce the lexicographic order on G by setting

$$p_1 \ll q_1 \ll p_2 \ll \cdots < 1,$$

where $a \ll b$ means that $a^n < b$ for all $n > 0$. Respectively,

$$p_1^{-1} \gg q_1^{-1} \gg p_2^{-1} \gg \cdots > 1.$$

We set

$$G_n = \langle p_n^{\lambda_n},\ q_n^{\mu_n},\ p_{n+1}^{\lambda_{n+1}},\ \dots,\ bb\rangle, \quad G_{(m)} = \langle p_1^{\lambda_1},\ q_1^{\mu_1},\ \dots,\ q_m^{\lambda_m}\rangle.$$

Obviously, G_n is isomorphic to G.

For given G and F, we construct a set $\mathcal{A}$ of Maltsev–Neumann series. Elements $a \in \mathcal{A}$ has the form

$$a = \sum_{g \in H_a} a(g)g, \quad H_a \in G \text{ is well ordered}, \quad a(g) \in F \setminus \{0\},$$

the set H_a is denoted by $\operatorname{supp} a$. Choose a subset A of $\mathcal{A}$ such that

$$A = \{a \in \mathcal{A} \mid \operatorname{supp} a \subset G_{(n(a))}\}.$$

Respectively, we set

$$A_n = \{a \in A \mid \operatorname{supp} a \subset G_n\}, \quad A_{(n)} = \{a \in A \mid \operatorname{supp} a \subset G_{(n)}\}.$$

The set A constructed is exactly the universe of the desired algebraic system. For the series on A_n, we define ordinary addition and multiplication, and also derivations $\left(\dfrac{\partial}{\partial p_n}, \dfrac{\partial}{\partial q_n}\right)$. Derivatives of the elements $g \in G_n$ with respect to p_1 and q_1 are elements of A_n. There are several formulas for these derivations (we omit them here). Following [126, 144], we define a multiplication $*$ on A as follows:

$$a, b \in A, \quad a * b = \sum_{i \geqslant 0} \frac{1}{i!} \frac{\partial^i a}{\partial q_1^i} \frac{\partial^i b}{\partial p_1^i}.$$

The multiplication $*$ is well defined and associative (cf. [126]). Then the system $\langle A, +, *, ||\rangle$ is an associative algebra with valuation. That this is a skew field follows from the fact that $a * x = 1$ has a solution in A. Moreover, A does not satisfy any generalized polynomial identity, i.e., for every nontrivial generalized polynomial $S(x) \in A * F[x] \setminus A$, there exists an element $a \in A$ such that $S(a) \neq 0$ (cf. [126, Lemma 1.3]).

We introduce the following notion, which generalizes the concept of an homogeneous polynomial in $A * F[x]$.

Definition 1.1.33. An operator

$$S_n(x) = \sum_{\imath, \jmath} f_{\imath,\jmath} x^{(i_1, j_1)} \dots x^{(i_k, j_k)},$$

where $\imath = (i_1 \dots i_k)$, $\jmath = (j_1 \dots j_k)$, $f_{\imath,\jmath} \in A_n$, and x is a common element in A_n, is called a *homogeneous operator* over A_n if the following conditions hold:

(i) there exists m such that $f_{\imath,\jmath} \in A_{(m)}$ for all $\imath, \jmath$;

(ii) for any $g \in G_n$ and $x \in A_n$, the following inequality holds only for finitely many summands in $S_n(x)$:

$$|f_{\imath,\jmath} x^{(i_1, j_1)} \dots x^{(i_k, j_k)}| \leqslant g;$$

(iii) all summands have the same degree $\deg S_n(x) = k$ over x.

In [126], Kolesnikov solved the equation $|S(x)| = g$ and found that $S_1(x) = f_1$. His proof contains a modification of original Makar-Limanov's proof, which is useful since it compensates for this loss by making the argument for algebraic closedness much easier.

In [127], Kolesnikov shows that each polynomial equation containing more than one homogeneous component over such a skew field necessarily has a nonzero solution. Precisely, he proved the following assertion.

Proposition 1.1.34 (see [127, Theorem 1])**.** *Let $S_i(x)$, $i = 1, \dots, n$, be homogeneous operators over A, where $n \geqslant 1$, and $T(x)$ be a homogeneous operator such that $\deg S_i < \deg T$. Then the equation $\sum_i S(x) = T(x)$ has a solution $x \in A$, $x \neq 0$.*

1.1.6.2. Algebraically closed skew field in the sense of matrices. Another conception of algebraic closedness is associated with the notion of singular eigenvalues of matrices. The definitions are given in [64].

Let D be a skew field with center k. Denote by $M_n(D)$ the ring of all $(n \times n)$-matrices over D. A matrix $A \in M_n(D)$ is said to be *singular* if there exists a nonzero column $u \in D^n$ such that $Au = 0$. A square matrix is singular if and only if it is not invertible. The property of being singular for a matrix is preserved under left or right multiplication by an invertible matrix, in particular, under elementary transformations of columns with coefficients from the skew field on the right, and of rows on the left.

An element $\lambda \in D$ is called a *singular eigenvalue* of A if $A - \lambda I$ is a singular matrix. It is worth mentioning that singular eigenvalues of matrices are not always preserved under similarity transformations, but central eigenvalues are invariant in this sense.

The following definition of algebraically closed skew field is due to P. Cohn (see [64]).

Definition 1.1.35. A skew field D is said to be *algebraically closed* in the sense of Cohn (notation AC) if every square matrix over D has a singular eigenvalue in this skew field. A skew field D is said to be *fully algebraically closed* (notation FAC) if every matrix $A \in M_n(D)$, which is not similar to a triangular matrix over the center of D, has a nonzero singular eigenvalue in D.

The definition of FAC skew field is equivalent to the following definition.

Definition 1.1.36. A skew field D is fully algebraically closed if every matrix $A \in M_n(D)$, which is not nilpotent, has a nonzero singular eigenvalue in D.

Consequently, if A is similar to a triangular matrix over the center of D, then either it is nilpotent or has a nonzero eigenvalue. Conversely, if A is nilpotent, then it is similar to its canonical form containing only 1 on a secondary diagonal.

Definition 1.1.37. We say that D is an AC_n (respectively, FAC_n) skew field if every (non-nilpotent) matrix $A \in M_m(D)$, $m \leqslant n$, has a nonzero eigenvalue in D.

Proposition 1.1.38 (see [127, Theorem 2])**.** *Let D be an FAC_n skew field and $a_i, b_i, c \in D$, $i = 1, \dots, n$. Then the equation*

$$L_n(x) = \sum_{i=1}^{n} a_i x b_i = c$$

has a solution $x = x_0 \in D$ if $L_n(x) \equiv 0$ for all $x \in D$.

1.2. Algebra Automorphisms and Quantization

In this section, $\mathbb{F}$ denotes a ground field with zero characteristic.

One of the main objects of the theory of polynomial mappings are Ind-schemes whose points are automorphisms of various algebras with polynomial identities, for example, algebras of commutative polynomials $\mathbb{F}[x_1, \dots, x_n]$ of n variables, free algebras $\mathbb{F}\langle x_1, \dots, x_n\rangle$ of n generators, some selected quotients, as well as algebras with additional structures; a famous example is the polynomial algebra equipped with the Poisson bracket. This area of research has its roots in the well-known Jacobian conjecture. Due to the relatively recent progress of Belov-Kanel and Kontsevich [41,42] and Y. Tsuchimoto [190, 191]), as well as earlier studies, a significant place in the scientific program regarding the Jacobian conjecture has come to be occupied by questions related to the quantization of classical algebras.

The study of geometry and topology of Ind-schemes of automorphisms, development of the approximation theory of symplectomorphisms by tame symplectomorphisms, as well as the construction of a correspondence between plane algebraic curves and holonomic modules (over the corresponding the Weyl algebra), are the basis of the approach to the conjecture of Belov-Kanel and Kontsevich on automorphisms of the Weyl algebra formulated by A. Elishev, A. Kanel-Belov, and J. T. Yu in [115,116] (cf. also [86, 234]).

