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VOLTERRA OPERATOR INCLUSIONS IN THE THEORY OF GENERALIZED NEURAL FIELD MODELS WITH CONTROL. I

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We obtained conditions for solvability of Volterra operator inclusions and continuous dependence of the solutions on a parameter. These results were implemented to investigation of generalized neural field equations involving control.
Key words: Volterra operator inclusions; neural field equations; control; existence of solutions; continuous dependence on parameters

1. Introduction.

The human brain cortex is the top layer of the hemispheres, of $2-4$ mm thick, involving about 10^9 neurons having 60×10^{12} connections [1]. The brain cortex is responsible for such higher functions of the human brain as e.g. memory, reasoning, thought, and language [2], [3]. The basic unit of the brain cortex is the neuron. It consists of dendrites, cell body (soma), and axon. The dendrites receive electrical signals from other neurons and propagate them to the soma. If the total sum of the input electrical potential in the soma exceeds a certain threshold value, the neuron produces the burst of the output electrical signal, which then propagates along the axon to other neurons. Thus, a natural way (see e.g [4]) of studying electrical activity in the neocortex is the framework of cortical networks.

The most well-known representative of such models is the Hopfield network model [5]

$$\dot{z}_i(t) = -z_i(t) + \sum_{j=1}^{N} \omega_{ij} f\Big(z_j(t)\Big),\ t \geq 0,\ i = 1, ..., N. \tag{1.1}$$

Here z_i is the electrical activity of the i-th neuron in the network, ω_{ij} is the connection strength between the i-th and j-th neurons, the non-negative function f gives the firing rate $f(z)$ of a neuron with activity z.

However, since the number of neurons and synapses in even a small piece of cortex is immense, a suitable modeling approach is to take a continuum limit of the neural networks and, thus, consider so-called neural field models of the brain cortex (rigorous justification of this limit procedure using the notion of parameterized measure is given in [6]). The most well-known and simplest model describing the macro-level neural field dynamics is the Amari model [7]

$$\partial_t u(t,x) = -u(t,x) + \int_{\Omega} \omega(x-y) f(u(t,y))dy, \tag{1.2}$$

$$t \geq 0,\, x \in \Omega \subseteq R^n.$$

Here $u(t,x)$ denotes the activity of a neural element u at time t and position x. The connectivity function ω determines the coupling strength between the elements and the non-negative function f gives the firing rate $f(u)$ of a neuron with activity u. Neurons at a position

x and time t are said to be active if $f(u(t,x))>0$. Typically f is a smooth function that has sigmoidal shape.

One of the key objects in the neuroscience community is the so-called bump-solutions, i.e. solutions satisfying the following condition

$$\lim_{|x|\to\infty} w(t,x)=0,\ t\in[a,\infty). \tag{1.3}$$

This type of solutions corresponds to the electrical brain activity that is prevalent during its normal functioning, encoding visual stimuli [8], representing head direction [9], and maintaining persistent activity states in working memory [10], [11].

The models of the type (1.1) are important in studies of cortical gain control or pharmacological manipulations [12]. The problems of therapy of Epilepsy, Parkinson's disease, and other disorders of the central nervous system has been recently investigated in [1]–[17]. The modeling frameworks in [1]–[17] incorporate brain electrical stimulation, which is considered as control, and the corresponding optimization problems. The unique solvability of such models and continuous dependence of the solutions obtained on the control involved in the modeling equations has been recently examined in [18], [19]. These works employed the theory of abstract Volterra operators in complete metric spaces and Banach spaces in order to establish the main results on controllable neural field equations. The present paper extends the results of [18], [19]. Here, we deal with controllable neural field equations where the whole right-hand side is parameterized. Generalizing the model (1.2) and adding control to it, we get

$$w(t,x)=\int_a^t\int_{\mathbb{R}^m} f(t,s,x,y,w(s,y),u(t,s,x,y),\lambda)dyds, \tag{1.4}$$

$$t\in[a,\infty),\ x\in\mathbb{R}^m,$$

with respect to the unknown continuous function $w:[a,\infty)\times\mathbb{R}^m\to\mathbb{R}^n$, which is spatially localized, i.e. satisfies (1.3). The function $u:[a,\infty)\times[a,\infty)\times\mathbb{R}^m\times\mathbb{R}^m\to U$ (U – compact subset of R^k) is a control which is assumed to be essentially bounded. Here λ is a parameter from some metric space Λ.

We can derive the following inclusion arising with respect to the control taking it's values in U with parametrization from Λ:

$$w(t,x)\in(F(w,\lambda))(t,x), \tag{1.5}$$

$$(F(w,\lambda))(t,x)=\int_a^t\int_{\mathbb{R}^m} f(t,s,x,y,w(s,y),U,\lambda)dyds,$$

$$t\in[a,\infty),\ x\in\mathbb{R}^m,\ \lambda\in\Lambda.$$

In order to approach the latter problem, we investigate solvability an parametric dependence properties of Volterra operator inclusions in the next section.

2. Volterra operator inclusions with parameter.

Let $\mathbb{R}^m$ be the m-dimensional real vector space with the norm $|\cdot|$. Let W be a metric space with the distance ρ_W. We denote by $B_W(w,r)$ an open ball of the radius r centered at $w\in W$. We denote $\Omega(W)$ to be the set of all non-empty closed subsets of W.

