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Two parameter C_0-semigroups of linear operators on locally convex spaces

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Abstract. The purpose of this paper is to study two parameter (resp. n-parameter) exponentially equicontinuous C_0-semigroups of continuous linear operators on sequentially complete locally convex Hausdorff spaces. In particular, we demonstrate the Hille–Yosida theorem for two parameter (resp. n-parameter) exponentially equicontinuous C_0-semigroups of continuous linear operators on sequentially complete locally convex Hausdorff spaces. Moreover, the n-parameter C_0-semigroups of continuous linear operators on Banach spaces are studied.

Keywords: C_0-semigroups of continuous linear operators, two parameter semigroups of continuous linear operators, locally convex spaces

Mathematics Subject Classification: 47D03.

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НАУЧНАЯ СТАТЬЯ

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Двухпараметрические C_0-полугруппы линейных операторов на локально выпуклых пространствах

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Аннотация. Целью данной работы является изучение двухпараметрических (соответственно n-параметрических) экспоненциально равностепенно непрерывных C_0-полугрупп непрерывных линейных операторов на секвенциально полных локально выпуклых хаусдорфовых пространствах. В частности, мы демонстрируем теорему Хилле–Иосиды для двухпараметрических (соответственно n-параметрических) экспоненциально равностепенно непрерывных C_0-полугрупп непрерывных линейных операторов на секвенциально полных локально выпуклых хаусдорфовых пространствах. Кроме того, изучаются n-параметрические C_0-полугруппы непрерывных линейных операторов на банаховых пространствах.

Ключевые слова: C_0-полугруппы непрерывных линейных операторов, двухпараметрические полугруппы непрерывных линейных операторов, локально выпуклые пространства

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1. Introduction

Abdelaziz [1] obtained a criterion for commutativity of the semigroups in terms of the generators and their domains. His criterion used to demonstrate the converse of a theorem of Hille and Phillips related to n-parameter semigroups of continuous linear operators on a Banach space. Recently, Al-Sharif and Khalil [2] gave a new definition of the infinitesimal generator for two parameter semigroups that gave the results in Trotter [3] and Abdelaziz [1] as special cases.

Semigroups of continuous linear operators in locally convex spaces were studied by several authors [4–7]. As an application of C_0-semigroups of continuous linear operators on sequentially complete locally convex Hausdorff spaces is the abstract Cauchy problem for differential equations on a sequentially complete locally convex Hausdorff space X given by

$$\begin{cases} \dfrac{du(t)}{dt} = Au(t), \quad t \in \mathbb{R}_+, \\ u(0) = x, \end{cases}$$

where $A : D(A) \subset X \to X$ is a densely defined closed linear operator on a sequentially complete locally convex Hausdorff space X and $x \in X$.

Throughout this paper, X is a sequentially complete locally convex Hausdorff space over the field of complex numbers $\mathbb{C}$ under the family of seminorms Γ_X and $\mathcal{L}(X)$ denotes the collection of continuous linear operators on X.

2. Preliminaries

In this section, we recall some preliminaries.

Definition 2.1. [8, p. 116] Let X be a vector space over $\mathbb{K}(=\mathbb{R}$ or $\mathbb{C})$. A function $p : X \to \mathbb{R}_+$ is called seminorm if

(i) For any $x \in X$ and for all $\lambda \in \mathbb{K}$, $p(\lambda x) = |\lambda| p(x)$,

(ii) For all $x, y \in X$, $p(x + y) \leq p(x) + p(y)$.

Definition 2.2. [8, Definition 4.1.2] Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let E be a subset of a vector space X over $\mathbb{K}$. Then

(i) E is said to be balanced if $\lambda x \in E$ for all $x \in E$ and for any $\lambda \in \mathbb{K}$ such that $|\lambda| \leq 1$,

(ii) E is said to be absorbent or absorbing if for all $x \in X$, there is a positive number r such that $\lambda x \in E$ for all $|\lambda| \leq r$.

Definition 2.3. [8, (a) of p. 6] A topological vector space X is said to be Hausdorff if for any two distinct points $x, y \in X$, there exist a neighbourhood U of x and a neighbourhood V of y such that $U \cap V = \emptyset$.

Definition 2.4. [9, Definition 2] A complex linear topological vector space X is called a locally convex space, if any if its open sets contains a convex, balanced and absorbing open set.

Theorem 2.1. [8, (a) of Theorem 5.5.1] *A locally convex space X is Hausdorff if and only if for all non-zero $x \in X$, there is a continuous seminorm p on X such that $p(x) \neq 0$.*

D e f i n i t i o n 2.5. [10, p. 294] Let X be a sequentially complete locally convex space. A one-parameter family $(J(t))_{t\in\mathbb{R}_+}$ of continuous linear operators on X is an exponentially equicontinuous C_0-semigroup if

(i) $J(0)=I$,

(ii) For any $t,s\in\mathbb{R}_+$, $J(t+s)=J(t)J(s)$,

(iii) $\lim\limits_{h\to 0^+} J(h)x=x$ for any $x\in X$,

(iv) There exists a real number $\omega>0$ such that $\{e^{-\omega t}J(t);\, t\in\mathbb{R}_+\}$ is equicontinuous.

The linear operator A defined by

$$D(A)=\big\{x\in X:\lim_{t\to 0^+}\frac{J(t)x-x}{t}\ \text{exists}\big\}$$

and

$$Ax=\lim_{t\to 0^+}\frac{J(t)x-x}{t}\ \text{ for any } x\in D(A),$$

is called the infinitesimal generator of the semigroup $(J(t))_{t\in\mathbb{R}_+}$.

Theorem 2.2. [11, Theorem 1] *Let A be a closed linear densely defined operator on a sequentially complete locally convex Hausdorff space X. Then in order that A be the infinitesimal generator of an equicontinuous C_0-semigroup $(J(s))_{s\in\mathbb{R}_+}$ it is necessary and sufficient that the family*

$$\{\lambda^n R(\lambda,A)^n;\ \lambda>0,\ n\in\mathbb{N}\} \tag{2.1}$$

is equicontinuous where $R(\lambda,A)=(\lambda I-A)^{-1}$.