1.2.1. Jacobian conjecture. One of the most well-known unsolved problems in the theory of polynomials in several variables is the so-called *Jacobian conjecture* formulated in 1939 by O. H. Keller (see [125]). Let $\mathbb{F}$ be the main field. For a fixed positive integer n, consider n polynomials

$$f_1(x_1, \dots, x_n), \quad \dots, \quad f_n(x_1, \dots, x_n)$$

of n variables $x_1, \dots, x_n$. Any such system of polynomials defines a unique image *endomorphism* of the algebra $\mathbb{F}[x_1, \dots, x_n]$ as follows:

$$F : \mathbb{F}[x_1, \dots, x_n] \to \mathbb{F}[x_1, \dots, x_n],$$
$$F \leftrightarrow (F(x_1), \dots, F(x_n)) \equiv (f_1(x_1, \dots, x_n), \dots, f_n(x_1, \dots, x_n).$$

The $\mathbb{F}$-endomorphism F of polynomial algebra is determined by its action on the set of generators. Let $J(F)$ denote the *Jacobian* (the determinant of the Jacobi matrix) of the map F:

$$J(F) = \det \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_n}{\partial x_1} & \dots & \dfrac{\partial f_n}{\partial x_n} \end{bmatrix}$$

Conjecture 1.2.1 (Jacobian conjecture JC_n)**.** Let the characteristic of the base field $\mathbb{F}$ be equal to zero. If the Jacobian $J(F)$ of the endomorphism F is equal to a nonzero constant (that is, it belongs to the set $\mathbb{F}^*$), then F is an automorphism.

An elementary exercise is to verify the statement that automorphisms of polynomial algebras always have a nonzero Jacobian constant. Conjecture 1.2.1 is thus a partially inverse statement of this property. It is also easy to see that if a polynomial endomorphism F is invertible, then the inverse will also be a polynomial endomorphism.

The Jacobian conjecture is trivial for $n = 1$. On the other hand, when the field $\mathbb{F}$ has positive characteristic, the Jacobian conjecture formulated as Conjecture 1.2.1 is invalid even in the case $n = 1$. Indeed, if $\operatorname{char} \mathbb{F} = p$ and $n = 1$, we can take $\varphi(x) = x - x^p$. The Jacobian of this mapping is equal to unity, but it is not invertible.

Despite the apparent simplicity of wording and context, the Jacobian conjecture is one of the most difficult open questions of modern algebraic geometry. This problem has become the subject of numerous studies and has greatly contributed to the development of related fields of algebra, algebraic geometry, and mathematical physics, which are also of independent interest.

The literature on the Jacobian conjecture, its analogs, and related problems is quite extensive. A detailed discussion of the results established in the context of the Jacobian conjecture is beyond the scope of this work. Below we give a brief overview of some results directly related to the Jacobian conjecture (i.e., for the algebra of polynomials in commuting variables). Among studies of topics similar to the Jacobian conjecture in associative algebra, it is worth noting the work of W. Dicks [73] and W. Dicks and J. Lewin [74] on an analog of the Jacobian conjecture for free associative algebras, the proof by U. U. Umirbaev of an analog of the Jacobian Conjecture for the free metabelian algebra (see [193]), and the deep and extremely significant works of A. V. Yagzhev [217–220] (see also [35]).

1.2.2. Some results related to the Jacobian conjecture. While the general case of the Jacobian conjecture (or even the Jacobian conjecture on the plane) remains an open problem, various partial results are known. We recall several such results.

In [215], S. S. Wang established the Jacobian conjecture for the case of endomorphisms defined by polynomials of degree 2. Also, H. Bass, E. H. Connell, and D. Wright (see [21]) showed that the general case of the Jacobian conjecture would follow from the special case of the Jacobian conjecture for the so-called endomorphisms of homogeneous cubic type, which are defined as mappings of the form

$$(x_1, \dots, x_n) \mapsto (x_1 + H_1, \dots, x_n + H_n),$$

where the polynomials H_k are homogeneous of degree 3.

Moreover, L. M. Drużkowski proved (see [83]) that the previous hypothesis can be weakened, by considering as H_k only polynomials that are cubes of linear homogeneous polynomials.

In the works of M. de Bondt and A. van den Essen (see [69, 70]) and Drużkowski (see [84]), it was shown that it suffices to establish the Jacobian conjecture for endomorphisms of homogeneous cubic type with a symmetric Jacobi matrix.

As above, we assume that the polynomial endomorphism F is given by the set of the images of generators:

$$F \leftrightarrow (F(x_1), \dots, F(x_n)) \equiv (F_1, \dots, F_n).$$

Then F is invertible if and only if the algebras

$$\mathbb{F}[x_1, \dots, x_n] \quad \text{and} \quad \mathbb{F}[F_1, \dots, F_n]$$

are isomorphic. In [125], Keller considered a rational analog of thte given criterion, i.e., the case of an isomorphism of function fields

$$\mathbb{F}(x_1, \dots, x_n) \quad \text{and} \quad \mathbb{F}(F_1, \dots, F_n)$$

and the invertibility following from the existence of an isomorphism is established by L. A. Campbell (see [61]). A generalization of Keller's original result to the case where $\mathbb{F}(x_1, \dots, x_n)$ is a Galois extension of the field $\mathbb{F}(F_1, \dots, F_n)$ (see also the works [171] of M. Razar and [216] of D. Wright generalizing the result mentioned).

In addition, some efforts were aimed at testing the fulfillment of the Jacobian conjecture for all endomorphisms defined by polynomials of degree not higher than some fixed number. T. T. Moh [161] performed a similar test for polynomials of two variables of degree not exceeding 100.

Despite the existence of the results described above (as well as some other similar theorems), the general case of the Jacobian conjecture remains not only open, but, apparently, at the moment irrefutable.

On the other hand, there are situations in which mappings, by their geometric properties close to polynomial endomorphisms, are nevertheless not invertible. S. Yu. Orevkov (see [164]) pointed to the following reformulation of the Jacobian conjecture, leading to a similar situation. Let l be an infinitely distant line in the complex projective plane $\mathbb{C}P^2$, U be its tubular neighborhood, f_1 and f_2 are meromorphic functions on U, holomorphic on $U \backslash l$ and defining a locally bijective mapping

$$F : U \backslash l \to \mathbb{C}^2.$$

The Jacobian conjecture is equivalent to the statement about the injectivity of mappings of this kind. Orevkov constructed the following example.

Theorem 1.2.2 (S. Yu. Orevkov [164]). *There is a smooth, noncompact, complex analytic surface $\tilde{X}$ on which there is a smooth curve $\tilde{L}$ isomorphic to the projective line, with the self-intersection index $+1$ and two functions f_1 and f_2 meromorphic on $\tilde{X}$ and holomorphic on $\tilde{X} \backslash \tilde{L}$ such that the mapping defined by*

$$F : \tilde{X} \backslash \tilde{L} \to \mathbb{C}^2$$

is locally bijective but not injective.

As was noted in [164], if $\tilde{U}$ is a tubular neighborhood of the curve $\tilde{L}$, then the pairs (U, l) (as above) and $(\tilde{U}, \tilde{L})$ are diffeomorphic, which implies the existence of a smooth immersion in a two-dimensional complex exterior space of a ball which is geometrically similar to a polynomial map and noninvertible. Also, if the pairs (U, l) and $(\tilde{U}, \tilde{L})$ were biholomorphic to each other, then from Orevkov's example one would derive the existence of a counterexample to the Jacobian conjecture. This consideration allows one to conclude that one can be fairly certain about the negativity of the Jacobian conjecture.

In his classic work [8], Anick developed a theory of approximation of polynomial automorphisms by tame automorphisms. In connection with Anick's theorem on approximation, the Jacobian conjecture can be reduced to solving the question for the invertibility of limits of sequences of tame automorphisms. Moreover, a symplectic analogue of Anick's theorem gives a natural (requiring nontrivial Abelian extensions) idea to solve the lifting type problems.

The central question in the approach to the Jacobian conjecture type problems based on approximation by sequences of tame automorphisms is the proof of the polynomial nature of the resulting limit. While in the case of the lifting of symplectomorphisms the proof of the correctness of the construction seems to be possible (a significant role in it is played by the invertibility of the sequence limits, which is obviously not the case for the Jacobian conjecture). In the context of the Jacobian conjecture there is no clarity in the matter, and considerations following from [164], indicate possible significant obstacles.

The Jacobian conjecture was studied by the methods of covering groups by S. Yu. Orevkov [163,165] and A. G. Vitushkin [205, 206]. The Jacobian conjecture was also the subject of significant works [134, 135] of Vik. S. Kulikov.

A number of other difficult problems in the theory of polynomial automorphisms are closely connected with the Jacobian conjecture and with affine algebraic geometry. These problems are important in the general mathematical context. For example, a special case of the classical Abyankar–Sataye conjecture [168, 232] posits isomorphisms of all embeddings of the complex affine line into three-dimensional space (in other words, it is a conjecture about the possibilities of formally algebraic definition of the knot).