Let an equivalence relation $\sim$ be defined on W. For any two equivalence classes $\overline{w}^1$, $\overline{w}^2$, we introduce

$$d(\overline{w}^1,\overline{w}^2)=\inf_{w^1\in\overline{w}^1,\ w^2\in\overline{w}^2}\rho(w^1,w^2). \tag{2.1}$$

If for any $\varepsilon > 0$ and any $\overline{w}^1, \overline{w}^2 \in W/\sim$, $w^1 \in \overline{w}^1$ one can find $w^2 \in \overline{w}^2$ such that $d(\overline{w}^1, \overline{w}^2) \geq$ $\geq \rho(w^1, w^2) - \varepsilon$, then (2.1) defines metric in $W/\sim$.

We put in correspondence to each $\gamma \in [0,1]$ the equivalence relation $\upsilon(\gamma)$. We assume that the family of equivalence relations $\upsilon = \{\upsilon(\gamma), \gamma \in [0,1]\}$ satisfy the following conditions:

υ_0) $\gamma = 0$ corresponds to the relation $\upsilon(0) = W^2$ (any two elements are $\upsilon(0)$-equivalent);

υ_1) $\gamma = 1$ corresponds to equality relation (any two distinct elements are not $\upsilon(1)$-equivalent);

υ) if $\gamma_1 > \gamma_2$, then $\upsilon(\gamma_1) \subseteq \upsilon(\gamma_2)$ (any $\upsilon(\gamma_1)$-equivalent elements are $\upsilon(\gamma_2)$-equivalent);

Definition 2.1. A set-valued map $\Psi : W \to \Omega(W)$ is said to be a *Volterra set-valued map on the family* υ if for any $\gamma \in [0,1]$ and any $w^1, w^2 \in W$ the fact that $(w^1, w^2) \in \upsilon(\gamma)$ implies $(\Psi w^1, \Psi w^2) \in \upsilon(\gamma)$, which means that $(\widetilde{w}^1, \widetilde{w}^2) \in \upsilon(\gamma)$ for any $\widetilde{w}^1 \in \Psi w^1$ and $\widetilde{w}^2 \in \Psi w^2$.

For any $w \in W$, let us denote $\overline{w}_\gamma$ to be the $\upsilon(\gamma)$-equivalence class of w.

Hereinafter we assume that (W, ρ_W) is a complete metric space with the equivalence relation υ satisfying $\upsilon_0), \upsilon_1), \upsilon)$. Moreover, we assume that for each $\gamma \in (0,1)$, the corresponding equivalence class is closed and the quotient set $W/\upsilon(\gamma)$ is a complete metric quotient space with the distance $d_{W/\upsilon(\gamma)}(\overline{w}^1, \overline{w}^2) = \inf\limits_{w^1 \in \overline{w}^1_\gamma,\ w^2 \in \overline{w}^2_\gamma} \rho_W(w^1, w^2)$.

Below we cite some important properties of single-valued Volterra operators (see [20]) that can be naturally extended to Volterra set-valued maps.

1. Choose an arbitrary set $\Gamma \subset [0,1]$, $\{0,1\} \subset \Gamma$, and for any decreasing (or any increasing) sequence $\{\gamma_i\}$, it holds true that $\lim\limits_{i\to\infty} \gamma_i \in \Gamma$. Let $\omega = \{\upsilon(\gamma), \gamma \in \Gamma\}$. We define the mapping $\eta : [0,1] \to \Gamma$ as $\eta(\gamma) = \inf\{\xi \in \Gamma\}, \xi \geq \gamma$ $(\eta(\gamma) = \inf\{\xi \in \Gamma\}, \xi \leq \gamma)$, and put in correspondence to any γ the equivalence relation $\upsilon(\eta(\gamma))$. If the set-valued map $\Psi : W \to \Omega(W)$ is a Volterra mapping on the family υ, then it is a (set-valued) Volterra on its subfamily ω.

2. If for some $\gamma_0 \in (0,1)$, $w \in W$ it holds true that $\Psi w \cap \overline{w}_{\gamma_0} \neq \emptyset$, then the set $\overline{w}_{\gamma_0}$ is invariant with respect to the Volterra set-valued mapping $\Psi : W \to \Omega(W)$ and the relation $\upsilon(\gamma)$ can be considered only on the elements of $\overline{w}_{\gamma_0} \subset W$. The set $\overline{w}_{\gamma_0}$ is a complete metric space with respect to the metric of the whole space W. Thus, the family of the equivalence relations satisfying the conditions $\upsilon_0, \upsilon_1, \upsilon$ is also defined on $\overline{w}_{\gamma_0}$. The quotient set $\overline{w}_{\gamma_0}/\upsilon(\gamma)$, $\gamma \leq \gamma_0$, consists of the unique element. If $\gamma > \gamma_0$, the quotient set $\overline{w}_{\gamma_0}/\upsilon(\gamma)$ is a complete metric space. Moreover, the fact that $\Psi : W \to \Omega(W)$ is a Volterra set-valued map on the family υ implies that the restriction $\Psi_{\gamma_0} : \overline{w}_{\gamma_0} \to \Omega(\overline{w}_{\gamma_0})$ of Ψ is a set-valued Volterra map on the family υ.