Theorem 2.3. [11, Theorem 2] *Let A be a closed linear densely defined operator on a sequentially complete locally convex Hausdorff space X. Then in order that A be the infinitesimal generator of a continuous semigroup $(J(s))_{s\in\mathbb{R}_+}$ with $\{e^{-\omega s}J(s);\ s\in\mathbb{R}_+\}$ is equicontinuous for some $\omega>0$ it is necessary and sufficient that the family*

$$\{(\lambda-\omega)^n R(\lambda,A)^n;\ \lambda>\omega,\ n\in\mathbb{N}\} \tag{2.2}$$

is equicontinuous where $R(\lambda,A)=(\lambda I-A)^{-1}$.

We continue with the following definitions.

D e f i n i t i o n 2.6. [12, Definition 2.8] Let X be a locally convex space. A two-parameter family $(J(s,t))_{(s,t)\in\mathbb{R}_+\times\mathbb{R}_+}$ of continuous linear operators on X is said to be a two-parameter semigroup if

(i) $J(0,0)=I$,

(ii) For every t_1, t_2, s_1 and $s_2\in\mathbb{R}_+$, $J(s_1+s_2,t_1+t_2)=J(s_1,t_1)J(s_2,t_2)$.

D e f i n i t i o n 2.7. [12, Definition 2.9] Let X be a sequentially complete locally convex space. A two-parameter semigroup $(J(s,t))_{(s,t)\in\mathbb{R}_+\times\mathbb{R}_+}$ on X is a two-parameter strongly continuous semigroup or a two parameter C_0-semigroup if $\lim\limits_{(s,t)\to(0^+,0^+)} J(s,t)x=x$ for all $x\in X$.

Definition 2.8. [12, Definition 2.10] Let X be a sequentially complete locally convex Hausdorff space. A two-parameter C_0-semigroup $(J(s,t))_{(s,t)\in\mathbb{R}_+\times\mathbb{R}_+}$ on X is equicontinuous if for any continuous seminorm p on X, there exists a continuous seminorm q on X such that for all $x\in X$ and for each $(s,t)\in\mathbb{R}_+^2$, $p(J(s,t)x)\le q(x)$.

Definition 2.9. [12, Definition 2.11] Let X be a sequentially complete locally convex Hausdorff space. A two-parameter C_0-semigroup $(J(s,t))_{(s,t)\in\mathbb{R}_+\times\mathbb{R}_+}$ on X is exponentially equicontinuous if there exist $\omega_1,\omega_2>0$ such that $\{e^{-\omega_1 s-\omega_2 t}J(s,t);\ s,t\in\mathbb{R}_+\}$ is equicontinuous.

3. Main results

In this section, we start with the following remark.

Remark 3.1. Let X be a sequentially complete locally convex Hausdorff space. Let $(J(s,t))_{(s,t)\in\mathbb{R}_+\times\mathbb{R}_+}$ be a two parameter exponentially equicontinuous C_0-semigroup on X. Then for all $x\in X$, the map $(s,t)\mapsto J(s,t)x$ is continuous on $\mathbb{R}_+\times\mathbb{R}_+$ into X. Let A_1 and A_2 be two linear operators defined by

$$D(A_1)=\{x\in X:\lim_{h\to 0^+}\frac{J(h,0)x-x}{h}\ \text{exists in}\ X\},$$

$$D(A_2)=\{x\in X:\lim_{h\to 0^+}\frac{J(0,h)x-x}{h}\ \text{exists in}\ X\},$$

and

$$A_1x=\lim_{h\to 0^+}\frac{J(h,0)x-x}{h}=\frac{\partial J(s,t)}{\partial s}\Big|_{(s,t)=(0,0)},\ \text{for each}\ x\in D(A_1),$$

$$A_2x=\lim_{h\to 0^+}\frac{J(0,h)x-x}{h}=\frac{\partial J(s,t)}{\partial t}\Big|_{(s,t)=(0,0)},\ \text{for each}\ x\in D(A_2).$$

It is easy to see that A_1 and A_2 are the infinitesimal generators of the one parameter semigroups $(J(s,0))_{s\in\mathbb{R}_+}$ and $(J(0,t))_{t\in\mathbb{R}_+}$ respectively.

Theorem 3.1. *Let $(J(t))_{t\in\mathbb{R}_+}$ be an exponentially equicontinuous C_0-semigroup on X, where X is a sequentially complete locally convex Hausdorff space. Then for all $x\in X$ and for each $t\in\mathbb{R}_+$,*

$$\lim_{h\to 0^+}\frac{1}{h}\int_t^{t+h}J(u)x\,du=J(t)x.$$

Proof. Let $x\in X$, $t\in\mathbb{R}_+$ and $\varepsilon>0$. From $(J(u))_{u\in\mathbb{R}_+}$ is an exponentially equicontinuous C_0-semigroup on X, then for all $\varepsilon>0$, there exists $\delta>0$ such that for each $u\in(0,\delta)$, we have

$$p(J(u)x-x)<\varepsilon\ \text{for all}\ x\in X\ \text{and for each}\ p\in\Gamma_X.$$

Hence for all $h\in(0,\delta)$, $x\in X$ and for any $p\in\Gamma_X$,

$$p\Big(\frac{1}{h}\int_0^h J(u)x\,du-x\Big)=p\Big(\frac{1}{h}\int_0^h[J(u)x-x]du\Big)\le\frac{1}{h}\int_0^h p\big(J(u)x-x\big)du<\frac{1}{h}\int_0^h\varepsilon\,du=\varepsilon.$$

Thus

$$\lim_{h\to 0^+}\frac{1}{h}\int_0^h J(u)x\,du=x.$$

Since $(J(u))_{u\in\mathbb{R}_+}$ is continuous on X, we have

$$\lim_{h\to 0^+}\frac{1}{h}\int_t^{t+h}J(u)xdu=J(t)\Big(\lim_{h\to 0^+}\frac{1}{h}\int_0^h J(u)xdu\Big)=J(t)x,$$

for all $t\in\mathbb{R}_+$ and for each $x\in X$. □

We extend the Corollary 2.5 of [13] in a sequentially complete locally convex Hausdorff space as follows.

Corollary 3.1. *Let X be a sequentially complete locally convex Hausdorff space, and let $(J(t))_{t\in\mathbb{R}_+}$ be an exponentially equicontinuous C_0-semigroup of continuous linear operators on X with infinitesimal generator A. Then A is a closed operator with dense domain in X.*

We extend the Theorem 2.6 of [13] in a sequentially complete locally convex Hausdorff space as follows.