1.2.3. Ind-schemes and varieties of automorphisms. One of the essential areas of algebraic geometry, the development of which was motivated by the Jacobian Conjecture is the theory of infinite-dimensional algebraic groups. The main reference is the seminal article of I.R. Shafarevich [177], in which he defined concepts that allowed one to study questions about some natural infinite-dimensional groups – for example, the group of automorphisms of an algebra of polynomials in several variables – using tools from algebraic geometry. In particular, Shafarevich defines *infinite-dimensional varieties* as inductive limits of directed systems of the form

$$\{X_i, f_{ij},\ i, j \in I\},$$

where X_i are algebraic varieties (more generally, algebraic sets) over a field $\mathbb{F}$, and the morphisms f_{ij} (defined for $i \leqslant j$) are closed embeddings. The inductive limit of a system of topological spaces carries a natural topology, and therefore the natural questions about connectivity and irreducibility arise, which were also studied in [177].

Following generally accepted terminology, we will call the direct limit of systems of varieties and closed embeddings an Ind-*variety*, and the corresponding limits of systems of schemes and morphisms of schemes an Ind-*scheme*.

The Jacobian Conjecture has the following elementary connection with Ind-schemes. Since the algebra of polynomials $\mathbb{F}[x_1, \ldots, x_n]$ can be endowed with a natural $\mathbb{Z}$-grading in total degree deg, which is defined as the appropriate monoid homomorphism by the requirement $\deg x_i = 1$, we can define the degree of endomorphism φ: namely, if $\varphi = (\varphi(x_1), \ldots, \varphi(x_n))$ defined by its action on algebra generators, then the degree $\deg \varphi$ is the maximum value of degree on the polynomials $\varphi(x_1), \ldots, \varphi(x_n)$. It defines an increasing filtration

$$\operatorname{End}^{\leqslant N} \mathbb{F}[x_1, \ldots, x_n],\ N \geqslant 0$$

on the set $\operatorname{End} \mathbb{F}[x_1, \ldots, x_n]$ of endomorphisms of the polynomial algebra. Points

$$\operatorname{End}^{\leqslant N} \mathbb{F}[x_1, \ldots, x_n]$$

are endomorphisms of degree at most N. It is easy to see that the algebraic sets

$$\operatorname{End}^{\leqslant N} \mathbb{F}[x_1, \ldots, x_n]$$

are isomorphic to affine spaces of appropriate dimension. The coordinates of the point φ are the coefficients of the polynomials $\varphi(x_1), \ldots, \varphi(x_n)$, and for

$$\operatorname{End} \mathbb{F}[x_1, \ldots, x_n]$$

these coordinates are not connected by any relations.

The total degree filtration also enables endowing the sets of automorphisms with the Zariski topology as follows (see also [177]): if φ is a polynomial automorphism, then consider a set of polynomials

$(\varphi(x_1), \ldots, \varphi(x_1), \varphi^{-1}(x_1), \ldots, \varphi^{-1}$
$(x_n))$, the images of generators under the action of the automorphism and its inverse. The coefficients of these polynomials serve as coordinates of φ as a point of some affine space.

Define the subsets

$$\operatorname{Aut}^{\leqslant N} \mathbb{F}[x_1, \ldots, x_n] = \{\varphi \in \operatorname{Aut} \mathbb{F}[x_1, \ldots, x_n] : \deg \varphi, \deg \varphi^{-1} \leqslant N\}$$

as sets of automorphisms such that all coefficients of polynomials in the presentation above for degrees greater than n are zero.

The sets $\operatorname{Aut}^{\leqslant N} \mathbb{F}[x_1, \ldots, x_n]$ are algebraic sets. Indeed, the identities that define the points $\operatorname{Aut}^{\leqslant N}$ are derived from the identity

$$\varphi \circ \varphi^{-1} = \mathrm{Id}$$

and, it is easy to see, are specified by polynomials.

Now let $\mathfrak{J}^{\leqslant N}$ denote a subset of

$$\operatorname{End}^{\leqslant N} \mathbb{F}[x_1, \ldots, x_n],$$

whose points are endomorphisms with a Jacobian equal to a nonzero constant.

Then Conjecture 1.2.1 can be clearly reformulated as follows

$$\forall \varphi \in \mathfrak{J}^{\leqslant N} \Rightarrow \varphi \in \operatorname{Aut} \mathbb{F}[x_1, \ldots, x_n], \ \forall N, \ \text{for} \ \operatorname{char} \mathbb{F} = 0.$$

1.2.4. Conjecture of Dixmier and quantization. J. Dixmier [75] in his seminal study of Weyl algebras found a connection between the Jacobian Conjecture and the following Conjecture. Let $W_{n,\mathbb{F}}$ denote the *n-th Weyl algebra* over the field $\mathbb{F}$ defined as the quotient algebra of the free algebra

$$F_{2n} = \mathbb{F}\langle a_1, \ldots, a_n, b_1, \ldots, b_n\rangle$$

of $2n$ generators by the two-sided ideal I_W, generated by polynomials

$$a_i a_j - a_j a_i, \ \ b_i b_j - b_j b_i, \ \ b_i a_j - a_j b_i - \delta_{ij} \ \ (1 \leqslant i, j \leqslant n),$$

where δ_{ij} is the Kronecker symbol. The Dixmier Conjecture states:

Conjecture 1.2.3 (Dixmier Conjecture, DC_n)**.** Let $\operatorname{char} \mathbb{F} = 0$. Then $\operatorname{End} W_{n,\mathbb{F}} = \operatorname{Aut} W_{n,\mathbb{F}}$.

In other words, the Dixmier Conjecture asks whether every endomorphism of the Weyl algebra over a field of characteristic zero is in fact an automorphism.

The Dixmier Conjecture for n variables, DC_n, implies the Jacobian Conjecture JC_n for n variables (see, for example, [202]). Significant progress in recent years in the study of Conjecture 1.2.1 has been achieved by Kanel-Belov (Belov) and Kontsevich [42] — and independently by Tsuchimoto [191] (see also [190]) — in the form of the following theorem.

Theorem 1.2.4 (A. Ya. Kanel-Belov and M. L. Kontsevich [42], Y. Tsuchimoto [191])**.** *JC_{2n} implies DC_n.*

In particular, Theorem 1.2.4 implies the stable equivalence of the Jacobian Conjecture and the Dixmier Conjecture, i.e., the equivalence of conjectures JC_∞ and DC_∞, where JC_∞ denotes the conjunction corresponding conjectures for all finite n.

Theorem 1.2.4 laid the foundation for the research into the Jacobian Conjecture based on the study of the behavior of varieties of endomorphisms and automorphisms of algebras under deformation quantization. The principal reference in this direction is an article by Kanel-Belov and Kontsevich [41], in it, several conjectures concerning Ind-varieties of automorphisms of the corresponding algebras are formulated. The main Conjecture is called the Kontsevich Conjecture and is as follows.

Conjecture 1.2.5 (Kontsevich Conjecture, [41])**.** Let $\mathbb{F} = \mathbb{C}$ be the field of complex numbers. The automorphism group $\operatorname{Aut} W_{n,\mathbb{C}}$ of the n-th Weyl algebra over $\mathbb{C}$ is isomorphic to the automorphism group $\operatorname{Aut} P_{n,\mathbb{C}}$ of the so-called n-th (commutative) Poisson algebra $P_{n,\mathbb{C}}$:

$$\operatorname{Aut} W_{n,\mathbb{C}} \simeq \operatorname{Aut} P_{n,\mathbb{C}}.$$

We have to note that the first solution of Kontsevich Conjecture 1.2.5 is given by Christopher Dodd in [76].

The algebra $P_{n,\mathbb{C}}$ is by definition the polynomial algebra

$$\mathbb{C}[x_1, \dots, x_n, p_1 \dots, p_n]$$

of $2n$ variables, equipped with the Poisson bracket, i.e., a bilinear operation $\{\ ,\ \}$, which is a Lie bracket satisfying the Leibniz rule and acting on generators of the algebra in the following way:

$$\{x_i, x_j\} = 0, \ \ \{p_i, p_j\} = 0, \ \ \{p_i, x_j\} = \delta_{ij}.$$

Endomorphisms of the algebra P_n are endomorphisms of the algebra of polynomials that preserve the Poisson bracket (which we sometimes call the Poisson structure). Elements of $\operatorname{Aut} P_{n,\mathbb{C}}$ are called polynomial symplectomorphisms. The choice of name is due to the existence of an (anti-) isomorphism between the group $\operatorname{Aut} P_{n,\mathbb{C}}$ and the group of polynomial symplectomorphisms of the affine space $\mathbb{A}^{2n}$.