3. For each $\gamma \in (0,1)$, we define the canonical projection $\Pi_\gamma : \Omega(W) \to \Omega(W/\upsilon(\gamma))$ as $\Pi_\gamma \mathfrak{W} =$ $= \bigcup\limits_{w \in \mathfrak{W}} \overline{w}_\gamma$, $\mathfrak{W} \in \Omega(W)$. For a set-valued Volterra mapping $\Psi : W \to \Omega(W)$ on the family υ, we define the map $\Psi_\gamma : W/\upsilon(\gamma) \to \Omega(W/\upsilon(\gamma))$ as $\Psi_\gamma \overline{w}_\gamma = \Pi_\gamma \Psi w$, where w is an arbitrary element of $\overline{w}_\gamma$. Choose an arbitrary $\gamma_0 \in (0,1)$. The family $\upsilon(\gamma_0)$ generates the corresponding equivalence relation on $W/\upsilon(\gamma_0)$. Let $\xi \in (0, \gamma_0)$, and let the elements $w^1, w^2 \in W$ be $\upsilon(\xi)$-equivalent. Then any $w^{1'} \in \overline{w}^1_{\gamma_0}$, $w^{2'} \in \overline{w}^2_{\gamma_0}$ are also $\upsilon(\xi)$-equivalent, which defines the notion of equivalence of the classes $\overline{w}^1_{\gamma_0}$ and $\overline{w}^2_{\gamma_0}$. Namely, the classes $\overline{w}^1_{\gamma_0}$ and $\overline{w}^2_{\gamma_0}$ are $\overline{\upsilon}_{\gamma_0}(\sigma)$-equivalent ($\sigma \in (0,1)$), if there exist (which, actually, means "any") $w^1 \in \overline{w}^1_{\gamma_0}$, $w^2 \in \overline{w}^2_{\gamma_0}$ satisfying the equivalence relation $\upsilon(\xi)$, $\xi = \gamma_0 \sigma$. Thus, the family $\overline{\upsilon}_{\gamma_0} = \{\overline{\upsilon}_{\gamma_0}(\sigma)\}$ of equivalence relations is defined on $W/\upsilon(\gamma_0)$. The quotient set $\Big(W/\upsilon(\gamma_0)\Big)/\overline{\upsilon}_{\gamma_0}$ with the distance

$$d(W_{\gamma_0\sigma}, W_{\gamma_0\sigma}) = \inf_{\overline{w}^1_{\gamma_0} \in W_{\gamma_0\sigma}, \overline{w}^2_{\gamma_0} \in W_{\gamma_0\sigma}} d_{W/\upsilon(\gamma_0\sigma)}(\overline{w}^1_{\gamma_0\sigma}, \overline{w}^2_{\gamma_0\sigma}) = \inf_{w^1 \in \overline{w}^1_{\gamma_0\sigma}, w^2 \in \overline{w}^2_{\gamma_0\sigma}} \rho_W(w^1, w^2)$$

is isometric to $W/\upsilon(\gamma_0\sigma)$ and, hence, is a complete metric space as well. If the set-valued map $\Psi : W \to \Omega(W)$ is a Volterra map on the family υ, then the operator $\Psi_{\gamma_0} : W/\upsilon(\gamma_0) \to \Omega(W/\upsilon(\gamma_0))$ is a (set-valued) Volterra operator on the family $\overline{\upsilon}_{\gamma_0}$

Below we introduce the notion of local contraction for set-valued maps, which allows to investigate the solvability and parametric dependence properties of operator inclusions.

We consider the following inclusion

$$w \in \Psi w, \tag{2.2}$$

where $\Psi : W \to \Omega(W)$ is a Volterra set-valued map on the family υ of equivalence relations.

D e f i n i t i o n 2.2. We define a $\upsilon(\gamma)$*-local solution* of the inclusion (2.2), $\gamma \in (0,1)$, to be an equivalence class $\overline{w}_\gamma \in W/\upsilon(\gamma)$, that satisfies the inclusion $\overline{w}_\gamma \in \Psi_\gamma \overline{w}_\gamma$. Identifying the element w, satisfying (2.2), with it's $\upsilon(1)$-equivalence class $\overline{w}$, we consider it a *global solution* to the inclusion (2.2). We define a $\upsilon(\gamma)$*-maximally extended solution* to (2.2), $\gamma \in (0,1)$, to be a map putting in correspondence to each $\xi \in (0,\gamma)$ a $\upsilon(\xi)$-local solution $\overline{w}_\xi$, and satisfying the following two conditions:

- for any η, ξ, $0<\eta<\xi<\gamma$, it holds $\overline{w}_\xi \subseteq \overline{w}_\eta$ (where $\overline{w}_\xi$ is a restriction of $\overline{w}_\gamma$);
- for any $w^0 \in W$ it holds $\lim\limits_{\xi \to \gamma - 0} d(w_\xi, w^0_\xi) = \infty$.

For any $\gamma \in (0,1)$, we denote by $\mathbb{S}_\gamma$ and $\mathbb{S}$ the sets of $\upsilon(\gamma)$-local solutions and global solutions to (2.2), respectively.

Let $h_{W/\upsilon(\gamma)}$ be the Hausdorff metric in the space of all non-empty closed subsets of the metric space $(W/\upsilon(\gamma), d_{W/\upsilon(\gamma)})$.

D e f i n i t i o n 2.3. We define a Volterra on the system υ set-valued map $\Psi : W \to \Omega(W)$ to be *locally contracting at a point* $\gamma \in [0,1)$ on the system υ, if for any $\overline{w}_\gamma \in W/\upsilon(\gamma)$, one can find: an element $w^0 \in \overline{w}_\gamma$ and $q<1$ such that for any $r>0$ there exists $\delta>0$ such that for all $\overline{w}^1_{\gamma+\delta}, \overline{w}^2_{\gamma+\delta} \in B_{W/\upsilon(\gamma+\delta)}(\overline{w}^0_{\gamma+\delta}, r)$ $(\overline{w}^0_{\gamma+\delta} \in \Psi_{\gamma+\delta} w^0)$, satisfying in the case $\gamma>0$ for any $\xi \in (0,\gamma)$ the inclusion $\overline{w}^1_{\gamma+\delta}, \overline{w}^2_{\gamma+\delta} \subset \overline{w}^0_\xi$, where $(\overline{w}^0_\xi \subset \Psi_\xi(w^0,\lambda))$, it holds true that

$$h_{W/\upsilon(\gamma+\delta)}(\Psi_{\gamma+\delta}\overline{w}^1_{\gamma+\delta}, \Psi_{\gamma+\delta}\overline{w}^2_{\gamma+\delta}) \le q d_{W/\upsilon(\gamma+\delta)}(\overline{w}^1_{\gamma+\delta}, \overline{w}^2_{\gamma+\delta}).$$

