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Existence of a coincidence point in a critical case when the covering constant and the Lipschitz constant are equal

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Abstract. We consider two mappings acting between metric spaces and such that one of them is covering and the other satisfies the enhanced Lipschitz property. It is assumed here that the covering constant and the Lipschitz constant of these mappings are equal. We prove the result of the existence of a coincidence point of single-valued mappings in the case when the series of iterations of the function that provides execution of the enhanced Lipschitz property converges. We prove the similar result for set-valued mappings. We provide examples of functions for which the series of their iterations converges or diverges.

Keywords: covering mapping, coincidence point, series of iterations

Mathematics Subject Classification: 54H25, 47H04.

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НАУЧНАЯ СТАТЬЯ

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Существование точки совпадения в критическом случае, когда константа накрывания и константа Липшица совпадают

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Аннотация. Рассматриваются два отображения, действующие между метрическими пространствами, и такие, что одно из них является накрывающим, а второе удовлетворяет усиленному условию Липшица. При этом предполагается, что константа накрывания и константа Липшица у этих отображений совпадают. Доказывается результат о существовании точки совпадения однозначных отображений в случае, когда ряд из итераций функции, обеспечивающей выполнение усиленного условия Липшица, сходится. Доказывается аналогичный результат для многозначных отображений. Приводятся примеры функций, для которых ряд из их итераций сходится или расходится.

Ключевые слова: накрывающее отображение, точка совпадения, ряд из итераций

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Introduction

In this paper, we consider the problem of the existence of a coincidence point of α-covering mapping and β-Lipschitz mapping acting between metric spaces $X = (X, \rho_X)$ and $Y = (Y, \rho_Y)$ in a critical case when $\alpha = \beta$. Without loss of generality we will consider only the case $\alpha = \beta = 1$. In case when $\alpha > \beta$ theorems 1 and 2 about the existence of a coincidence point from [1] take place. But in case when $\alpha = \beta$ this theorems do not work. Without making new assumptions about Φ and Ψ a coincidence point may not exist. This is evidenced by the following simple example $X = Y = \mathbb{R}$, $\Phi(x) \equiv x$, $\Psi(x) \equiv x + 1$.

Throughout the following γ and δ denote continuous non-decreasing functions acting from $[0, +\infty)$ to $[0, +\infty)$ with the property $\gamma(t) < t$, $\delta(t) < t$ for all $t > 0$.

Let us make the following assumption on Φ :

$$\rho_Y(\Phi(x_1), \Phi(x_2)) \leqslant \gamma(\rho_X(x_1, x_2)) \qquad \forall x_1, x_2 \in X, \tag{0.1}$$

where the function γ is such that the functional series

$$A_\gamma(t) = t + \gamma(t) + \gamma(\gamma(t)) + \dots \tag{0.2}$$

converges. Series (0.2) is called the iteration series corresponding to the function γ.

The main feature of this paper is that we consider mappings Φ and Ψ for which the series (0.2) converges and $\gamma(t) < t$ for all $t > 0$. We show that under these assumptions a coincidence point of Φ and Ψ exists. Some of the constructions below are similar to constructions in [1].

For convenience we denote $\gamma^k(t) := \underbrace{\gamma(\dots(\gamma}_{k}(t)))$ as the value of k-th iteration of the function γ at the point t, and also $\gamma^0(t) := t$. Then we have $A_\gamma(t) = \sum_{k=0}^{\infty} \gamma^k(t)$.

Lemma 0.1. *The convergence set of a functional series* (0.2) *is the whole closed half-line* $[0, +\infty)$ *or just the point* $\{0\}$.

P r o o f. We show that the situation when A_γ converges at $t_1 > 0$ and at the same time diverges at $t_2 > 0$ is impossible. If $t_1 > t_2$ then we note that $A_\gamma(t_1) \geqslant A_\gamma(t_2)$. Indeed, since γ is non-decreasing and $t_1 > t_2$, we have term-by-term estimation $\gamma^k(t_1) \geqslant \gamma^k(t_2)$ for all numbers $k \geqslant 0$. Hence, due to the assumption $A_\gamma(t_1) < \infty$ and the majorant test of convergence, we have $A_\gamma(t_2) < \infty$, which leads to a contradiction. If $t_1 < t_2$ then we firstly note that the sequence $\gamma^k(t_2) \to 0$. Indeed, since $\gamma(t) < t$ for all $t > 0$, this sequence is non-increasing, i. e. is monotonic. Since it is bounded below by zero and above by t_2, there exists a limit $L = \lim_{k\to\infty} \gamma^k(t_2) \geqslant 0$. If $L > 0$ then we pass to the limit in the obvious equality $\gamma^{k+1}(t_2) = \gamma(\gamma^k(t_2))$ at $k \to \infty$ and get $L = \gamma(L) < L$ which leads to a contradiction. Hence, $L = 0$. This means that there exists a number $N \geqslant 1$ such that $\gamma^N(t_2) \leqslant t_1$. So,