The Kontsevich Conjecture is true for $n = 1$. The proof of this result is a direct description of automorphism groups $\operatorname{Aut} P_{1,\mathbb{C}}$ and $\operatorname{Aut} W_{1,\mathbb{C}}$, contained in the classical works of H.W. Jung [103], Van der Kulk [203], Dixmier [75] and Makar-Limanov [142] (see also [141]). Namely, consider the following transformation groups: the group G_1 is a semidirect product

$$\mathrm{SL}(2, \mathbb{C}) \rtimes \mathbb{C}^2,$$

whose elements are called special affine transformations, and the group G_2 by definition consists of the following "triangular" substitutions:

$$(x, p) \mapsto (\lambda x + F(p), \lambda^{-1} p), \ \ \lambda \in \mathbb{C}^\times, \ \ F \in \mathbb{C}[t].$$

Then the automorphism group of the algebra $P_{1,\mathbb{C}}$ [103] is isomorphic to the quotient group of the free product of the groups G_1 and G_2 by their intersection. Dixmier [75] and, later, Makar-Limanov [142] showed that if in the description above one replaces the commuting Poisson generators with their quantum (Weyl) analogues, one obtains a description of the group of automorphisms of the first Weyl algebra $W_{1,\mathbb{C}}$.

Remark 1.2.6. The theorems of Jung, van der Kulk, Dixmier and Makar-Limanov also mean that all automorphisms of the polynomial algebra of two variables and the first Weyl algebra W_1 are *tame* (we provide the definition of the concept of tame automorphism, in the Subsection 1.2.5). Also, Makar-Limanov [141] and A. Czerniakiewicz [66,67] proved that all automorphisms of the free algebra $\mathbb{K}\langle x, y\rangle$ are tame.

In view of these circumstances, the case of two variables is to be considered exceptional. However, the Jacobian Conjecture is a difficult open problem even in this case.

Recently, Kanel-Belov, together with Elishev and Yu, have suggested a proof of the general case of the Kontsevich Conjecture [115, 116]. An independent proof of a closely related result (based on a study of the properties of holonomic $\mathcal{D}$-modules) was proposed by C. Dodd [76].

In contrast to the Jacobian Conjecture, which is an extremely difficult problem, in the study of the Kontsevich Conjecture there are several possible approaches. First of all, in [41], Kanel-Belov and Kontsevich have formulated several generalizations of Conjecture 1.2.5. In [42] and [191], which is devoted to the proof of Theorem 1.2.4, the construction of homomorphisms

$$\phi : \operatorname{Aut} W_{n,\mathbb{C}} \to \operatorname{Aut} P_{n,\mathbb{C}}$$

and

$$\phi : \operatorname{End} W_{n,\mathbb{C}} \to \operatorname{End} P_{n,\mathbb{C}}$$

involved in the construction, from a counterexample to DC_n, of an irreversible endomorphism with a single Jacobian, has been presented. A straightforward strengthening of Conjecture 1.2.5 is the statement that the homomorphism ϕ realizes the isomorphism of the Kontsevich Conjecture. Also, namely, in Chapter 8 of [41], an approach to solve the problem of lifting of polynomial symplectomorphisms to automorphisms of the Weyl algebra (i.e., constructing a homomorphism inverse to ϕ) was discussed. Conjecture 5 of [41], along with Conjecture 6, which is a weaker form of Conjecture 1.2.5, make up the essential contents of the construction proposed in [41]. To solve the problem of lifting of symplectomorphisms in the sense of these conjectures, it is necessary to study the properties of $\mathcal{D}$-modules, (left) modules over the Weyl algebra. The work of Dodd [76] is based on this approach.

For an arbitrary commutative ring R one can define the Weyl algebra $W_{n,R}$ over R, by just replacing $\mathbb{C}$ by R in the definition.

The algebra $W_{n,R}$ considered as an R-module is free with basis

$$\hat{x}^\alpha := \hat{x}_1^{\alpha_1} \dots \hat{x}_{2n}^{\alpha_{2n}}, \ \alpha = (\alpha_1, \dots, \alpha_{2n}) \in \mathbb{Z}_{\geqslant 0}^{2n} \ .$$

We define an increasing filtration (the Bernstein filtration) on the algebra $W_{n,R}$ by

$$W_{n,R}^{\leqslant N} := \left\{ \sum_\alpha c_\alpha \hat{x}^\alpha \,|\, c_\alpha \in R, \ c_\alpha = 0 \text{ for } |\alpha| := \alpha_1 + \dots + \alpha_{2n} > N \right\} \ .$$

This filtration induces a filtration on the automorphism group:

$$\operatorname{Aut}^{\leqslant N} W_{n,R} := \{ f \in \operatorname{Aut}(W_{n,R}) |\, f(\hat{x}_i), f^{-1}(\hat{x}_i) \in W_{n,R}^{\leqslant N} \ \ \forall i = 1, \dots, 2n \} \ .$$

The following functor on commutative rings:

$$R \mapsto \operatorname{Aut}^{\leqslant N}(W_{n,R}) \ ,$$

is representable by an affine scheme of finite type over $\mathbb{Z}$. We denote this scheme by

$$\underline{\operatorname{Aut}}^{\leqslant N}(W_n) \ .$$

Conjecture 1.2.5 says that groups of points $\underline{\operatorname{Aut}}(W_{n,\mathbb{C}})$ and $\underline{\operatorname{Aut}}(P_{n,\mathbb{C}})$ are isomorphic. We expect that the isomorphism should preserve the filtration by degrees, compatible with stabilization embeddings, and should be a constructible map for any given term of filtration, defined over $\mathbb{Q}$:

Conjecture 1.2.7 (see [41])**.** There exists a family $\phi_{n,N}$ of constructible one-to-one maps

$$\phi_{n,N} : \underline{\operatorname{Aut}}^{\leqslant N}(W_{n,\mathbb{Q}}) \to \underline{\operatorname{Aut}}^{\leqslant N}(P_{n,\mathbb{Q}})$$

compatible with the inclusions increasing indices N and n, and with the group structure.

Now let R be a finitely generated smooth commutative algebra over $\mathbb{Z}$, and $g \in \operatorname{Aut}(P_{n,R})$ be a symplectomorphism defined over R. Let us denote by $M_{g,p}$ any bimodule over $W_{n,R/pR}$ corresponding to the Morita autoequivalence (the interested reader is reffered to [41]), then we have the following conjecture:

Conjecture 1.2.8. For any finitely generated smooth commutative algebra R over $\mathbb{Z}$ and any $g \in \operatorname{Aut}(P_{n,R})$ for all sufficiently large p, the bimodule $M_{g,p}$ is a free rank one left $A_{n,R/pR}$-module.

In the rest of this subsection, we would like to mention some conjectures from [129] and for more information on these conjectures, the interested reader is referred to [129].

Let X be a smooth affine algebraic variety over field $\mathbb{F}$ of zero characteristic, $\dim X = n$. The ring $\mathcal{D}(X)$ of differential operators is $\mathbb{F}$-algebra of operators acting on $\mathcal{O}(X)$, generated by functions and derivations:

$$f \mapsto gf, \ f \mapsto \xi(f), \ \ g \in \mathcal{O}(X), \xi \in \Gamma(X, T_{X/\mathsf{Spec}\,\mathbb{F}}) \ .$$

Algebra $\mathcal{D}(X)$ carries the filtration $\mathcal{D}(X) = \cup_{k \geqslant 0} \mathcal{D}_{\leqslant k}(X)$ by the degree of operators, the associated graded algebra is canonically isomorphic to the algebra of functions on T^*X. In geometric terms, the grading comes from the dilation by $\mathbb{G}_m$ along the fibers of the cotangent bundle.

Let M be a finitely generated $\mathcal{D}(X)$-module, and choose a finite-dimensional subspace $V \subset M$ generating M. Then consider the filtration

$$M_{\leqslant k} := \mathcal{D}_{\leqslant k}(X) \cdot V \subset M, \ \ k \geqslant 0 \ .$$

The associated graded module op $gr(M)$ is a finitely generated $\mathcal{O}(T^*X)$-module.

Noetherianity of $\mathcal{D}(X)$ implies that M is the cokernel of a morphism of free finitely generated $\mathcal{D}(X)$-modules. Therefore, there exists a finitely generated ring $R \subset \mathbb{F}$ such that variety X, embedding i and module M have models X_R, i_R, M_R over $\mathsf{Spec}\,R$.

A finitely generated module M is called holonomic if and only if the dimension of its support is exactly n.

Conjecture 1.2.9. For holonomic M the support $\mathsf{supp}_{p,v} M_R$ is Lagrangian for sufficiently large p and any v.

The conjecture 1.2.9, was solved by Thomas Bitoun in his PhD thesis in 2010 [54] and in 2013 Michel Van den Bergh [89] gave an alternative proof of this conjecture.