Theorem 3.2. *Let $(J(t))_{t\in\mathbb{R}_+}$ and $(S(t))_{t\in\mathbb{R}_+}$ be two exponentially equicontinuous C_0-semigroups of continuous linear operators on X, where X is a sequentially complete locally convex Hausdorff space. Suppose that both has the same generator A. Then $J(t)=S(t)$ for any $t\in\mathbb{R}_+$.*

We define the infinitesimal generator of a two parameter semigroup in a locally convex space as follows.

Definition 3.1. Let X be a locally convex space. Let $(J(s,t))_{(s,t)\in\mathbb{R}_+\times\mathbb{R}_+}$ be a two-parameter semigroup on X. The infinitesimal generator of the two parameter semigroup $(J(s,t))_{s,t\in\mathbb{R}_+}$ is the derivative of J at $(0,0)$.

From the definition of the infinitesimal generator, we have.

Theorem 3.3. *Let X be a sequentially complete locally convex Hausdorff space. If $(J(s,t))_{(s,t)\in\mathbb{R}_+\times\mathbb{R}_+}$ is a two parameter exponentially equicontinuous C_0-semigroup on X. Then the infinitesimal generator of the two parameter exponentially equicontinuous C_0-semigroup $(J(s,t))_{(s,t)\in\mathbb{R}_+\times\mathbb{R}_+}$ is the linear transformation $L:\mathbb{R}_+\times\mathbb{R}_+\to\mathcal{L}(D(A_1)\cap D(A_2),X)$ defined by for all $x\in D(A_1)\cap D(A_2)$ and $(a,b)\in\mathbb{R}_+\times\mathbb{R}_+$, $L(a,b)x=(A_1,A_2)\begin{pmatrix}a\\ b\end{pmatrix}x=aA_1x+bA_2x$. Furthermore if $x\in D(A_1)\cap D(A_2)$, we have for all $(a,b)\in\mathbb{R}_+\times\mathbb{R}_+$, $DJ(s,t)\begin{pmatrix}a\\ b\end{pmatrix}x=(A_1,A_2)\begin{pmatrix}a\\ b\end{pmatrix}J(s,t)x$, where A_1 and A_2 are the infinitesimal generators of the one parameter exponentially equicontinuous C_0-semigroups $(J(s,0))_{s\in\mathbb{R}_+}$ and $(J(0,t))_{t\in\mathbb{R}_+}$ on X respectively.*

Proof. Let $x\in D(A_1)\cap D(A_2)$ and $(s,t)\in\mathbb{R}_+\times\mathbb{R}_+$. Then $DJ(s,t)|_{(s,t)=(0,0)}$ the derivate of $(J(s,t))_{(s,t)\in\mathbb{R}_+\times\mathbb{R}_+}$ at $(0,0)$ as a function of two variables exists if there exists a linear transformation L from $\mathbb{R}_+\times\mathbb{R}_+$ into $\mathcal{L}(D(A_1)\cap D(A_2),X)$ such that $J(s,t)=L(s,t)+R(s,t)$ where

$$\lim_{(s,t)\to(0^+,0^+)}\frac{p(R(s,t)x)}{\|(s,t)\|}=0\ \text{for all}\ p\in\Gamma_X.$$

Let A_1, A_2 be infinitesimal generators of the one parameter semigroups $(J(s,0))_{s\in\mathbb{R}_+}$ and $(J(0,t))_{t\in\mathbb{R}_+}$ respectively. Since $(J(s,0))_{s\in\mathbb{R}_+}$ and $(J(0,t))_{t\in\mathbb{R}_+}$ are exponentially equicontinuous C_0-semigroups on X, then for any continuous seminorm p on X, there exist a continuous seminorm q on X and $\omega_1, \omega_2 > 0$ such that for all $x \in X$ and for each $(s,t) \in \mathbb{R}_+^2$,

$$p(J(s,0)x) \leq e^{\omega_1 s} q(x) \text{ and } p(J(0,t)x) \leq e^{\omega_2 t} q(x). \tag{3.1}$$

Setting $J = J(s,t) - J(0,0) - (A_1,\ A_2)\begin{pmatrix} s \\ t \end{pmatrix}$. Then for any continuous seminorm p on X, there exists a continuous seminorm q on X such that

$$\begin{aligned} p(Jx) &= p(J(s,t)x - x - sA_1x - tA_2x) \\ &= p(J(s,0)J(0,t)x - J(s,0)x - tA_2x + J(s,0)x - x - sA_1x) \\ &= p\Big(tJ(s,0)\Big(\frac{J(0,t)x - x}{t} - A_2x\Big) + t\big(J(s,0)A_2x - A_2x\big) + s\Big(\frac{J(s,0)x - x}{s} - A_1x\Big)\Big) \\ &\leq \Big\{|t|e^{\omega_1 s} q\Big(\frac{J(0,t)x - x}{t} - A_2x\Big) + |t|p\big(J(s,0)A_2x - A_2x\big);\ |s|p\Big(\frac{J(s,0)x - x}{s} - A_1x\Big)\Big\} \end{aligned}$$

by (3.1). Divide both sides by $\|(s,t)\| = \sqrt{s^2 + t^2}$, it follows that

$$\lim_{(s,t)\to(0^+,0^+)} \frac{p(Jx)}{\|(s,t)\|} = 0 \text{ for all } p \in \Gamma_X.$$

Hence $DJ(s,t)|_{(s,t)=(0,0)} = (A_1, A_2)$ as a linear transformation on $\mathbb{R}_+ \times \mathbb{R}_+$ is the derivative of the two parameter semigroup $(J(s,t))_{(s,t)\in\mathbb{R}_+\times\mathbb{R}_+}$. Thus the linear transformation $L = (A_1, A_2)$ is the infinitesimal generator of the two parameter semigroup $(J(s,t))_{(s,t)\in\mathbb{R}_+\times\mathbb{R}_+}$. Let $u = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}_+ \times \mathbb{R}_+$ and $x \in D(A_1) \cap D(A_2)$. Then it is easy to see that

$$\begin{aligned} DJ(s,t)\begin{pmatrix} a \\ b \end{pmatrix} x = \Big(\frac{\partial J(s,t)}{\partial s}, \frac{\partial J(s,t)}{\partial t}\Big)\begin{pmatrix} a \\ b \end{pmatrix} x &= \Big(a\frac{\partial J(s,t)}{\partial s} + b\frac{\partial J(s,t)}{\partial t}\Big)x \\ &= \Big(a\Big(\frac{\partial J(s,0)}{\partial s}\Big)J(0,t) + b\Big(\frac{\partial J(0,t)}{\partial t}\Big)J(s,0)\Big)x. \end{aligned}$$