$$A_\gamma(t_2) = \sum_{k=0}^{N-1} \gamma^k(t_2) + \sum_{k\geqslant N} \gamma^k(t_2) \leqslant \sum_{k=0}^{N-1} \gamma^k(t_2) + A_\gamma(t_1) < \infty,$$

which contradicts the assumption of divergence of $A_\gamma(t_2)$. □

Due to Lemma 0.1, we will further simply write that series (0.2) converges for the function γ without specifying the point at which this convergence takes place.

Lemma 0.2. *Let there exist* $T > 0 : \gamma(t) \leqslant \delta(t)$ *for all* $t \in (0, T)$ *and let the series* (0.2) *converge for the function* δ. *Then it also converges for the function* γ.

P r o o f. The proof follows obviously from the Weierstrass test of convergence for series. □

Let us now consider a case of single-valued mappings.

1. Case of single-valued mappings

Recall the definition of the coincidence point of two single-valued mappings. A point $\xi \in X$ is called a coincidence point of two single-valued mappings $\Phi : X \to Y$ and $\Psi : X \to Y$ if

$$\Phi(\xi) = \Psi(\xi).$$

Let a number $\alpha > 0$ be given. Then a mapping $\Psi : X \to Y$ is called α*-covering* if

$$B^Y(\Psi(x), \alpha r) \subset \Psi(B^X(x, r)) \quad \forall x \in X \; \forall r > 0.$$

(In case when Ψ is set-valued the definition remains the same.)

The following statement takes place.

Theorem 1.1. *Let* $X = (X, \rho_X)$ *be a complete metric space,* $Y = (Y, \rho_Y)$ *be a metric space,* $\Phi : X \to Y$ *and* $\Psi : X \to Y$ *be mappings such that* Ψ *is* 1*-covering and continuous and* Φ *satisfies the inequality* (0.1) *with a function* γ *for which the series* (0.2) *converges. Then* Φ *and* Ψ *have a coincidence point.*

P r o o f. We will construct the sequence $\{x_n\} \subset X$ with the property $\Psi(x_{n+1}) = \Phi(x_n)$ for all numbers $n \geqslant 1$ by induction. We choose the point x_0 arbitrary. Let $r_0 = \rho_Y(\Phi(x_0), \Psi(x_0))$. Since Ψ is 1-covering, $\Psi(B^X(x_0, r_0)) \supset B^Y(\Psi(x_0), r_0) \ni \Phi(x_0)$. So there exists a point $x_1 \in B^X(x_0, r_0) \colon \Psi(x_1) = \Phi(x_0)$. The point x_1 is built.

Let the points $x_0, \ldots, x_j$ be already built. Put $r_j = \rho_Y(\Psi(x_j), \Phi(x_j))$. Since Ψ is 1-covering, $\Psi(B^X(x_j, r_j)) \supset B^Y(\Psi(x_j), r_j) \ni \Phi(x_j)$. So there exists a point $x_{j+1} \in B^X(x_j, r_j)$: $\Psi(x_{j+1}) = \Phi(x_j)$. The point x_{j+1} is built.

Put $\rho_j = \rho_X(x_j, x_{j-1})$ for $j \geqslant 1$ and $\rho_0 = \rho_Y(\Phi(x_0), \Psi(x_0))$. Then for each point x_j we have the estimation

$$\begin{aligned} \rho_j = \rho_X(x_j, x_{j-1}) \leqslant r_{j-1} = \rho_Y(\Psi(x_{j-1}), \Phi(x_{j-1})) &= \rho_Y(\Phi(x_{j-2}), \Phi(x_{j-1})) \\ &\leqslant \gamma(\rho_X(x_{j-2}, x_{j-1})) = \gamma(\rho_{j-1}) \end{aligned}$$