D e f i n i t i o n 2.4. We define a Volterra set-valued map $\Psi : W \to \Omega(W)$ to be *locally contracting on the system* υ, if it is locally contracting for any $\gamma \in [0,1)$ with the constants q и $\delta(r)$ independent of $\gamma \in [0,1)$.

T h e o r e m 2.1. *Let the set-valued map* Ψ *is a locally contracting Volterra map on the system* υ.

Then the inclusion (2.2) *has a local solution and each local solution is extendable to a global or maximally extended solution.*

P r o o f. We construct the solution in the following way. We choose $r_1 = (1-q)^{-1}\rho_W(w^0, \Psi w^0) + 1$ and find all $\delta>0$ that satisfy the theorem condition with $r=r_1$. For $\delta_1 = \frac{1}{2}\sup\{\delta\}$, we have

$$h_{W/\upsilon(\delta_1)}(\Psi\overline{w}^1, \Psi\overline{w}^2) \le q d_{W/\upsilon(\delta_1)}(\overline{w}^1, \overline{w}^2)$$

at any $\overline{w}^1, \overline{w}^1 \subset B_{W/\upsilon(\delta_1)}(\overline{w}^0_{\delta_1}, r_1)$, which implies $\Psi_{\delta_1}(B_{W/\upsilon(\delta_1)}(\overline{w}^0_{\delta_1}, r_1)) \subset B_{W/\upsilon(\delta_1)}(\overline{w}^0_{\delta_1}, r_1)$. By the Nadler theorem (see e.g. [21]), the mapping Ψ_{δ_1} has a fixed point $\overline{w}_{\delta_1}$ in the ball $B_{W/\upsilon(\delta_1)}(\overline{w}^0_{\delta_1}, r_1)$. This fixed point is a $\upsilon(\delta_1)$-local solution to (2.2).

Choose some solution $\overline{w}_{\delta_1}$ to the equation (2.2) and the corresponding radius $r_2 = (1-q)^{-1} d_{W/\upsilon(\delta_1)}(\Psi(\overline{w}_{\delta_1}, w^0)$. We find all possible $\delta>0$ that satisfy the theorem condition with $r=r_2$. For $\delta_2 = \frac{1}{2}\sup\{\delta\}$ at any $\overline{w}^1, \overline{w}^2 \subset B_{W/\upsilon(\delta_1+\delta_2)}(\overline{w}^0_{\delta_1}, r_2) \cap \overline{w}_{\delta_1}$ we have

$$h_{W/\upsilon(\delta_1+\delta_2)}(\Psi\overline{w}^1, \Psi\overline{w}^2) \le q d_{W/\upsilon(\delta_1+\delta_2)}(\overline{w}^1, \overline{w}^2).$$

According to the Nadler theorem there exists a fixed point $w_{\delta_1+\delta_2}$ of the mapping $\Psi_{\delta_1+\delta_2}$ in $B_{W/\upsilon(\delta_1+\delta_2)}(\overline{w}^0_{\delta_1+\delta_2}, r_2)$. This fixed point is a $\upsilon(\delta_1+\delta_2)$-local solution to (2.2) extending the chosen $\upsilon(\delta_1)$-local solution $\overline{w}_{\delta_1}$. Next, let us choose arbitrary $\upsilon(\delta_1+\delta_2)$-local solution $w_{\delta_1+\delta_2}$, the corresponding $r_3=(1-q)^{-1}d_{W/\upsilon(\delta_1+\delta_2)}(\Psi\overline{w}_{\delta_1+\delta_2}, w^0)$, find all possible $\delta>0$ that satisfy the theorem condition with $r=r_3$ and repeat the procedure, etc.

If the distances from the obtained local solutions to the element $w^0\in W$ are uniformly bounded by some $\mathfrak{C}\in R$, then for $r=\mathfrak{C}+1$ due to the local contractivity of the multi-valued operator $F(\cdot,\lambda): W\to\Omega(W)$ we find δ such that $\delta_i\geq\frac{\delta}{2}$ at each of the steps described above. Therefore, in a finite number of steps we will obtain a unique global solution to (2.2). But if such $\mathfrak{C}$ does not exist, then the number of steps becomes infinite. As a result, we obtain a unique maximally extended solution to (2.2).

We now consider the following inclusion

$$w\in F(w,\lambda), \tag{2.3}$$

parameterized by $\lambda\in\Lambda$. We assume that for any $\lambda\in\Lambda$, the corresponding set-valued map $F(\cdot,\lambda): W\to\Omega(W)$ is a Volterra map on the family υ and $F(\cdot,\lambda_0)=\Psi$ for some $\lambda_0\in\Lambda$.

At each $\lambda\in\Lambda$, we naturally apply Definition 2.2 to the inclusion (2.3). For any $\lambda\in\Lambda$ and $\gamma\in(0,1)$, we denote by $\mathbb{S}_\gamma(\lambda)$ and $\mathbb{S}(\lambda)$ the sets of $\upsilon(\gamma)$-local solutions and global solutions to (2.3), respectively, corresponding to $\lambda\in\Lambda$.