By Remark 3.1, we get

$$\begin{aligned} DJ(s,t)\begin{pmatrix} a \\ b \end{pmatrix} x &= \Big(aA_1J(s,0)J(0,t) + bA_2J(0,t)J(s,0)\Big)x \\ &= (aA_1J(s,t) + bA_2J(s,t))x = (A_1, A_2)\begin{pmatrix} a \\ b \end{pmatrix} J(s,t)x. \end{aligned}$$

□

Definition 3.2. Let $(J(s,t))_{(s,t)\in\mathbb{R}_+\times\mathbb{R}_+}$ be a two parameter semigroup of continuous linear operators on a sequentially complete locally convex Hausdorff space X. For $u = (a,b) \in \mathbb{R}_+^2$, the directional derivative D_uJ of $J(s,t)$ at $(0,0)$ in the direction of $u = (a,b)$ is defined as

$$Domain(D_uJ) = \{x \in X : \lim_{h\to 0^+} \frac{J(ah,bh)x - x}{h} \text{ exists in } X\} \tag{3.2}$$

and for all $x \in Domain(D_uJ)$,

$$D_uJx = \lim_{h\to 0^+} \frac{J(ah,bh)x - x}{h}.$$

P r o p o s i t i o n 3.1. *Let* $(J(s,t))_{(s,t)\in\mathbb{R}_+\times\mathbb{R}_+}$ *be a two parameter exponentially equicontinuous* C_0*-semigroup of continuous linear operators on a sequentially complete locally convex Hausdorff space* X. *Then for* $u=(a,b)\in\mathbb{R}^2_+, D_uJ = aA_1 + bA_2$ *is the infinitesimal generator of the one parameter product semigroup* $J(at,0)J(0,bt)$, *where* A_1, A_2 *are the infinitesimal generators of the one parameter exponentially equicontinuous* C_0*-semigroups* $(J(s,0))_{s\in\mathbb{R}_+}$ *and* $(J(0,t))_{t\in\mathbb{R}_+}$ *respectively.*

P r o o f. Let $(a,b)\in\mathbb{R}^2_+$ and $x\in Domain(D_uJ)$, we get

$$\begin{aligned}
D_uJx &= \lim_{h\to 0^+} \frac{J(ah,bh)x - x}{h} = \lim_{h\to 0^+} \frac{J(ah,0)J(0,bh)x - x}{h}\\
&= \lim_{h\to 0^+} \frac{J(ah,0)J(0,bh)x - J(ah,0)x + J(ah,0)x - x}{h}\\
&= \lim_{h\to 0^+} \Big(bJ(ah,0)\frac{J(0,bh)x - x}{bh} + a\Big(\frac{J(ah,0)x - x}{ah}\Big)\Big) = aA_1x + bA_2x.
\end{aligned}$$

□

Theorem 3.4. *Let* X *be a sequentially complete locally convex Hausdorff space, and let* $(J(s,t))_{(s,t)\in\mathbb{R}_+\times\mathbb{R}_+}$ *be a two parameter exponentially equicontinuous* C_0*-semigroup of continuous linear operators on* X *with infinitesimal generator* (A_1, A_2). *Then the followings hold:*

(i) $DJ(s,t)\begin{pmatrix} a\\ b\end{pmatrix} x = (A_1, A_2)\begin{pmatrix} a\\ b\end{pmatrix} J(s,t)x$ *for all* $(a,b)\in\mathbb{R}^2_+$ *and all* $x \in D\Big((A_1,A_2)\begin{pmatrix} a\\ b\end{pmatrix}\Big)$, *where* A_1, A_2 *are the infinitesimal generators of the one parameter exponentially equicontinuous* C_0*-semigroups* $(J(s,0))_{s\in\mathbb{R}_+}$ *and* $(J(0,t))_{t\in\mathbb{R}_+}$ *respectively.*

(ii) *For any* $x\in X$ *and for each* $(s,t)\in\mathbb{R}^2_+$, *we have*

$$\lim_{(h,k)\to(0^+,0^+)} \frac{1}{hk}\int_t^{t+h}\int_s^{s+k} J(u,v)x\,du\,dv = J(s,t)x.$$

(iii) *For each* $x\in X$ *and for all* $t,s\in\mathbb{R}_+$, *we have*

$$\int_0^t\int_0^s J(u,v)x\,du\,dv \in D\Big((A_1,A_2)\begin{pmatrix} a\\ b\end{pmatrix}\Big).$$

P r o o f. (i) Let $(a,b)\in\mathbb{R}^2_+$ and $(s,t)\in\mathbb{R}^2_+$. Since $(J(s,t))_{(s,t)\in\mathbb{R}_+\times\mathbb{R}_+}$ is a function of two variables, we have

$$\begin{aligned}
DJ(s,t)\begin{pmatrix} a\\ b\end{pmatrix} x &= \Big(\frac{\partial J(s,t)}{\partial s}, \frac{\partial J(s,t)}{\partial t}\Big)\begin{pmatrix} a\\ b\end{pmatrix} x = \Big(a\frac{\partial J(s,t)}{\partial s} + b\frac{\partial J(s,t)}{\partial t}\Big)x\\
&= \Big(a\Big(\frac{\partial J(s,0)}{\partial s}\Big)J(0,t) + b\Big(\frac{\partial J(0,t)}{\partial t}\Big)J(s,0)\Big)x.
\end{aligned}$$

By Remark 3.1, we get

$$DJ(s,t)\begin{pmatrix}a\\b\end{pmatrix}x=\Big(aA_1J(s,0)J(0,t)+bA_2J(0,t)J(s,0)\Big)x$$
$$=(aA_1J(s,t)+bA_2J(s,t))x=(A_1,A_2)\begin{pmatrix}a\\b\end{pmatrix}J(s,t)x.$$

(ii) Set for all $x\in X$ and for each $(s,t)\in\mathbb{R}_+^2$,

$$J_1(s,t)x=\lim_{(h,k)\to(0^+,0^+)}\frac{1}{hk}\int_t^{t+h}\int_s^{s+k}J(u,v)xdudv.$$