(if $j > 1$) and the estimation $\rho_1 \leqslant \rho_Y(\Phi(x_0), \Psi(x_0)) = r_0 = \rho_0$ (if $j = 1$). Note that $\rho_j \leqslant \gamma(\rho_{j-1}) \leqslant \ldots \leqslant \gamma^{j-1}(\rho_1)$ for any $j \geqslant 1$. For arbitrary numbers $m < n$ we have the estimation

$$\begin{aligned} \rho_X(x_m, x_n) &\leqslant \rho_X(x_m, x_{m+1}) + \rho_X(x_{m+1}, x_{m+2}) + \ldots + \rho_X(x_{n-1}, x_n) \\ &= \rho_{m+1} + \rho_{m+2} + \ldots + \rho_n \leqslant \rho_{m+1} + \gamma(\rho_{m+1}) + \ldots + \gamma^{n-m-1}(\rho_{m+1}) \\ &\leqslant \gamma^m(\rho_1) + \gamma^{m+1}(\rho_1) + \ldots + \gamma^{n-1}(\rho_1) \leqslant A_\gamma(\rho_1). \end{aligned}$$

Here we use the triangle inequality for the metric ρ_X and the fact of convergence of the series (0.2). Due to this estimation we get that the sequence $\{x_n\}$ is fundamental. Since the metric space X is complete, the sequence $\{x_n\}$ converges to some point $\xi \in X$. Passing to the limit in the equation $\Psi(x_n) = \Phi(x_{n-1})$ at $n \to \infty$, we get $\Psi(\xi) = \Phi(\xi)$. □

Let us now proceed to set-valued case.

2. Case of set-valued mappings

We assume that $X = (X, \rho_X)$ and $Y = (Y, \rho_Y)$ are metric spaces, $\Phi : X \rightrightarrows Y$ and $\Psi : X \rightrightarrows Y$ are set-valued mappings, i. e. $\Phi(x), \Psi(x) \subset Y$ for all $x \in X$ and the sets $\Phi(x), \Psi(x)$ are closed in the space Y in relation to metric ρ_Y.

We assume that the mapping Φ obeys the inequality

$$h(\Phi(x_1), \Phi(x_2)) \leqslant \gamma(\rho_X(x_1, x_2)) \tag{2.1}$$

where $h(\cdot, \cdot)$ is the Hausdorff distance defined by the equality

$$h(K_1, K_2) = \inf\{r > 0 : B^Y(K_1, r) \supset K_2,\ B^Y(K_2, r) \supset K_1\}$$

for arbitrary bounded sets $K_1, K_2 \subset Y$. Here $B^Y(K, r) = \{y \in Y : \operatorname{dist}(y, K) \leqslant r\}$ is the r-neighbourhood of the set K and $\operatorname{dist}(y, K)$ is the distance between point y and set K. We assume that the mapping Ψ is 1-covering and its graph $\operatorname{gph}\Psi = \{(x, y) \in X \times Y : y \in \Psi(x)\}$ is closed in relation to the metric $\rho((x_1, y_1), (x_2, y_2)) = \rho_X(x_1, x_2) + \rho_Y(y_1, y_2)$ defined on the Cartesian product $X \times Y$.

Recall the definition of a coincidence point of two set-valued mappings. A point $\xi \in X$ is called a coincidence point of set-valued mappings $\Phi : X \rightrightarrows Y$ and $\Psi : X \rightrightarrows Y$ if

$$\Phi(\xi) \cap \Psi(\xi) \neq \emptyset.$$

Let us formulate and prove the lemma about set-valued mappings.

Lemma 2.1. *Let $X = (X, \rho_X)$ and $Y = (Y, \rho_Y)$ be metric spaces. Let the set-valued mappings $\Phi : X \rightrightarrows Y$, $\Psi : X \rightrightarrows Y$ be compact-valued and such that Ψ is 1-covering and Φ satisfies (2.1). Then there exist sequences $\{x_n\} \subset X$ and $\{y_n\} \subset Y$ such that*

$$\rho_X(x_1, x_0) \leqslant \operatorname{dist}(\Phi(x_0), \Psi(x_0)), \tag{2.2}$$

$$y_i \in \Psi(x_i) \cap \Phi(x_{i-1}) \quad \forall i \geqslant 1, \tag{2.3}$$

$$\rho_X(x_i, x_{i-1}) \leqslant \gamma(\rho_X(x_{i-1}, x_{i-2})) \quad \forall i \geqslant 2, \tag{2.4}$$

$$\rho_Y(y_i, y_{i-1}) \leqslant \gamma(\rho_X(x_{i-1}, x_{i-2})) \quad \forall i \geqslant 2. \tag{2.5}$$