D e f i n i t i o n 2.5. Let for any $\lambda\in\Lambda$, the set-valued map $F(\cdot,\lambda): W\to\Omega(W)$ be a Volterra map. We define a Volterra set-valued map $F: W\times\Lambda\to\Omega(W)$ to be *uniformly locally contracting on the system* υ, if it is locally contracting for any $\gamma\in[0,1)$ and $\lambda\in\Lambda$ with the constants q and $\delta(r)$ independent of $\gamma\in[0,1)$ and $\lambda\in\Lambda$.

T h e o r e m 2.2. *Let the following two conditions be satisfied:*

1) *The set-valued maps* $F(\cdot,\lambda): W\to\Omega(W)$, $\lambda\in\Lambda$ *are uniformly locally contracting on the system* υ.

2) *For any* $w\in W$ *and some* $\lambda_0\in\Lambda$, *the set-valued map* $\Psi: W\times\Lambda\to\Omega(W)$ *is lower semi-continuous (in the Hausdorff metric) at* (w,λ_0).

Then for any $\lambda\in\Lambda$, *the inclusion* (2.3) *has a local solution and each local solution is extendable to a global or maximally extended solution.*

If the inclusion (2.3) *has a global solution* $\overline{w}_0=w_0$ *at* $\lambda=\lambda_0$, *then for any* λ *(sufficiently close to* λ_0*), the inclusion* (2.3) *also has a global solution* $w=w(\lambda)$. *Moreover, the set-valued map* $\lambda\mapsto\mathbb{S}(\lambda)$ *is lower semi-continuous at* λ_0.

If the inclusion (2.3) *has a maximally extended solution* $\overline{w}_{0\zeta}$ *at* $\lambda=\lambda_0$, *then for any* $\gamma\in(0,\zeta)$, *the inclusion* (2.3) *has a local solution* $\overline{w}_\gamma=\overline{w}_\gamma(\lambda)$. *Moreover, the set-valued map* $\lambda\mapsto\mathbb{S}_\gamma(\lambda)$ *is lower semi-continuous at* λ_0.

P r o o f. The solvability of the inclusion (2.3) for any $\lambda\in B_\Lambda(\lambda_0,\varrho_0)$ follows from Theorem 1.1.

We prove the continuous dependence of the sets of solutions on the parameter λ. Consider the case when the inclusion (2.3) has global solution. Choose an arbitrary global solution $w_0=w(\lambda_0)\in W$ at $\lambda=\lambda_0$. Choose an arbitrary $\varepsilon>0$. Let us find $\delta>0$ satisfying Definition 1.3 at $r_1=\rho(w_0,w^0)+1$, $\gamma=0$ and any $\lambda\in B_\Lambda(\lambda_0,\varrho_0)$. For $k=[\frac{1}{\delta}]+1$ denote $\Delta_l=l\delta$, $l=1,2,\ldots,k$. Since the condition 2) holds true, for any $\varepsilon>0$ one can find $\sigma_1>0$ and $\varrho_1>0$ such that for each $\lambda\in B_\Lambda(\lambda_0,\varrho_1)$ we have

$$h_W(F(\varpi,\lambda),F(w_0,\lambda_0))<\frac{(1-q)\varepsilon}{6}$$

for all $\varpi \in B_W(w_0, \sigma_1)$. Assume that $\sigma_1 < \frac{(1-q)\varepsilon}{6}$. Let us find $\sigma_2 > 0$ and ϱ_2 such that for arbitrary $\lambda \in B_\Lambda(\lambda_0, \varrho_2)$ it holds that

$$h_{W/\upsilon(\Delta_{k-1})}(F_{\Delta_{k-1}}(\varpi_{\Delta_{k-1}}, \lambda), F_{\Delta_{k-1}}(\overline{w}_{0\Delta_{k-1}}, \lambda_0)) < \frac{(1-q)\sigma_1}{6}$$

for all $\varpi_{\Delta_{k-1}} \in B_{W/\upsilon(\Delta_{k-1})}(\overline{w}_{0\Delta_{k-1}}, \sigma_2)$. Assume that $\sigma_2 < \frac{(1-q)\sigma_1}{6}$, $\varrho_2 \le \varrho_1$. There exist $\sigma_3 > 0$ and ϱ_3 such that for any $\lambda \in B_\Lambda(\lambda_0, \varrho_3)$ it holds true that

$$h_{W/\upsilon(\Delta_{k-2})}(F_{\Delta_{k-2}}(\varpi_{\Delta_{k-2}}, \lambda), F_{\Delta_{k-2}}(\overline{w}_{0\Delta_{k-2}}, \lambda_0)) < \frac{(1-q)\sigma_2}{6}$$

for any $\varpi_{\Delta_{k-2}} \in B_{W/\upsilon(\Delta_{k-2})}(\overline{w}_{0\Delta_{k-2}}, \sigma_3)$; $\sigma_3 < \frac{(1-q)\sigma_2}{6}$, $\varrho_3 \le \varrho_2$ etc. We perform k iterations and at the last step find σ_k and ϱ_k , $0 < \sigma_k < \frac{(1-q)\sigma_{k-1}}{6}$, $\varrho_k \le \varrho_{k-1}$.