Utilizing Theorem 3.1, we have for each $(s,t)\in\mathbb{R}_+^2$ and for all $x\in X$,

$$\begin{aligned}J_1(s,t)x&=\lim_{(h,k)\to(0^+,0^+)}\frac{1}{hk}\int_t^{t+h}\int_s^{s+k}J(0,v)J(u,0)xdudv\\&=\lim_{(h,k)\to(0^+,0^+)}\frac{1}{h}\int_t^{t+h}J(0,v)\frac{1}{k}\int_s^{s+k}J(u,0)xdudv\\&=\lim_{h\to0^+}\frac{1}{h}\int_t^{t+h}J(0,v)\lim_{k\to0^+}\frac{1}{k}\int_s^{s+k}J(u,0)xdudv\\&=\lim_{h\to0^+}\frac{1}{h}\int_t^{t+h}J(0,v)J(s,0)xdv=J(0,t)J(s,0)x=J(s,t)x.\end{aligned}$$

(iii) Let $x\in X$ and $(a,b),(s,t)\in\mathbb{R}_+^2$. Put $J_2(s,t)x=(aA_1+bA_2)\int_0^t\int_0^s J(u,v)xdudv$. Hence

$$\begin{aligned}J_2(s,t)x&=(aA_1+bA_2)\int_0^t\int_0^s J(u,v)xdudv\\&=(aA_1+bA_2)\int_0^t J(u,0)\Big(\int_0^s J(0,v)xdv\Big)du\\&=aA_1\int_0^t J(u,0)\Big(\int_0^s J(0,v)xdv\Big)du+bA_2\int_0^t J(u,0)\Big(\int_0^s J(0,v)xdv\Big)du\\&=a\int_0^s J(0,v)\Big(J(t,0)x-x\Big)dv+b\int_0^t J(u,0)\Big(J(0,s)x-x\Big)du.\end{aligned}$$

Then

$$\int_0^t\int_0^s J(u,v)xdudv\in D\Big((A_1,A_2)\begin{pmatrix}a\\b\end{pmatrix}\Big)$$

for all $(a,b),(s,t)\in\mathbb{R}_+\times\mathbb{R}_+$ and for any $x\in X$. □

Theorem 3.5. *Let X be a sequentially complete locally convex Hausdorff space, and let $(J(s,t))_{(s,t)\in\mathbb{R}_+\times\mathbb{R}_+}$ be a two parameter exponentially equicontinuous C_0-semigroup of continuous linear operators on X with infinitesimal generator (A_1,A_2). Then $(A_1,A_2)\begin{pmatrix}a\\b\end{pmatrix}$ is a closed operator with dense domain in X for any $(a,b)\in\mathbb{R}_+\times\mathbb{R}_+$.*

P r o o f. Since $(J(s,t))_{(s,t)\in\mathbb{R}_+\times\mathbb{R}_+}$ is an exponentially equicontinuous C_0-semigroup on X, then $(J(s,0))_{s\in\mathbb{R}_+}$ and $(J(0,t))_{t\in\mathbb{R}_+}$ are one parameter C_0-semigroups on X. Utilizing Theorem 3.1, it follows that their infinitesimal generators A_1 and A_2 are both closed in X. Then for all $u=(a,b)\in\mathbb{R}_+\times\mathbb{R}_+$,

$$(A_1,A_2)\begin{pmatrix}a\\b\end{pmatrix}x=\lim_{h\to0}\frac{J(ah,bh)x-x}{h}=\lim_{h\to0}\frac{J_1(h)x-x}{h},$$

where $J_1(h)=J(hu)$. Since $(J(s,t))_{(s,t)\in\mathbb{R}_+\times\mathbb{R}_+}$ is an exponentially equicontinuous C_0-semigroup on X, then $(J_1(t))_{t\in\mathbb{R}_+}$ is a C_0-semigroup on X. From Theorem 3.1, $(A_1,A_2)\begin{pmatrix}a\\b\end{pmatrix}x=aA_1x+bA_2x$ is a closed operator. Being the infinitesimal generator of a C_0-semigroup, aA_1+bA_2 has a dense domain in X. □

Theorem 3.6. *Let X be a sequentially complete locally convex Hausdorff space, and let $(J(s,t))_{(s,t)\in\mathbb{R}_+\times\mathbb{R}_+}$ and $(S(s,t))_{(s,t)\in\mathbb{R}_+\times\mathbb{R}_+}$ be two parameter exponentially equicontinuous C_0-semigroups of continuous linear operators on X. Suppose that both has the same generator (A_1,A_2). Then $J(s,t)=S(s,t)$ for any $s,t\in\mathbb{R}_+$.*

P r o o f. Since A_1 is the infinitesimal generator of the one parameter C_0-semigroups $(J(s,0))_{s\in\mathbb{R}_+}$ and $(S(s,0))_{s\in\mathbb{R}_+}$, then from Theorem 3.2, $J(s,0)=S(s,0)$ for any $s\in\mathbb{R}_+$. Similarly $J(0,t)=S(0,t)$ for any $t\in\mathbb{R}_+$. □

Let $A\in\mathcal{L}(X)$, then the resolvent set $\rho(A)$ of A is defined by

$$\rho(A)=\{\lambda\in\mathbb{C}:(\lambda I-A)^{-1}\in\mathcal{L}(X)\}. \tag{3.3}$$

Set for all $\lambda\in\rho(A)$, $R(\lambda,A)=(\lambda I-A)^{-1}$, then $R(\lambda,A)$ is called the resolvent of A. For more details, see [7, p. 236].

Let A_1 and A_2 be two linear operators on a sequentially complete locally convex Hausdorff space X, then the domain of the composite operator A_1A_2 is

$$D(A_1A_2)=\{x\in D(A_2):A_2x\in D(A_1)\}.$$

We consider the following conditions on A_1 and A_2 :

(a) $D(A_1A_2)\cap D(A_1)=D(A_2A_1)\cap D(A_2)=D\neq\{0\}$.

(b) $D(A_2(\lambda_0-A_1))\subset D(A_2A_1)$ for some $\lambda_0\in\rho(A_1)$.

(c) $A_1A_2x=A_2A_1x$ for any $x\in D$.