We expect that in the case $\dim X > 1$ also there exists a notion of a logarithmic family of effective Lagrangian cycles in T^*X, and the arithmetic support should always belong to such a family. In the special case when a Lagrangian cycle is a *smooth* closed Lagrangian variety $L \subset T^*X$ (taken with multiplicity one) we expect a more clearer picture of what is the logarithmic family:

Definition 1.2.10. A smooth logarithmic family of smooth Lagrangian subvarieties in T^*X is a pair $(S, \mathcal{L})$ where S is a smooth variety over $\mathbb{F}$ and $\mathcal{L} \subset T^*X \times S$ is a smooth closed submanifold such that its projection to S is smooth, all fibers $\mathcal{L}_s$, $s \in S$ are Lagrangian, and the following property holds. For any $s \in S$ the natural map

$$T_s S \to \Gamma(\mathcal{L}_s, (T_X)_{|\mathcal{L}_s}/T_{\mathcal{L}_s}) = \Gamma(\mathcal{L}_s, T^*_{\mathcal{L}_s})$$

identifies $T_s S$ with the space of 1-forms on $\mathcal{L}_s$ with logarithmic singularities[1].

Conjecture 1.2.11. For a smooth closed Lagrangian $L \subset T^*X$ there exists a smooth logarithmic family $(S, \mathcal{L})$ with base point $s_0 \in S$ such that $\mathcal{L}_{s_0} = L$. Also, any two such families coincide with each other in the vicinity of s_0.

We also have the following conjectures. For more information on these conjectures we refer to [129]:

Conjecture 1.2.12. For any smooth closed connected Lagrangian subvariety L in T^*X over $\mathbb{F} = \mathbb{C}$ such that $H_1(L(\mathbb{C}), \mathbb{Z}) = 0$ there exists a unique holonomic $\mathcal{D}_X$-module $M = M_L$ with the arithmetic support equal to L taken with multiplicity 1. Moreover, op $Ext^1(M, M) = 0$.

Conjecture 1.2.13. For a differential operator $P \in \mathcal{D}(X)$, the support at prime p of $\mathcal{D}(X)/\mathcal{D}(X) \cdot (P + \lambda \cdot 1)$ is divisible by p^{n-1} for generic constant λ if and only if P belongs to a quantum integrable system, i.e., P belongs to a finitely generated commutative $\mathbb{F}$-subalgebra of $\mathcal{D}(X)$ of Krull dimension $n = \dim X$.

Conjecture 1.2.14. There exists a homomorphism from the group $BirSympl_{n,\mathbb{F}}$ of birational symplectomorphisms the algebraic torus $\mathbb{G}^{2n}_{m,\mathbb{F}}$ endowed with the standard symplectic form $\sum_{i,j \leqslant 2n} \omega_{ij} (x_i^{-1} dx_i) \wedge (x_j^{-1} dx_j)$, to the group of outer automorphisms of the skew field of fractions of the quantum torus. Also, the semiclassical limit as $q \to 1$ exists and gives the identity map from the group of birational symplectomorphisms the group of birational symplectomorphisms the algebraic torus to itself.

In a very recent paper, Edward Witten and Davide Gaiotto re-examine quantization via branes with the goal of understanding its relation to geometric quantization [138].

1.2.5. Tame automorphisms. An automorphism $\varphi \in \operatorname{Aut} \mathbb{F}[x_1, \ldots, x_N]$ is said to be *elementary* if it has the form

$$\varphi = (x_1, \ldots,\ x_{k-1},\ ax_k + f(x_1, \ldots, x_{k-1},\ x_{k+1},\ \ldots,\ x_N),\ x_{k+1},\ \ldots,\ x_N)$$

with $a \in \mathbb{F}^\times$. Observe that linear invertible changes of variables – that is, transformations of the form

$$(x_1,\ \ldots,\ x_N) \mapsto (x_1,\ \ldots,\ x_N)A, \quad A \in \mathrm{GL}(N, \mathbb{F})$$

are realized as compositions of elementary automorphisms.

The subgroup of $\operatorname{Aut} \mathbb{F}[x_1, \ldots, x_N]$ generated by all elementary automorphisms is the group $\operatorname{TAut} \mathbb{F}[x_1, \ldots, x_N]$ of so-called *tame automorphisms.*

[1] All such forms are automatically closed.

Let $P_n(\mathbb{F}) = \mathbb{F}[x_1, \dots, x_n, p_1, \dots, p_n]$ be the polynomial algebra in $2n$ variables with Poisson structure. It is clear that for an elementary automorphism

$$\varphi \in \operatorname{Aut} \mathbb{F}[x_1, \dots, x_n, p_1, \dots, p_n]$$

to be a symplectomorphism, it must either be a linear symplectic change of variables—that is, a transformation of the form

$$(x_1, \dots, x_n, p_1, \dots, p_n) \mapsto (x_1, \dots, x_n, p_1, \dots, p_n)A$$

with $A \in \operatorname{Sp}(2n, \mathbb{F})$ a symplectic matrix, or an elementary transformation of one of two following types:

$$(x_1, \dots, x_{k-1}, x_k + f(p_1, \dots, p_n), x_{k+1}, \dots, x_n, p_1, \dots, p_n)$$

or

$$(x_1, \dots, x_n, p_1, \dots, p_{k-1}, p_k + g(x_1, \dots, x_n), p_{k+1}, \dots, p_n).$$

Note that in both cases we do not include translations of the affine space into our consideration, so we may safely assume the polynomials f and g to be at least of height one.

The subgroup of $\operatorname{Aut} P_n(\mathbb{F})$ generated by all such automorphisms is the group $\operatorname{TAut} P_n(\mathbb{F})$ of *tame symplectomorphisms*. One similarly defines the notion of tameness for the Weyl algebra $W_n(\mathbb{F})$, with tame elementary automorphisms having the exact same form as for $P_n(\mathbb{F})$.

The automorphisms which are not tame are called *wild*. It is unknown at the time of writing whether the algebras W_n and P_n have any wild automorphisms in characteristic zero for $n > 1$; however, for $n = 1$ all automorphisms are known to be tame [103, 141, 142, 203]. On the other hand, the celebrated example of Nagata

$$(x + (x^2 - yz)x,\ y + 2(x^2 - yz)x + (x^2 - yz)^2 z,\ z)$$

provides a wild automorphism of the polynomial algebra $\mathbb{F}[x, y, z]$.

It is known, due to Kanel-Belov and Kontsevich [41, 42], that for $\mathbb{F} = \mathbb{C}$ the groups

$$\operatorname{TAut} W_n(\mathbb{C}) \text{ and } \operatorname{TAut} P_n(\mathbb{C})$$

are isomorphic. The homomorphism between the tame subgroups is obtained by means of non-standard analysis and involves certain non-constructible entities, such as free ultrafilters and infinite prime numbers. Recent effort [40, 115] has been directed to proving the homomorphism's independence of such auxiliary objects, with limited success.

1.2.6. Approximation by tame automorphisms. Tsuchimoto [190, 191], and independently Kanel-Belov and Kontsevich [42], found a deep connection between the Jacobian Conjecture and a celebrated Conjecture of Dixmier [75] on endomorphisms of the Weyl algebra, which is stated as Conjecture 1.2.3.

The correspondence between the two open problems, in the case of algebraically closed $\mathbb{F}$, is based on the existence of a composition-preserving map

$$\operatorname{End} W_n(\mathbb{F}) \to \operatorname{End} \mathbb{F}[x_1, \dots, x_{2n}]$$

which is a homomorphism for the corresponding automorphism groups. Furthermore, the mappings that belong to the image of this homomorphism preserve the canonical symplectic form on $\mathbb{A}^{2n}_{\mathbb{F}}$. Accordingly, Kontsevich and Kanel-Belov [41] formulated several conjectures on the correspondence between automorphisms of the Weyl algebra W_n and the Poisson algebra P_n (which is the polynomial algebra $\mathbb{F}[x_1, \dots, x_{2n}]$ endowed with the canonical Poisson bracket) in characteristic zero. In particular, there is

Conjecture 1.2.15. The automorphism groups of the n-th Weyl algebra and the polynomial algebra in $2n$ variables with Poisson structure over the rational numbers are isomorphic:

$$\operatorname{Aut} W_n(\mathbb{Q}) \simeq \operatorname{Aut} P_n(\mathbb{Q}).$$

Relatively little is known about the case $\mathbb{F} = \mathbb{Q}$, and the proof techniques developed in [41] rely heavily on model-theoretic objects such as infinite prime numbers (in the sense of non-standard analysis). That in turn requires the base field $\mathbb{F}$ to be of characteristic zero and algebraically closed (effectively $\mathbb{C}$ by the Lefschetz principle). However, even the seemingly easier analogue of the above Conjecture, the case $\mathbb{F} = \mathbb{C}$, is known (and positive) only for $n = 1$.

In the case $n = 1$, the affirmative answer to the Kontsevich Conjecture, as well as positivity of several isomorphism statements for algebras of a similar nature, relies on the fact that all automorphisms of the algebras in question are tame. Groups of tame automorphisms are rather interesting objects. Anick [8] has proved that the group of tame automorphisms of $\mathbb{F}[x_1, \ldots, x_N]$ is dense (in power series topology) in the subspace of all endomorphisms with non-zero constant Jacobian. This fundamental result enables one to reformulate the Jacobian Conjecture as a statement on invertibility of limits of tame automorphism sequences.