Let $\overline{w}_{0\Delta_1}$ denote a $\upsilon(\Delta_1)$ -local solution to the inclusion (2.3) at $\lambda = \lambda_0$, that is a fixed point of the multi-valued mapping $F_{\Delta_1}(\cdot, \lambda_0) : W/\upsilon(\Delta_1) \to \Omega(W/\upsilon(\Delta_1))$. If $h_{W/\upsilon(\Delta_1)}(\varpi_{\Delta_1}, \overline{w}_{0\Delta_1}) < \sigma_k$, then

$$h_{W/\upsilon(\Delta_1)}(F_{\Delta_1}(\varpi_{\Delta_1}, \lambda), F_{\Delta_1}(\overline{w}_{0\Delta_1}, \lambda_0)) < \frac{(1-q)\sigma_{k-1}}{6}$$

for all $\lambda \in B_\Lambda(\lambda_0, \varrho_k)$.

Taking into account the condition 1) , we get for any natural number m that

$$h_{W/\upsilon(\Delta_1)}(F^m_{\Delta_1}(\overline{w}_{0\Delta_1}, \lambda), w_{0\Delta_1}) \le h_{W/\upsilon(\Delta_1)}(F^m_{\Delta_1}(\overline{w}_{0\Delta_1}, \lambda), F^{m-1}_{\Delta_1}(\overline{w}_{0\Delta_1}, \lambda)) + \ldots$$

$$\ldots + h_{W/\upsilon(\Delta_1)}(F_{\Delta_1}(\overline{w}_{0\Delta_1}, \lambda), \overline{w}_{0\Delta_1}) \le (q^{m-1} + \ldots + q + 1)\frac{(1-q)\sigma_{k-1}}{6} \le \frac{\sigma_{k-1}}{6}.$$

Due to the convergence of the sequential approximations $F^m_{\Delta_1}(\overline{w}_{0\Delta_1}, \lambda)$ to the fixed point set $\mathbb{S}_{\Delta_1}(\lambda)$ of the multi-valued operator $F_{\Delta_1}(\cdot, \lambda) : W/\upsilon(\Delta_1) \to \Omega(W/\upsilon(\Delta_1))$, we obtain the relation $h_{W/\upsilon(\Delta_1)}(\mathbb{S}_{\Delta_1}(\lambda), \overline{w}_{0\Delta_1}) \le \frac{\sigma_{k-1}}{6}$ for each $\lambda \in B_\Lambda(\lambda_0, \varrho_k)$. Further, let $\overline{w}_{0\Delta_2}$ be a $\upsilon(\Delta_2)$ -local solution to the inclusion (2.3) at $\lambda = \lambda_0$. Choose some $\overline{w}_{\Delta_1} \in \mathbb{S}_{\Delta_1}(\lambda)$. Then, for all $\lambda \in B_\Lambda(\lambda_0, \varrho_k)$, $\varrho_k \le \varrho_{k-1}$ and any $\varpi_{\Delta_2} \in B_{W/\upsilon(\Delta_2)}(\overline{w}_{0\Delta_2}, \sigma_{k-1}) \bigcap \overline{w}_{\Delta_1}$ we get

$$h_{W/\upsilon(\Delta_2)}(F_{\Delta_2}(\varpi_{\Delta_2}, \lambda), \overline{w}_{0\Delta_2}) = h_{W/\upsilon(\Delta_1)}(F_{\Delta_2}(\varpi_{\Delta_2}, \lambda), F_{\Delta_2}(\overline{w}_{0\Delta_2}, \lambda_0)) < \frac{(1-q)\sigma_{k-2}}{6}.$$

Then

$$h_{W/\upsilon(\Delta_2)}(F_{\Delta_2}(\varpi_{\Delta_2}, \lambda), \varpi_{\Delta_2}) < \sigma_{k-1} + \frac{(1-q)\sigma_{k-2}}{6} < \frac{(1-q)\sigma_{k-2}}{3}.$$

For all $m = 1, 2, \ldots$ we have

$$h_{W/\upsilon(\Delta_2)}(F^m_{\Delta_2}(\varpi_{\Delta_2}, \lambda), \varpi_{\Delta_2}) \le h_{W/\upsilon(\Delta_2)}(F^m_{\Delta_2}(\varpi_{\Delta_2}, \lambda), F^{m-1}_{\Delta_2}(\varpi_{\Delta_2}, \lambda)) + \ldots$$

$$\ldots + h_{W/\upsilon(\Delta_2)}(F_{\Delta_2}(\varpi_{\Delta_2}, \lambda), \varpi_{\Delta_2}) \le (q^{m-1} + \ldots + q + 1)\frac{(1-q)\sigma_{k-2}}{3} \le \frac{\sigma_{k-2}}{3}.$$

Taking into account the convergence of the approximations $F^m_{\Delta_2}(u_{\Delta_2}, \lambda)$ to $\mathbb{S}_{\Delta_2}(\lambda)$, for any $\overline{w}_{\Delta_2} \in$ $\in \mathbb{S}_{\Delta_2}(\lambda)$, we obtain

$$h_{W/\upsilon(\Delta_2)}(\overline{w}_{\Delta_2}, \overline{w}_{0\Delta_2}) \le h_{W/\upsilon(\Delta_2)}(\overline{w}_{\Delta_2}, F^m_{\Delta_2}(\varpi_{\Delta_2}, \lambda)) +$$

$$+ h_{W/\upsilon(\Delta_2)}(F^m_{\Delta_2}(\varpi_{\Delta_2}, \lambda), \varpi_{\Delta_2}) + h_{W/\upsilon(\Delta_2)}(\varpi_{\Delta_2}, \overline{w}_{0\Delta_2}) \le \frac{\sigma_{k-2}}{3} + \sigma_{k-1} \le \frac{\sigma_{k-2}}{2}.$$