Lemma 3.1. *Let X be a sequentially complete locally convex Hausdorff space. Suppose that A_1 and A_2 satisfy $(a),(b)$ and (c), hence*

(i) *$D=R(\lambda,A_1)R(\mu,A_2)X$ and $R(\lambda,A_1),R(\mu,A_2)$ commute for any $\lambda\in\rho(A_1)$ and $\mu\in\rho(A_2)$,*

(ii) *$A_2R(\lambda,A_1)x=R(\lambda,A_1)A_2x$, $x\in D$ and $\lambda\in\rho(A_1)$.*

P r o o f. Let λ_0 be as in (b). It is easy to see that

$$D=D(A_2(\lambda_0-A_1))=R(\lambda_0,A_1)R(\mu,A_2)X.$$

Let $x \in X$ and set $z = R(\lambda_0, A_1)R(\mu, A_2)x$, then z is in D. Using (c),

$$x = (\lambda_0 I - A_1)(\mu I - A_2)z,$$

then $R(\lambda_0, A_1), R(\mu, A_2)$ commute. Since $D(A_1) = R(\lambda_0, A_1)X = R(\lambda, A_1)X$ for all $\lambda \in \rho(A_1)$, we obtain $D = R(\mu, A_2)R(\lambda, A_1)X$. Now (i) follows by the preceding argument with λ_0 replaced by λ. Next, we note by (i) that $R(\lambda, A_1)D \subset D$. Hence from (c), we have

$$(\mu - A_2)x = (\mu - A_2)(\lambda - A_1)R(\lambda, A_1)x = (\lambda - A_1)(\mu - A_2)R(\lambda, A_1)x, \quad x \in D. \tag{3.4}$$

Then for all $x \in D(A_2)$,

$$(\mu - A_2)x = (\lambda - A_1)R(\lambda, A_1)(\mu - A_2)x, \quad x \in D. \tag{3.5}$$

From (3.4) and (3.5), it follows (ii). □

Let us recall that the generator of an exponentially equicontinuous C_0-semigroup $(J(t))_{t\in\mathbb{R}_+}$ satisfying for any $p \in \Gamma_X$, there exist $\omega \in \mathbb{R}_+$ and $q \in \Gamma_X$ such that for all $x \in X$, $t \in \mathbb{R}_+$, $p(J(t)x) \leq e^{\omega t}q(x)$ is a densely defined closed linear operator such that its resolvent operator given by

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t}J(t)xdt, \quad x \in X, \quad Re(\lambda) > \omega$$

and

$$J(t)x = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t}R(\lambda, A)xd\lambda, \quad x \in X, \tag{3.6}$$

where $\gamma > \max\{0, \omega\}$.

Theorem 3.7. *Let X be a sequentially complete locally convex Hausdorff space, and let $(J(t))_{t\in\mathbb{R}_+}$ and $(S(t))_{t\in\mathbb{R}_+}$ be two exponentially equicontinuous C_0-semigroups on X with generators A_1 and A_2 respectively. Then $(J(t))_{t\in\mathbb{R}_+}$ and $(S(t))_{t\in\mathbb{R}_+}$ commute if and only if A_1 and A_2 satisfy conditions (a), (b) and (c).*

P r o o f. First we demonstrate that D is dense in X. It suffices to prove that it is dense in $D(A_1)$ since the latter is dense in X. Let $x \in D(A_1)$, hence $x = R(\lambda, A_1)y$ for some $y \in X$ and $\lambda \in \rho(A_1)$. Since $D(A_2)$ is dense in X, there is a sequence $(y_n)_n$ in $D(A_2)$ such that $y_n \to y$ as $n \to \infty$. So $R(\lambda, A_1)y_n \to R(\lambda, A_1)y = x$, where $R(\lambda, A_1)y_n \in D$ from Lemma 3.1.

Sufficiency. Let $x \in D$ and $\gamma > \max\{0, \omega\}$. Utilizing (3.6), (ii) of Lemma 3.1 and A_2 is closed, we have

$$J(t)A_2x = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} A_2(e^{\lambda t}R(\lambda, A)x)d\lambda = A_2J(t)x.$$

In particular $J(t)$ maps D into $D(A_2)$ and since $R(\mu, A_2)$ maps D into itself (Lemma 3.1), $J(t)R(\mu, A_2)$ maps D into $D(A_2)$. Hence for $x \in D$,

$$(\mu I - A_2)J(t)R(\mu, A_2)x = J(t)(\mu I - A_2)R(\mu, A_2)x = J(t)x$$

then $J(t)$ and $R(\mu, A_2)$ commute on D. From (3.6), we have $J(t)S(t)x = S(t)J(t)x$ for any $x \in D$ and $t \in \mathbb{R}_+$. Since D is dense, we obtain $J(t)S(t)x = S(t)J(t)x$ for any $x \in X$ and $t \in \mathbb{R}_+$.

Necessity. Suppose that $J(t)S(t) = S(t)J(t)$ for any $t \in \mathbb{R}_+$. Let $A_2(h) = h^{-1}(S(t) - I)$, then $A_2(h)$ and $J(t)$ commute for any sufficiently small h. From A_2 is closed and letting $h \to 0^+$, we have $J(t)A_2x = A_2J(t)x$ for all $x \in D(A_2)$. By (3.6), we have $R(\lambda, A_1)A_2x = A_2R(\lambda, A_1)x$. In particular, $R(\lambda, A_1)D(A_2) \subset D(A_2)$. Hence (b) holds. To demonstrate (a) and (c), let $x \in D(A_1A_2) \cap D(A_1)$, hence

$$A_1A_2x = \lim_{h\to 0^+} A(h)A_2x = \lim_{h\to 0^+} A_2A(h)x = A_2A_1x,$$

where $A(h) = h^{-1}(J(h) - I)$. So $A_1x \in D(A_2)$, then $x \in D(A_2A_1) \cap D(A_2)$. The converse inclusion follows from symmetry. □

Now, we try to see the form of the resolvent of two parameter semigroups.