Another interesting problem is to ask whether all automorphisms of a given algebra are tame [**?**, 66, 67, 103, 203]. For instance, this is the case [141, 145] for $\mathbb{F}[x, y]$, the free associative algebra $\mathbb{F}\langle x, y\rangle$ and the free Poisson algebra $\mathbb{F}\{x, y\}$. It is also the case for free Lie algebras (a result of Cohn). On the other hand, tameness is no longer the case for $\mathbb{F}[x, y, z]$ (the wild automorphism example is provided by the well-known Nagata automorphism, cf. [181]).

In 1983, Anick's approximation theorem was established for polynomial automorphisms. We obtain the approximation theorems for polynomial symplectomorphisms and Weyl algebra automorphisms. We focus on *the problem of lifting of symplectomorphisms*:

> Can an arbitrary symplectomorphism in dimension $2n$ be lifted to an automorphism of the n-th Weyl algebra in characteristic zero?

The lifting problem is the milestone in the Kontsevich Conjecture. The use of tame approximation is advantageous due to the fact that tame symplectomorphisms correspond to Weyl algebra automorphisms: in fact [41], the tame automorphism subgroups are isomorphic when $\mathbb{F} = \mathbb{C}$.

The problems formulated above, as well as other statements of similar flavor, outline behavior of algebra-geometric objects when subject to quantization. Conversely, quantization (and anti-quantization in the sense of Tsuchimoto) provides a new perspective for the study of various properties of classical objects. Many of such properties have a distinctly K-theoretic nature. The lifting problem is a subject of a thorough study of V.A. Artamonov [9–12], one of the main results of which is the proof of an analogue of the Serre-Quillen-Suslin theorem for metabelian algebras. The possibility of lifting of (commutative) polynomial automorphisms to automorphisms of metabelian algebra is a well-known result of Umirbaev, cf. [197]. The metabelian lifting property was instrumental in Umirbaev's solution of Anick's Conjecture (which says that a specific automorphism of the free algebra $\mathbb{F}\langle x, y, z\rangle$, $\operatorname{char}\mathbb{F} = 0$ is wild). In addition, there is also a series of well-known papers [181–183].

In this thesis, we establish the approximation property for polynomial symplectomorphisms and comment on the lifting problem of polynomial symplectomorphisms and Weyl algebra automorphisms. In particular, the main results discussed here are as follows.

Theorem 1.2.16. *Let* $\varphi = (\varphi(x_1), \ldots, \varphi(x_N))$ *be an automorphism of the polynomial algebra* $\mathbb{F}[x_1, \ldots, x_N]$ *over a field* $\mathbb{F}$ *of characteristic zero, such that its Jacobian*

$$\mathrm{J}(\varphi) = \det\left[\frac{\partial\varphi(x_i)}{\partial x_j}\right]$$

is equal to 1. *Then there exists a sequence* $\{\psi_k\} \subset \operatorname{TAut}\mathbb{F}[x_1, \ldots, x_N]$ *of tame automorphisms converging to* φ *in formal power series topology.*

Anick [8] proved this tame approximation theorem for polynomial automorphisms.

Theorem 1.2.17. *Let $\sigma = (\sigma(x_1), \ldots, \sigma(x_n), \sigma(p_1), \ldots, \sigma(p_n))$ be a symplectomorphism of $\mathbb{F}[x_1, \ldots, x_n, p_1, \ldots, p_n]$ with unit Jacobian. Then there exists a sequence $\{\tau_k\} \subset \operatorname{TAut} P_n(\mathbb{F})$ of tame symplectomorphisms converging to σ in formal power series topology.*

Theorem 1.2.18. *Let $\mathbb{F} = \mathbb{C}$ and let $\sigma : P_n(\mathbb{C}) \to P_n(\mathbb{C})$ be a symplectomorphism over complex numbers. Then there exists a sequence*

$$\psi_1, \psi_2, \ldots, \psi_k, \ldots$$

of tame automorphisms of the n-th Weyl algebra $W_n(\mathbb{C})$, such that their images σ_k in $\operatorname{Aut} P_n(\mathbb{C})$ converge to σ.

We are mainly interested in the last theorem. As we shall see, sequences of tame symplectomorphisms lifted to automorphisms of Weyl algebra (either by means of the isomorphism of [41], or explicitly through deformation quantization $P_n(\mathbb{C}) \to P_n(\mathbb{C})[[\hbar]]$) are such that their limits may be thought of as power series in Weyl algebra generators. If we could establish that those power series were actually polynomials, then the Dixmier Conjecture would imply the Kontsevich Conjecture (with $\mathbb{Q}$ replaced by $\mathbb{C}$). Conversely, approximation by tame automorphisms provides a possible means to attack the Dixmier Conjecture (and, correspondingly, the Jacobian Conjecture).

1.2.7. Holonomic $\mathcal{D}$-modules and Lagrangian submanifolds.

The following general Conjecture holds ([41], see also [129]).

Conjecture 1.2.19. Let X be a smooth variety. There is a one-to-one correspondence between (irreducible) holonomic $\mathcal{D}(X)$-modules and Lagrangian subvarieties T^*X of the corresponding dimension.

Kontsevich [129] introduces the general definition of the holonomic $\mathcal{D}$-module as follows. Let X be a smooth affine algebraic variety of dimension n over the field $\mathbb{K}$. Consider the $\mathbb{K}$-algebra $\mathcal{D}(X)$ of differential operators, the algebra of operators, acting on the ring $\mathcal{O}(X)$ generated by functions and $\mathbb{K}$-derivations:

$$f \mapsto gf, \quad f \mapsto \xi(f), \quad g \in \mathcal{O}(X), \quad \xi \in \Gamma(X, T_{X/\operatorname{Spec}\mathbb{K}}).$$

The natural filtration is defined on the algebra

$$\mathcal{D}(X) = \cup_{k \geqslant 0} \mathcal{D}_{\leqslant k}(X),$$

with respect to the order of operators, the associated graded algebra is naturally isomorphic to the algebra of functions on the cotangent bundle T^*X. Let M be a finitely generated module over $\mathcal{D}(X)$, and V be a finite-dimensional subspace of elements generating M. It induces a filtration

$$M_{\leqslant k} = \mathcal{D}_{\leqslant k}(X)V \subset M, \quad k \geqslant 0,$$

such that the associated graded module $\operatorname{gr}(M)$ is finitely generated over $\mathcal{O}(T^*X)$. It is known (this result belongs to O. Gabber, see [129]) that its support

$$\operatorname{supp}(\operatorname{gr}(M)) \subset T^*X$$

is a coisotropic variety. In particular, the dimension of any of its irreducible components is not less than n. The support is independent of the choice of the subspace V (and is denoted in the original article [129] as $\operatorname{supp}(M)$).

A finitely generated module M is said to be *holonomic* if, by definition, the dimension of its support is n.

Conjecture 1.2.19 (which can also be called the Kontsevich Conjecture) generalizes Conjecture 1.2.5, as well as Conjectures 5 and 6 of [41] in the context of the lifting of symplectomorphisms. Namely, with any symplectomorphism one may naturally associate a Lagrangian subvariety (namely, its graph). On the other hand, holonomic $\mathcal{D}$-modules correspond to autoequivalences of Weyl algebra, from which in principle (taking into account Conjecture 5 of [41]) one can get a correspondence with automorphisms.

In this connection, the necessity to study the holonomic $\mathcal{D}$-modules is a natural consequence. The problems of lifting of polynomial symplectomorphisms in the case of low dimensions, namely, for $n = 1$, which corresponds to the well-known case of the Kontsevich Conjecture, has become the prime candidate for testing these new deep insights. Some progress in this direction has been achieved in [40] by Kanel-Belov and Elishev. The general case of arbitrary dimension was investigated by Kontsevich in [129] (see also [54]). Significant results on Conjecture 1.2.19 were obtained (according to our understanding) by Dodd [76].

Namely, Dodd devised the proof of the following result.

Theorem 1.2.20 (C. Dodd, [76])**.** *Let X be a smooth variety over $\mathbb{C}$, $L \subset T^*X$ be a Lagrangian subvariety of the cotangent space. Assume that*:

(1) *the projection $\pi : L \to X$ is a dominant mapping*;
(2) *the first singular homology group $H_1^{sing}(L, \mathbb{Z})$ is trivial;*
(3) *there exists a smooth projective compactification $\bar{L}$ of the variety L with trivial* $(0, 2)$*-Hodge cohomology.*

Then there exists a unique irreducible holonomic $\mathcal{D}(X)$-module M with constant arithmetic support[1]*, equal to L, with multiplicity* 1.