Using the convergence of the sequential approximations $F^m_{\Delta_3}(\varpi_{\Delta_3},\lambda)$ to the fixed point set $\mathbb{S}_{\Delta_3}(\lambda)$ of the multi-valued operator $F_{\Delta_3}(\cdot,\lambda): W/\upsilon(\Delta_3)\to\Omega(W/\upsilon(\Delta_3))$ for any $\overline{w}_{\Delta_2}\in\mathbb{S}_{\Delta_2}(\lambda)$, $\overline{\varpi}_{\Delta_3}\in$ $\in B_{W/\upsilon(\Delta_3)}(\overline{w}_{0\Delta_3},\sigma_{k-2})\bigcap\overline{w}_{\Delta_2}$ and each $\lambda\in B_\Lambda(\lambda_0,\varrho_k)$, $\varrho_k\leq\varrho_{k-1}$, we obtain the following estimate: $d_{W/\upsilon(\Delta_3)}(\overline{w}_{\Delta_3},\overline{w}_{0\Delta_3})\leq\frac{\sigma_{k-3}}{2}$. We, then, repeat this procedure. At the k-th step we prove in an analogous way that the inequality $\rho_W(w(\lambda),w_0)<\varepsilon$ holds true for some $w(\lambda)\in\mathbb{S}(\lambda)$ for all $\lambda\in B_\Lambda(\lambda_0,\varrho_k)$. Therefore, $h_W(\mathbb{S}(\lambda),w_0)\to 0$ as $\lambda\to\lambda_0$, and, thus, the set-valued map $\lambda\mapsto\mathbb{S}(\lambda)$ is lower semi-continuous at λ_0

Let now a solution $\overline{w}_{0\eta}$ to the inclusion (2.3) at $\lambda=\lambda_0$ be maximally extended. Fix arbitrary $\gamma\in(0,\eta)$ and let $\overline{w}_{0\gamma}$ denote the restriction of the solution $\overline{w}_{0\eta}$. For the equation $\overline{\varpi}_\gamma=F_\gamma(\overline{\varpi}_\gamma,\lambda_0)$ the element $\overline{w}_{0\gamma}\in W/\upsilon(\gamma)$ is a global solution. As is shown above, for all λ from some neighborhood of λ_0, the inclusions $\overline{\varpi}_\gamma\in F_\gamma(\overline{\varpi}_\gamma,\lambda)$ have global solutions $\overline{w}_\gamma(\lambda)$, and $h_{W/\upsilon(\gamma)}(\mathbb{S}_\gamma(\lambda),\overline{w}_{0\gamma})\to 0$ as $\lambda\to\lambda_0$. Therefore, the set-valued map $\lambda\mapsto\mathbb{S}_\gamma(\lambda)$ is lower semi-continuous at λ_0.

REFERENCES

1. *Shepherd G.M.* The synaptic organization of the brain // Oxford university press. 2004.

2. *Lui J.H., Hansen D.V., Kriegstein A.R.* Development and Evolution of the Human Neocortex // Cell. 2011. V. 146. № 1. P. 18–36.

3. *Swenson R.S.* Review of clinical and functional neuroscience, In: Ed. G.L. Holmes, Educational Review Manual in Neurology. Castle Connolly Graduate Medical Publishing. 2006.

4. *Graben P.B., Kurths J.* Simulating global properties of electroencephalograms with minimal random neural networks // Neurocomp. 2008. V. 71. № 4. P. 999–1007.

5. *Hopfield J.J.* Neural networks and physical systems with emergent collective computational properties // Proc. Nat. Acad. Sci. 1982. V. 79. P. 2554–2558.

6. *Burlakov E., Zhukovskiy E., Ponosov A., Wyller J.* Existence, uniqueness and continuous dependence on parameters of solutions to neural field equations // Memoirs on Differential Equations and Mathematical Physics. 2015. V. 65. P. 35–55.

7. *Amari S.* Dynamics of Pattern Formation in Lateral-Inhibition Type Neural Fields // Biol. Cybern. 1977. V. 27. P. 77–87.

8. *Sompolinsky H., Shapley R.* New perspectives on the mechanisms for orientation selectivity // Curr. Opin. Neurobiol. 1997. V. 5. P. 514–522.

9. *Taube J.S., Bassett J.P.* Persistent neural activity in head direction cells // Cereb. Cortex. 2003. V. 13. P. 1162–1172.

10. *Fuster J.M., Alexander G.* Neuron activity related to short-term memory // Science. 1971. V. 173. P. 652.

11. *Wang X-J.* Synaptic reverberation underlying mnemonic persistent activity // Trends Neurosci. 2001. V. 24. P. 455–463.

12. *Pinotsis D.A., Leite M., Friston K.J.* On conductance-based neural field models // Frontiers in Computational Neuroscience. 2013. V. 7. P. 158.

13. *Tass P.A.* A model of desynchronizing deep brain stimulation with a demand-controlled coordinated reset of neural subpopulations // Biological cybernetics. 2003. V. 89. P. 81–88.

14. *Suffczynski P., Kalitzin S., and Lopes Da Silva F.H.* Dynamics of non-convulsive epileptic phenomena modeled by a bistable neuronal network // Neuroscience. 2004. V. 126. P. 467–484.

15. *Kramer M.A., Lopour B.A., Kirsch H.E., and Szeri A.J.* Bifurcation control of a seizing human cortex // Physical Review E. 2006. V. 73 :41928.

16. *Schiff S.J.* Towards model-based control of Parkin- son's disease // Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences. 2010. V. 368. P. 2269–2308.

17. *Ruths J., Taylor P., Dauwels J.* Optimal Control of an Epileptic Neural Population Model // Proceedings of the International Federation of Automatic Control. Cape Town, 2014.