P r o o f. We will construct these sequences by induction. The point x_0 is taken arbitrary. Let $r_0 = \operatorname{dist}(\Phi(x_0), \Psi(x_0))$, then due to compact-valuing of Φ, Ψ we have $\exists x_1 \in B^X(x_0, r) :$ $\Psi(x_1) \cap \Phi(x_0) \neq \emptyset$. We take an arbitrary point $y_1 \in \Psi(x_1) \cap \Phi(x_0)$. Then the points x_1, y_1 satisfying (2.2)–(2.3) are built.

If $x_1 = x_0$ then we put $x_2 = x_3 = \ldots = x_1$. Assume that $x_1 \neq x_0$. Put $r_1 = \gamma(\rho_X(x_1, x_0))$, then from (2.1) we have the inequality $h(\Phi(x_0), \Phi(x_1)) \leqslant r_1$, hence, since Φ is compact-valued, we get $B^Y(\Phi(x_1), r_1) \supset \Phi(x_0) \ni y_1$. It means that $\exists y_2 \in \Phi(x_1) : \rho_Y(y_2, y_1) \leqslant r_1$, so $y_2 \in B^Y(\Psi(x_1), r_1)$ at $y_1 \in \Psi(x_1)$. Hence, there exists a point $x_2 \in B^X(x_1, r_1) : y_2 \in \Psi(x_2)$. The points x_2, y_2 satisfying (2.2)–(2.4) are built.

Let the points $x_0, y_0, x_1, y_1, \ldots, x_j, y_j$ be already built. If $x_j = x_{j-1}$ then we put $x_{j+1} = x_j$. Assume that $x_j \neq x_{j-1}$. Put $r_j = \gamma(\rho_X(x_j, x_{j-1}))$. Then due to (2.1) we have the inequality $h(\Phi(x_j), \Phi(x_{j-1})) \leqslant r_j$, and since Φ is compact-valued we get $B^Y(\Phi(x_j), r_j) \supset \Phi(x_{j-1}) \ni y_j$. It means that $\exists y_{j+1} \in \Phi(x_j) : \rho_Y(y_{j+1}, y_j) \leqslant r_j$, and hence, $y_j \in \Psi(x_j)$, so we conclude that $y_{j+1} \in B^Y(\Psi(x_j), r_j)$, and due to 1-covering of Ψ there exists a point $x_{j+1} \in B^X(x_j, r_j) :$ $y_{j+1} \in \Psi(x_{j+1})$. The points x_{j+1}, y_{j+1} satisfying (2.3)–(2.5) are built. □

Let us formulate and prove a result about the existence of a coincidence point of two set-valued mappings.

Theorem 2.1. *Let $X = (X, \rho_X)$ and $Y = (Y, \rho_Y)$ be metric spaces and the set-valued mappings $\Phi : X \rightrightarrows Y$ and $\Psi : X \rightrightarrows Y$ be compact-valued and such that Ψ is 1-covering and its graph $\operatorname{gph}\Psi$ is closed. Let Φ satisfy the inequality (2.1) with a function γ for which the series (0.2) converges. Let also at least one of the sets $\operatorname{gph}\Phi \subset X \times Y$, $\operatorname{gph}\Psi \subset X \times Y$ be a complete metric space. Then Φ and Ψ have a coincidence point.*

P r o o f. Due to Lemma 2.1, there exist sequences $\{x_n\}$, $\{y_n\}$ satisfying (2.2)–(2.5). Because of the inequalities (2.4), (2.5) and the convergence of the series (0.2), these sequences are fundamental.

Assume at first that $\operatorname{gph}\Psi$ is complete. Then the sequence $\{(x_n, y_n)\}$ with all its elements lying in $\operatorname{gph}\Psi$ is fundamental due to (2.3). Then we have the convergence $(x_n, y_n) \to (\xi, y) \in X \times Y$. Due to closeness of $\operatorname{gph}\Psi$ and (2.3), the point $(\xi, y) \in \operatorname{gph}\Psi$, and hence $y \in \Psi(\xi)$. On the other side, $y \in \Phi(\xi)$, as we can pass to the limit in (2.3) at $i \to \infty$ by taking into account upper semi-continuity of Φ (that immediately follows from (2.1)). We obtain that $y \in \Phi(\xi)$ and $y \in \Psi(\xi)$, which means that y is the coincidence point of Φ and Ψ.