This theorem partially solves the problem of finding sufficient conditions for the correspondence between holonomic modules and Lagrangian varieties as formulated in Conjecture 1.2.19. Dodd also notes that in the case when $X = \mathbb{A}^n$ is an affine space, condition 2 of Theorem 1.2.20 can be dropped. In this connection, there is the following corollary:

Corollary 1.2.21 (C. Dodd, [76])**.** *Let $L \subset T^*\mathbb{A}^m$ be a smooth Lagrangian subvariety satisfying conditions 2 and 3 of Theorem 1.2.20. Then there exists a unique irreducible holonomic $\mathcal{D}(\mathbb{A}^m)$-module M whose arithmetic support is L, with multiplicity* 1.

This result is closely related to the construction studied in [40].

As Dodd notes, Theorem 1.2.20 and Corollary 1.2.21 allow us to give a description of the Picard group $\mathrm{Pic}(W_{n,\mathbb{C}})$ of the Weyl algebra. Recall that the Picard group of an associative algebra is defined as a group of classes (modulo isomorphism) of invertible bimodules over a given algebra, with a group operation given by the tensor product of modules.

Consider polynomial symplectomorphisms of the variety $T^*\mathbb{A}^m$. It is easy to show that the graph of any symplectomorphism φ is a Lagrangian subvariety of L^φ in $T^*\mathbb{A}^{2m}$, isomorphic to $\mathbb{A}^{2m}$ and, therefore, satisfies cohomological conditions of Theorem 1.2.20. Applying Corollary 1.2.21, we obtain (uniquely identified) $\mathcal{D}(\mathbb{A}^{2m}) \simeq \mathcal{D}(\mathbb{A}^m) \otimes \mathcal{D}(\mathbb{A}^m)^{op}$-module M^φ corresponding to L^φ. One can check [76] that the inverse symplectomorphism φ^{-1} in such a construction corresponds to inverse bimodule.

From these considerations, Dodd obtains the following result.

Theorem 1.2.22 (C. Dodd, [76])**.** *There is an isomorphism of groups (over $\mathbb{C}$)*

$$\mathrm{Pic}(\mathcal{D}(\mathbb{A}^m)) \simeq \mathrm{Sympl}(T^*\mathbb{A}^m),$$

where $\mathrm{Sympl}(T^*\mathbb{A}^m)$ *denotes the group of polynomial symplectomorphisms (this group is a geometric analogue of the group* $\mathrm{Aut}\, P_{m,\mathbb{C}}$*).*

In the case $m = 1$, it is known (Dixmier, [75]) that $\mathrm{Pic}(\mathcal{D}(\mathbb{A}^1)) = \mathrm{Aut}(\mathcal{D}(\mathbb{A}^1))$,and the algebra $\mathcal{D}(\mathbb{A}^1)$ is isomorphic to the first Weyl algebra $W_{1,\mathbb{C}}$. This means that we are in the situation of Conjecture 1.2.5 for $m = 1$.

1.2.8. Tame automorphisms and the Quantization Conjectures. Dodd's constructions are deep in content and, apparently, prove the Kontsevich Conjecture on the correspondence between Lagrangian varieties and holonomic modules (more precisely, its essential part). On the other hand, starting from Theorem 1.2.22 we cannot immediately arrive at the general case of Conjecture 1.2.5.

[1]For the definition of arithmetic support, see [129].

The proof of Conjecture 1 of [41] requires a solution to the lifting problem of symplectomorphisms to automorphisms of the corresponding Weyl algebra.

One of the main results of [41] was the proof of the following homomorphism properties

$$\phi : \operatorname{Aut} W_{n,\mathbb{C}} \to \operatorname{Aut} P_{n,\mathbb{C}}$$

constructed in [41] and [191]. First, let φ be an automorphism of the polynomial algebra $\mathbb{F}[x_1, \dots, x_n]$. We call φ *elementary* if it has the form

$$\varphi = (x_1, \dots,\ x_{k-1},\ ax_k + f(x_1, \dots, x_{k-1},\ x_{k+1},\ \dots,\ x_n),\ x_{k+1},\ \dots,\ x_n).$$

In particular, automorphisms given by linear substitutions of generators are elementary.

Tame automorphisms of the algebra $P_{n,\mathbb{F}}$ are, by definition, compositions of those tame elementary automorphisms which preserve the Poisson bracket. Tame automorphisms of the Weyl algebra are defined $W_{n,\mathbb{F}}$ similarly.

The following theorem is proved in [41].

Theorem 1.2.23 (A. Kanel-Belov and M.L. Kontsevich, [41])**.** *The homomorphism*

$$\phi : \operatorname{Aut} W_{n,\mathbb{C}} \to \operatorname{Aut} P_{n,\mathbb{C}}$$

restricts to the isomorphism

$$\phi_{|\,\mathrm{TAut}} : \operatorname{TAut} W_{n,\mathbb{C}} \to \operatorname{TAut} P_{n,\mathbb{C}}$$

between subgroups of tame automorphisms.

In particular, due to the tame nature of automorphism groups of Weyl and Poisson algebras for $n = 1$, the homomorphism ϕ gives an isomorphism of the Kontsevich Conjecture between $\operatorname{Aut} W_{1,\mathbb{C}}$ and $\operatorname{Aut} P_{1,\mathbb{C}}$.

It is not known whether all automorphisms of the Poisson and Weyl algebras are tame for $n > 1$, or even stably tame (an automorphism is called *stably tame* if it becomes tame after adding dummy variables and extending the action on them by means of the identity automorphism). For the algebra of polynomials in three variables, the Nagata automorphism

$$(x, y, z) \mapsto (x - 2(xz + y^2)y - (xz + y^2)^2 z, y + (xz + y^2)z, z)$$

is wild (the famous result due to I.P. Shestakov and Umirbaev [181]).

Nevertheless, tame automorphisms turn out to play a significant role in the context of the Kontsevich Conjecture and the Jacobian Conjecture, due to the following reason. Anick [8] showed that the set of tame automorphisms of the algebra of polynomials $\mathbb{F}[x_1, \dots, x_n]$ ($n \geqslant 2$) is dense in the topology of formal power series in the space $\mathfrak{J}$ of polynomial endomorphisms with nonzero constant Jacobian. In particular, for any automorphism of a polynomial algebra there exists a sequence of tame automorphisms converging to it in this topology; in other words, Anick's theorem implies the existence of *approximations* of automorphisms, or *approximations* by tame automorphisms (and in general, endomorphisms with nonzero constant Jacobian). In view of Anick's theorem, the Jacobian Conjecture can be formulated as a problem of invertibility of limits of sequences of tame automorphisms (this is discussed in the conclusion of [8]). This formulation of the Jacobian Conjecture can be directly generalized to the case of a field of arbitrary characteristic, see more below as well as in [123].

Anick's results, together with Theorem 1.2.23, suggest the idea of solving the lifting problem of polynomial symplectomorphisms to automorphisms of the Weyl algebra (an alternative construction to that proposed in [41]). Namely, if there is a symplectic analogue of Anick's theorem, that is, if there is an approximation of polynomial symplectomorphisms by tame symplectomorphisms, then, taking a sequence of tame symplectomorphisms converging to a given point, we can take the sequence of their pre-images under the isomorphism $\phi_{|\,\mathrm{TAut}}$ and try to prove that its limit exists and is an automorphism of the Weyl algebra. The symplectic analogue of Anick's theorem was proved in [117]. The application of approximation theory to the lifting problem constitutes the main idea of the proof of Conjecture 1.2.5 in [116].

However, the direct application of the main result of [117] to the solution of the lifting problem does not achieve the desired result, since the homomorphism ϕ does not preserve the topology of formal power series (due to commutation relations in the Weyl algebra). In this connection, the naive approximation approach needs some modification. It turns out that such a modification is possible (see [116]). The nature of this modification is significant and is connected with the geometric properties of Ind-schemes of automorphisms of the corresponding algebras. *Therefore, the study of the geometry of* Ind-*schemes of automorphisms is justified in the framework of the Kontsevich Conjecture.*

1.2.9. Quantization of classical algebras. As already noted, the approach to the Jacobian Conjecture, using techniques from the theory of deformation quantization, namely, the approach based on stable equivalence between the Jacobian Conjecture and the Dixmier Conjecture as well as, to a somewhat lesser extent, the Kontsevich Conjecture, is currently one of the more promising approaches to finding a possible solution to the Jacobian Conjecture. However, as in questions of the geometric theory of Ind-schemes and infinite-dimensional algebraic groups, the issues arising in connection with the application of quantization methods, due to their nontriviality and depth, is a direction whose value may well be comparable with the value of a possible solution to the original problem.