18. *Burlakov E., Zhukovskiy E.S.* Existence, uniqueness and continuous dependence on control of solutions to generalized neural field equations // Vestnik Tambovskogo universiteta. Seriya Estestvennye i tekhnicheskie nauki – Tambov University Review. Series: Natural and Technical Sciences, 2015. V. 20. Iss. 1. P. 9–16.

19. *Burlakov E.O., Zhukovskiy E.S.* On well-posedness of generalized neural field equations with impulsive control // Izv. Vuzov. Mathematics. 2016 V. 5. P. 75–79.

20. *Zhukovskiy E.S.* Generalized Volterra operators in metric spaces // Vestnik Tambovskogo universiteta. Seriya Estestvennye i tekhnicheskie nauki – Tambov University Review. Series: Natural and Technical Sciences, 2009. V. 14. Iss. 3. P. 501–508.

21. *Borisovich Yu.G., Gelman B.D., Myshkis A.D., Obukhovskii V.V.* Introduction to the Theory of Multivalued Maps and Differential Inclusions. 2nd ed. Moscow: Librokom, 2011.

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ОПЕРАТОРНЫЕ ВКЛЮЧЕНИЯ ВОЛЬТЕРРЫ В ОБОБЩЕННЫХ МОДЕЛЯХ НЕЙРОПОЛЕЙ С УПРАВЛЕНИЕМ. I

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Получены условия разрешимости операторных включений Вольтерры и непрерывной зависимости решений от параметра. Результаты могут применяться к исследованию обобщенных моделей нейрополей с управлением.
Ключевые слова: операторные включения Вольтерры; модели нейрополей; управление; существование решений; непрерывная зависимость от параметров

СПИСОК ЛИТЕРАТУРЫ

1. *Shepherd G.M.* The synaptic organization of the brain // Oxford university press. 2004.
2. *Lui J.H., Hansen D.V., Kriegstein A.R.* Development and Evolution of the Human Neocortex // Cell. 2011. V. 146. № 1. P. 18–36.
3. *Swenson R.S.* Review of clinical and functional neuroscience, In: Ed. G.L. Holmes, Educational Review Manual in Neurology. Castle Connolly Graduate Medical Publishing. 2006.
4. *Graben P.B., Kurths J.* Simulating global properties of electroencephalograms with minimal random neural networks // Neurocomp. 2008. V. 71. № 4. P. 999–1007.
5. *Hopfield J.J.* Neural networks and physical systems with emergent collective computational properties // Proc. Nat. Acad. Sci. 1982. V. 79. P. 2554–2558.
6. *Burlakov E., Zhukovskiy E., Ponosov A., Wyller J.* Existence, uniqueness and continuous dependence on parameters of solutions to neural field equations // Memoirs on Differential Equations and Mathematical Physics. 2015. V. 65. P. 35–55.
7. *Amari S.* Dynamics of Pattern Formation in Lateral-Inhibition Type Neural Fields // Biol. Cybern. 1977. V. 27. P. 77–87.
8. *Sompolinsky H., Shapley R.* New perspectives on the mechanisms for orientation selectivity // Curr. Opin. Neurobiol. 1997. V. 5. P. 514–522.
9. *Taube J.S., Bassett J.P.* Persistent neural activity in head direction cells // Cereb. Cortex. 2003. V. 13. P. 1162–1172.
10. *Fuster J.M., Alexander G.* Neuron activity related to short-term memory // Science. 1971. V. 173. P. 652.
11. *Wang X-J.* Synaptic reverberation underlying mnemonic persistent activity // Trends Neurosci. 2001. V. 24. P. 455–463.

12 . *Pinotsis D.A., Leite M., Friston K.J.* On conductance-based neural field models // Frontiers in Computational Neuroscience. 2013. V. 7. P. 158.

13 . *Tass P.A.* A model of desynchronizing deep brain stimulation with a demand-controlled coordinated reset of neural subpopulations // Biological cybernetics. 2003. V. 89. P. 81–88.

14 . *Suffczynski P., Kalitzin S., and Lopes Da Silva F.H.* Dynamics of non-convulsive epileptic phenomena modeled by a bistable neuronal network // Neuroscience. 2004. V. 126. P. 467–484.

15 . *Kramer M.A., Lopour B.A., Kirsch H.E., and Szeri A.J.* Bifurcation control of a seizing human cortex // Physical Review E. 2006. V. 73 :41928.

16 . *Schiff S.J.* Towards model-based control of Parkin- son's disease // Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences. 2010. V. 368. P. 2269–2308.

17 . *Ruths J., Taylor P., Dauwels J.* Optimal Control of an Epileptic Neural Population Model // Proceedings of the International Federation of Automatic Control. Cape Town, 2014.

18 . *Burlakov E., Zhukovskiy E.S.* Existence, uniqueness and continuous dependence on control of solutions to generalized neural field equations // Вестник Тамбовского университета. Серия Естественные и технические науки. Тамбов, 2015. Т. 20. Вып. 1. С. 9–16.

19 . *Burlakov E.O., Zhukovskiy E.S.* On well-posedness of generalized neural field equations with impulsive control // Известия вузов. Математика. 2016 Т. 5. С. 75–79.

20 . *Zhukovskiy E.S.* Generalized Volterra operators in metric spaces // Вестник Тамбовского университета. Серия Естественные и технические науки. Тамбов, 2009. Т. 14. Вып. 3. С. 501–508.

21 . *Borisovich Yu.G., Gelman B.D., Myshkis A.D., Obukhovskii V.V.* Introduction to the Theory of Multivalued Maps and Differential Inclusions. 2nd ed. Moscow: Librokom, 2011.

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