Assume now that $\operatorname{gph}\Phi$ is complete. Then the sequence $\{(x_n, y_{n+1})\}$ with all its elements lying in $\operatorname{gph}\Phi$ because of (2.3) is fundamental. Then we have the convergence $(x_n, y_{n+1}) \to (\xi, y) \in \operatorname{gph}\Phi$, and hence $y \in \Phi(\xi)$. By passing to the limit in (2.3) at $i \to \infty$ and taking into account the closeness of $\operatorname{gph}\Psi$ we obtain $y \in \Psi(\xi)$. It means that $y \in \Phi(\xi)$ and $y \in \Psi(\xi)$, i. e. y is the coincidence point of Φ and Ψ. □

3. Examples

Here we provide some examples of functions γ for which the series (0.2) converges.

E x a m p l e 3.1. $\gamma(t) = \beta t$ where the number $\beta \in (0, 1)$. In this case the mapping Φ is Lipschitz with Lipschitz constant $\beta < 1$ and the series $A_\gamma(t) = \sum_{k=0}^{\infty} \beta^k t = t/(1-\beta)$.

E x a m p l e 3.2. $\gamma(t) = t/(1+t^\beta)$ where the number $\beta \in (0, 1)$. In this case, there exists a right derivative $\gamma'(0) = \lim_{t\to 0+0} \gamma(t)/t = \lim_{t\to 0+0} 1/(1+t^\beta) = 1$ and hence the mapping Φ is not Lipschitz with any constant $\beta < 1$. Let us show that the series A_γ converges. At first we will prove some statements.

Lemma 3.1. *For any number $\beta \in (0, 1)$ there exist numbers $\alpha > 1$ and $T > 0$ such that*

$$\frac{t}{1+t^\beta} \leqslant \frac{t}{\left(1+t^{\frac{1}{\alpha}}\right)^\alpha}$$

for all $t \in (0, T)$.

P r o o f. Consider the function $f(t) = t/(1+t^\beta) - t/\left(1+t^{\frac{1}{\alpha}}\right)^\alpha$. Its derivative

$$f'(t) = t^{\beta-1}\left(\left(1+t^{\frac{1}{\alpha}}\right)^{\alpha-1} t^{\frac{1}{\alpha}} - \beta\right).$$

Let $\alpha = (1+1/\beta)/2$, then we have $\alpha > 1$ and $\alpha\beta = (\beta+1)/2 < 1$, so $1/\alpha > \beta$ and $t^{\frac{1}{\alpha-\beta}} \to 0$ at $t \to 0+0$. Since $\left(1+t^{\frac{1}{\alpha}}\right)^{\alpha-1} \to 1$ at $t \to 0+0$, we have $\left(1+t^{\frac{1}{\alpha}}\right)^{\alpha-1} t^{\frac{1}{\alpha}-\beta} - \beta \to -\beta < 0$ at $t \to 0+0$, so there exists a number $T > 0$: $f'(t) < 0$ for any $t \in (0, T)$.

Since $f(0) = 0$, we can conclude that

$$\frac{t}{1+t^\beta} \leqslant \frac{t}{\left(1+t^{\frac{1}{\alpha}}\right)^\alpha}$$

for arbitrary $t \in (0, T)$, and it ends the proof. □

Lemma 3.2. *The series* (0.2) *converges for the function* $\delta(t) = t/\left(1+t^{\frac{1}{\alpha}}\right)^\alpha$ *for all* $\alpha > 1$.

P r o o f. We will show that if $\delta(t) = t/(1+t^{\frac{1}{\alpha}})^\alpha$ then $A_\delta(1) = \zeta(\alpha)$, where ζ denotes the Riemann zeta function. Indeed, we have $\delta^n(1) = 1/(1+n)^\alpha$ for any $n \geqslant 0$. Let us show it by induction on n. If $n = 0$ then $\delta^0(1) = 1 = 1/(0+1)^\alpha$. Suppose that $\delta^n(1) = 1/(1+n)^\alpha$ for any $n < k$. Then we have at $n = k$

$$\delta^k(1) = \delta(\delta^{k-1}(1)) = \delta\Big(\frac{1}{k^\alpha}\Big) = \frac{\dfrac{1}{k^\alpha}}{\left(1+\left(\dfrac{1}{k^\alpha}\right)^{\frac{1}{\alpha}}\right)^\alpha} = \frac{\dfrac{1}{k^\alpha}}{\left(1+\dfrac{1}{k}\right)^\alpha} = \frac{1}{(1+k)^\alpha}.$$