Theorem 3.8. *Let X be a sequentially complete locally convex Hausdorff space, and let $(J(s,t))_{(s,t)\in\mathbb{R}_+\times\mathbb{R}_+}$ be two parameter exponentially equicontinuous C_0-semigroup on X satisfying for any $p \in \Gamma_X$, there exist $\omega_1, \omega_2 > 0$ and $q \in \Gamma_X$ such that for all $x \in X$, $p(J(s,t)x) \leq e^{\omega_1 s+\omega_2 t}q(x)$ with infinitesimal generator (A_1, A_2). If in addition $\lambda \in \rho\Big((A_1, A_2)\begin{pmatrix} a\\ b\end{pmatrix}\Big)$ and $\lambda > \max\{\omega_1, \omega_2\}$, then*

$$R\Big(\lambda, (A_1, A_2)\begin{pmatrix} a\\ b\end{pmatrix}\Big)x = \int_0^\infty e^{-\lambda t}J(at, bt)xdt.$$

P r o o f. Let $x \in X$ and $\lambda > \max\{\omega_1, \omega_2\}$, define

$$R(\lambda)x = \int_0^\infty e^{-\lambda t}J(at, bt)xdt.$$

Since the map $t \mapsto J(at, bt)$ is continuous and $\lambda > \max\{\omega_1, \omega_2\}$, the integral exists as an improper Riemann integral and defines a continuous linear operator on X. Moreover, for $h > 0$,

$$\begin{aligned}
\frac{J(ah, bh) - I}{h}R(\lambda)x &= \frac{1}{h}\int_0^\infty e^{-\lambda t}(J(a(t+h), b(t+h))x - J(at, bt)x)dt\\
&= \frac{1}{h}\int_h^\infty e^{-\lambda(t-h)}J(at, bt)xdt - \frac{1}{h}\int_0^\infty e^{-\lambda t}J(at, bt)xdt\\
&= \frac{e^{\lambda h}}{h}\int_h^\infty e^{-\lambda t}J(at, bt)xdt - \frac{1}{h}\int_0^\infty e^{-\lambda t}J(at, bt)xdt\\
&= \frac{e^{\lambda h} - 1}{h}\int_0^\infty e^{-\lambda t}J(at, bt)xdt - \frac{e^{\lambda h}}{h}\int_0^h e^{-\lambda t}J(at, bt)xdt.
\end{aligned}$$

Letting $h \to 0^+$, we obtain

$$(A_1, A_2)\begin{pmatrix} a\\ b\end{pmatrix}R(\lambda)x = \lambda R(\lambda)x - x.$$

Hence $R(\lambda)x \in D\Big((A_1, A_2)\begin{pmatrix} a\\ b\end{pmatrix}\Big)$ for any $x \in X$. Then

$$\Big(\lambda I - (A_1, A_2)\begin{pmatrix} a\\ b\end{pmatrix}\Big)R(\lambda) = I.$$

Now, for $x \in D\Big((A_1, A_2)\begin{pmatrix} a \\ b \end{pmatrix}\Big)$, we obtain

$$R(\lambda)(A_1, A_2)\begin{pmatrix} a \\ b \end{pmatrix} x = \int_0^\infty e^{-\lambda t}(A_1, A_2)\begin{pmatrix} a \\ b \end{pmatrix} J(at, bt)x dt.$$

Moreover, $(A_1, A_2)\begin{pmatrix} a \\ b \end{pmatrix}$ is closed, then

$$R(\lambda)(A_1, A_2)\begin{pmatrix} a \\ b \end{pmatrix} x = (A_1, A_2)\begin{pmatrix} a \\ b \end{pmatrix} \int_0^\infty e^{-\lambda t} J(at, bt)x dt = (A_1, A_2)\begin{pmatrix} a \\ b \end{pmatrix} R(\lambda)x.$$

Thus

$$R(\lambda)\Big(\lambda I - (A_1, A_2)\begin{pmatrix} a \\ b \end{pmatrix}\Big)x = x$$

for all $x \in D\Big((A_1, A_2)\begin{pmatrix} a \\ b \end{pmatrix}\Big)$. Since $D\Big((A_1, A_2)\begin{pmatrix} a \\ b \end{pmatrix}\Big)$ is dense in X, we have

$$R(\lambda)\Big(\lambda I - (A_1, A_2)\begin{pmatrix} a \\ b \end{pmatrix}\Big) = I.$$

□

As an extension of Trotter's theorem [3] to sequentially complete locally convex spaces.

Theorem 3.9. *Let X be a sequentially complete locally convex Hausdorff space, and let $(T(t))_{t\in\mathbb{R}_+}$ and $(S(t))_{t\in\mathbb{R}_+}$ be two exponentially equicontinuous C_0-semigroups on X with generators A_1 and A_2 respectively. If the semigroups commute, then the product $U(t) = T(t)S(t)$ is a C_0-semigroup, whose generator is the closure of $A_1 + A_2$.*

R e m a r k 3.2. Let X be a sequentially complete locally convex Hausdorff space. Let $(T(t))_{t\in\mathbb{R}_+}$ and $(S(t))_{t\in\mathbb{R}_+}$ be two exponentially equicontinuous C_0-semigroups on X with generators A_1 and A_2 respectively. It is clear that if A_1 and A_2 satisfy (a), (b) and (c), hence the conclusion of Trotter's theorem holds.

Now, we demonstrate one of the main results of this paper (Generalization of the Hille–Yosida Theorem to two parameter semigroups).

Theorem 3.10. *Let X be a sequentially complete locally convex Hausdorff space. Let $(J(s,t))_{(s,t)\in\mathbb{R}_+\times\mathbb{R}_+}$ be two parameter exponentially equicontinuous C_0-semigroup on X satisfying for any $p \in \Gamma_X$, there exist $\omega_1, \omega_2 > 0$ and $q \in \Gamma_X$ such that for all $x \in X$, $p(J(s,t)x) \le e^{\omega_1 s + \omega_2 t} q(x)$. Then the linear transformation (A_1, A_2) on $\mathbb{R}_+^2$ into $\mathcal{L}(X)$ defined by*
$(A_1, A_2)\begin{pmatrix} s \\ t \end{pmatrix} x = sA_1x + tA_2x$ is the infinitesimal generator of $(J(s,t))_{(s,t)\in\mathbb{R}_+\times\mathbb{R}_+}$ on X if and only if the following statements hold:

(i) $(A_1, A_2)\begin{pmatrix} a \\ b \end{pmatrix}$ *is a closed operator and* $\overline{D\Big((A_1, A_2)\begin{pmatrix} a \\ b \end{pmatrix}\Big)} = X$ *for all* $(a, b) \in \mathbb{R}_+^2$.

(ii) $(\omega_i, \infty) \subseteq \rho(A_i)$ *for* $i = 1, 2$.