Analogues of JC and DC for algebras of quantum polynomials are not obvious and often do not admit a naive transfer of formulations (for example, E. Backelin [19] wrote about the q-quantum version of the Dixmier Conjecture). On the other hand, the well-known theorem of Umirbaev [197], showing the validity of an analogue of the Jacobian Conjecture for free metabelian algebras, may be considered as an argument in favor of the validity of the Jacobian Conjecture.

Significant development of algebra and non-commutative geometry of quantum polynomials has been achieved by Artamonov [9–12, 15]. In particular, he proved [12] the quantum-algebra analogue of the Serre Conjecture (Quillen–Suslin theorem) – the result which is extremely non-trivial even in the commutative case.

In connection with the Jacobian Conjecture, we mention the works of Dicks [73], Dicks and Lewin [74] as well as Yagzhev [217–220]. In a sense, they can be interpreted as works consistent with viewing the Jacobian problem as a problem related to quantization. Regarding the practical benefits of studying relationships induced by quantization-type correspondences, there are known examples of applications of the elements of the quantization procedure to some (previously proven by other means) problems of general algebra. An example [120, 233] is a new proof of Bergman's centralizer theorem **??** of the free associative algebra, based on the deformation quantization procedure, which we discuss in this work.

1.3. Torus Actions on Free Associative Algebras and the Białynicki-Birula Theorem

In the proof of the results concerning the geometry of Ind-schemes automorphisms, we use the famous A. Białynicki-Birula theorem [51, 52] on the linearizability of regular actions of a maximal torus on an affine space merits consists in the following.

Let $\mathbb{K}$ be the base field and let $\mathbb{K}^\times = \mathbb{K}\backslash\{0\}$ be the multiplicative group of the field, considered as an algebraic $\mathbb{K}$-group.

The n-dimensional algebraic $\mathbb{K}$-torus is the group $\mathbb{T}_n \simeq (\mathbb{K}^\times)^n$ (with obviously certain multiplication).

Definition 1.3.1. An n-dimensional algebraic $\mathbb{K}$-torus is the group $\mathbb{T}_n \simeq (\mathbb{K}^\times)^n$ (with obvious multiplication).

Denote by $\mathbb{A}^n$ the affine space of dimension n over $\mathbb{K}$.

Definition 1.3.2. A (left, geometric) torus action is a morphism

$$\sigma : \mathbb{T}_n \times \mathbb{A}^n \to \mathbb{A}^n.$$

that fulfills the usual axioms (identity and compatibility):

$$\sigma(1,x) = x, \ \ \sigma(t_1, \sigma(t_2, x)) = \sigma(t_1 t_2, x).$$

The action σ is *effective* if for every $t \neq 1$ there is an element $x \in \mathbb{A}^n$ such that $\sigma(t,x) \neq x$.

In [51], Białynicki-Birula proved the following two theorems, for $\mathbb{K}$ algebraically closed.

Theorem 1.3.3. *Any regular action of* $\mathbb{T}_n$ *on* $\mathbb{A}^n$ *has a fixed point.*

Theorem 1.3.4. *Any effective and regular action of* $\mathbb{T}_n$ *on* $\mathbb{A}^n$ *is a representation in some coordinate system.*

The notion of regular action means regularity in the sense of algebraic geometry (preservation of regular functions; Białynicki-Birula also considered birational actions in [51]). The last theorem states that any effective regular action of the maximal torus on an affine space is conjugate to a linear action (representation); in other words, such an action *admits linearization.*

An algebraic group action on $\mathbb{A}^n$ is the same as the corresponding action by automorphisms on the algebra $\mathbb{K}[x_1, \ldots, x_n]$ of coordinate functions. In other words, it is a group homomorphism

$$\sigma : \mathbb{T}_n \to \operatorname{Aut} \mathbb{K}[x_1, \ldots, x_n].$$

An action is effective if and only if $\operatorname{Ker} \sigma = \{1\}$.

The polynomial algebra is a quotient of the free associative algebra

$$F_n = \mathbb{K}\langle z_1, \ldots, z_n \rangle$$

by the commutator ideal I (it is the two-sided ideal generated by all elements of the form $fg - gf$). The definition of torus action on the free algebra is thus purely algebraic.

The following result has been established in [87, 88].

Theorem 1.3.5. *Suppose given an action* σ *of the algebraic* n*-torus* $\mathbb{T}_n$ *on the free algebra* F_n*. If* σ *is effective, then it is linearizable.*

The theory of algebraic group actions on varieties is a substantial part of the study of Ind-varieties. Among the significant works on this subject, the reader is well advised to consult the papers of T. Kambayashi and P. Russell [112], M. Koras and P. Russell [130], T. Asanuma [18], G. Schwartz [176] and H. Bass [20].

The group action linearity problem asks, generally speaking, whether any action of a given algebraic group on an affine space is linear in some suitable coordinate system (or, in other words, whether for any such action there exists an automorphism of the affine space such that it conjugates the action to a representation). This subject owes its existence largely to the classical work of A. Białynicki-Birula [51], who considered regular (i.e. by polynomial mappings) actions of the n-dimensional torus on the affine space $\mathbb{A}^n$ (over algebraically closed ground field) and proved that any faithful action is conjugate to a representation (or, as we sometimes say, linearizable). The result of Białynicki-Birula had motivated the study of various analogous instances, such as those that deal with actions of tori of dimension smaller than that of the affine space, or, alternatively, linearity conjectures that arise when the torus is replaced by a different sort of algebraic group. In particular, Białynicki-Birula himself [52] had proved that any effective action of $(n-1)$-dimensional torus on $\mathbb{A}^n$ is linearizable, and for a while it was believed [112] that the same was true for arbitrary torus and affine space dimensions. Eventually, however, the negation of this generalized linearity conjecture was established, with counter-examples due to Asanuma [18].

More recently, the linearity of effective torus actions has become a stepping stone in the study of geometry of automorphism groups. In the paper [123], the following result was obtained.

Theorem 1.3.6. *Let* $\mathbb{K}$ *be algebraically closed, and let* $n \geqslant 3$*. Then any* Ind*-variety automorphism* Φ *of the* Ind*-group* $\operatorname{Aut}(K[x_1, \ldots, x_n])$ *is inner.*

The notions of Ind-variety (or Ind-group in this context) and Ind-variety morphism were introduced by Shafarevich [177]: an Ind-variety is the direct limit of a system whose morphisms are closed embeddings. Automorphism groups of algebras with polynomial identities, such as the (commutative) polynomial algebra and the free associative algebra, are archetypal examples; the corresponding direct systems of varieties consist of sets $\mathrm{Aut}^{\leqslant N}$ of automorphisms of total degree less or equal to a fixed number, with the degree induced by the grading. The morphisms are inclusion maps which are obviously closed embeddings.

Theorem 1.3.6 is proved by means of tame approximation (stemming from the main result of [8]), with the following Proposition, originally due to E. Rips, constituting one of the key results.

Proposition 1.3.7. *Let $\mathbb{K}$ be algebraically closed and $n \geqslant 3$ as above, and suppose that Φ preserves the standard maximal torus action on the commutative polynomial algebra*[1]*. Then Φ preserves all tame automorphisms.*

The proof relies on the Białynicki-Birula theorem on the maximal torus action. In a similar fashion, the paper [123] examines the Ind-group $\mathrm{Aut}\,\mathbb{K}\langle x_1,\dots,x_n\rangle$ of automorphisms of the free associative algebra $\mathbb{K}\langle x_1,\dots,x_n\rangle$ in n variables, and establishes results completely analogous to Theorem 1.3.6 and Proposition 1.3.7. [2] In accordance with that, the free associative analogue of the Białynicki-Birula theorem was required.

Such an analogue is indeed valid, and we have established it in our notes [87, 88] on the subject. We will provide the proof of this result in the sequel.

Given the existence of a free algebra version of the Białynicki-Birula theorem, one may inquire whether various other instances of the linearity problem (such as the Białynicki-Birula theorem on the action of the $(n-1)$-dimensional torus on $\mathbb{K}[x_1,\dots,x_n]$) can be studied. As it turns out, direct adaptation of proof techniques from the commutative realm is sometimes possible. There are certain limitations, however. For instance, Białynicki-Birula's proof [52] of linearity of $(n-1)$-dimensional torus actions uses commutativity in an essential way. Nevertheless, a neat workaround of that hurdle can be performed when $n = 2$, as we show in this note. Also, a special class of torus actions (positive-root actions) turns out to be linearizable. Finally, some of the proof techniques developed by Asanuma [18] admit free associative analogues; this will allow us to prove the existence of non-linearizable torus actions in positive characteristic, in complete analogy with Asanuma's work.

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[1] That is, the action of the n-dimensional torus on the polynomial algebra $\mathbb{K}[x_1,\dots,x_n]$, which is dual to the action on the affine space.

[2] The free associative case was amenable to the above approach when $n > 3$.

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