Hence,

$$A_\delta(1) = \sum_{s=0}^{\infty} \delta^s(1) = \sum_{s=0}^{\infty} \frac{1}{(1+s)^\alpha} = \zeta(\alpha) < \infty.$$

Due to Lemma 0.1, the convergence of the series A_δ at $t = 1$ implies the convergence of A_δ at any point $t > 0$. □

Lemmas 3.1, 3.2 justify the convergence of the series A_γ for $\gamma(t) = t/(1+t^\beta)$ with arbitrary $\beta \in (0, 1)$.

R e m a r k 3.1. The result about a coincidence point of set-valued mappings for functions $\gamma(t) = t/(1+t^\beta)$ at $\beta \in (0, 1)$ was previously obtained in [2, § 4, Proposition 1].

Now we provide some examples of γ for which the series (0.2) diverges.

E x a m p l e 3.3. Consider the function $\gamma(t) = t/(1+Ct)$ where the constant $C > 0$. It is a straightforward task to ensure that the iteration value $\gamma^k(1) = 1/(1+kC)$ for all numbers $k \geqslant 0$. So we have

$$\begin{aligned} A_\gamma(1) = 1 + \frac{1}{1+C} + \ldots + \frac{1}{1+nC} + \ldots = 1 + \frac{1}{C}\Big(\frac{1}{1+\frac{1}{C}} + \ldots \frac{1}{n+\frac{1}{C}} + \ldots\Big) \\ \geqslant 1 + \frac{1}{C}\Big(\frac{1}{1+N} + \ldots \frac{1}{n+N} + \ldots\Big) = \infty, \end{aligned}$$

where the number $N = [1/C] + 1 > 1/C$ ($[x] = \max\{z \in \mathbb{Z} : z \leqslant x\}$ is the integer part of the number x). Due to the divergence of the harmonic series, the series A_γ diverges.

E x a m p l e 3.4. Suppose that the function γ satisfies the inequality $\gamma(t) \geqslant t - Ct^2$ for any $t \in (0, T)$ where $C, T > 0$ are some constants. Then

$$\exists T_1 > 0 : \ \forall t \in (0, T_1) \ \Rightarrow \ \gamma(t) \geqslant \frac{t}{1+2Ct}.$$

Indeed, we have

$$\gamma(t) \geqslant t - Ct^2 = \frac{t}{1+2Ct} + \frac{Ct^2(1-2Ct)}{1+2Ct} \geqslant \frac{t}{1+2Ct} \quad \forall t \in \Big(0, \frac{1}{2C}\Big),$$

and by choosing $T_1 = \min\{1/2C, T\}$ we get what required. Due to the previous example, the series (0.2) for the function $t/(1+2Ct)$ diverges, and hence due to Lemma 0.2 it diverges for the function γ as well.

Conclusion

In conclusion, let us formulate some questions whose answers are still unknown to the authors.

1. Is there a function γ that does not satisfy the condition

$$\exists T > 0 : \forall t \in (0, T) \Rightarrow \gamma(t - \gamma(t)) \leqslant \gamma(t) - \gamma(\gamma(t))$$

and such that the series (0.2) for it converges?

2. Will the statements of Theorems 1.1 and 2.1 remain true for coincidence points (but not for fixed points) if we omit the requirement of convergence of the series (0.2)?

3. Let a mapping $\Phi : X \to Y$ be such that for any function γ the inequality

$$\rho_Y(\Phi(x_1), \Phi(x_2)) \leqslant \gamma(\rho_X(x_1, x_2)) \quad \forall x_1, x_2 \in X$$

leads to $A_\gamma = \infty$. Does such a Φ exist?

Let a compact-valued mapping $\Phi : X \rightrightarrows Y$ be such that for any function γ the inequality

$$h(\Phi(x_1), \Phi(x_2)) \leqslant \gamma(\rho_X(x_1, x_2)) \quad \forall x_1, x_2 \in X$$

leads to $A_\gamma = \infty$. Does such a Φ exist?

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