(iii) *The families* $\{(\lambda-\omega_i)^n R(\lambda, A_i)^n;\ \lambda>\omega_i,\ n\in\mathbb{N}\}$, $i=1,2$, *are equicontinuous.*

(iv) A_i *satisfy conditions* $(a), (b)$ *and* (c) *for* $i=1,2$.

P r o o f. Let $(J(s,t))_{(s,t)\in\mathbb{R}_+\times\mathbb{R}_+}$ be two parameter exponentially equicontinuous C_0-semigroup on X. By Theorem 3.1, (i) holds. Since A_1 and A_2 are the infinitesimal generators of the one parameter semigroups $(J(s,0))_{s\in\mathbb{R}_+}$ and $(J(0,t))_{t\in\mathbb{R}_+}$ respectively, hence from Theorem 2.3, (ii) and (iii) hold. Since $(J(s,t))_{s,t\in\mathbb{R}_+}$ is a semigroup of operators on X, then $(J(s,0))_{s\in\mathbb{R}_+}$ and $(J(0,t))_{t\in\mathbb{R}_+}$ commute for all $s,t\in\mathbb{R}_+$. Using Theorem 3.7, (iv) is satisfied.

Conversely, let $(a,b)\in\mathbb{R}_+$. Then $(A_1,A_2)\begin{pmatrix}a\\ b\end{pmatrix}=aA_1+bA_2$. In particular if $(a,b)=(1,0)$ and from (i), it follows that A_1 is a closed operator and $\overline{D(A_1)}=X$. Similarly if $(a,b)=(0,1)$, we get A_2 is a closed operator and $\overline{D(A_2)}=X$. From (ii),(iii) and Theorem 2.3, then A_1 and A_2 are the infinitesimal generators of one parameter C_0-semigroups $(J(s,0))_{s\in\mathbb{R}_+}$ and $(J(0,t))_{t\in\mathbb{R}_+}$ respectively. Since A_1 and A_2 satisfy (iv), hence from Theorem 3.7, $J(0,t)$ and $J(s,0)$ commute for all $s,t\in\mathbb{R}_+$. Thus for all $s,t\in\mathbb{R}_+$, $J(s,t)=J(s,0)J(0,t)$ is a two parameter exponentially equicontinuous C_0-semigroup on X with infinitesimal generator (A_1,A_2). □

4. Genaration theorem for n-parameter semigroups of operators in locally convex spaces

In this section, we set $u=(s_1,\cdots,s_n)$, $v=(t_1,\ldots,t_n)\in\mathbb{R}_+^n$, then we have the following definition.

D e f i n i t i o n 4.1. Let X be a locally convex space. An n-parameter family $(J(u))_{u\in\mathbb{R}_+^n}$ of continuous linear operators on X is said to be an n-parameter semigroup if

(i) $J(0,\ldots,0)=I$ where $(0,\ldots,0)\in\mathbb{R}_+^n$,

(ii) For all $u,v\in\mathbb{R}_+^n$, $J(u+v)=J(u)J(v)$.

D e f i n i t i o n 4.2. Let X be a sequentially complete locally convex Hausdorff space. An n-parameter semigroup $(J(u))_{u\in\mathbb{R}_+^n}$ on X is an n-parameter strongly continuous semigroup or an n-parameter C_0-semigroup if $\lim_{u\to(0^+,\ldots,0^+)}J(u)x=x$ for all $x\in X$.

D e f i n i t i o n 4.3. Let X be a sequentially complete locally convex Hausdorff space. An n-parameter C_0-semigroup $(J(u))_{u\in\mathbb{R}_+^n}$ is equicontinuous if for any continuous seminorm p on X, there exists a continuous seminorm q on X such that for all $x\in X$ and for each $u\in\mathbb{R}_+^n$, $p(J(u)x)\le q(x)$.

D e f i n i t i o n 4.4. Let X be a sequentially complete locally convex Hausdorff space. An n-parameter C_0-semigroup $(J(u))_{u\in\mathbb{R}_+^n}$ is exponentially equicontinuous if there exist $\omega_1,\omega_2,\ldots,\omega_n>0$ such that $\{e^{-\sum_{k=1}^n\omega_k s_k}J(u);\ u\in\mathbb{R}_+^n\}$ is equicontinuous.

R e m a r k 4.1. Let X be a sequentially complete locally convex space. Let $(J(u))_{u\in\mathbb{R}_+^n}$ be an n-parameter semigroup on X, then for all $u=(s_1,\ldots,s_n)\in\mathbb{R}_+^n$,

$$J(u)=\prod_{k=1}^n J(u_k),\quad u_k=(0,\ldots,s_k,0\ldots)=s_k e_k,$$

where $(e_k)_{1\le k\le n}$ is the canonical basis of $\mathbb{R}^n$.

Definition 4.5. Let X be a locally convex space. Let $(J(u))_{u\in\mathbb{R}_+^n}$ be an n-parameter semigroup on X. The infinitesimal generator of the n-parameter semigroup $(J(u))_{u\in\mathbb{R}_+^n}$ is the derivative of J at $(0,\ldots,0)$.

Remark 4.2. Let X be a sequentially complete locally convex Hausdorff space. Let $(J(u))_{u\in\mathbb{R}_+^n}$ be an n-parameter exponentially equicontinuous C_0-semigroup on X. Then for any $x\in X$, the map $(s_1,\ldots,s_n)\mapsto J(s_1,\ldots,s_n)x$ is continuous on $\mathbb{R}_+^n$ into X. Let $(A_k)_{1\le k\le n}$ be a sequence of linear operators defined by for all $k\in\{1,\ldots,n\}$,

$$D(A_k)=\{x\in X:\lim_{h\to 0^+}\frac{J(he_k)x-x}{h}\ \text{exists in}\ X\}$$

and

$$A_kx=\lim_{h\to 0^+}\frac{J(he_k)x-x}{h}=\frac{\partial J(u)}{\partial s_k}\Big|_{u=(0,\ldots,0)}\ \text{for each}\ x\in D(A_k).$$

It is easy to see that $A_1,\ldots,A_n$ are the infinitesimal generators of the one parameter semigroups $(J(s_1e_1))_{s_1\in\mathbb{R}_+}$, $\ldots$, $(J(s_ne_n))_{s_n\in\mathbb{R}_+}$ respectively. From the definition of the infinitesimal generator, we have.

Theorem 4.1. *Let X be a sequentially complete locally convex Hausdorff space, and let $(J(u))_{u\in\mathbb{R}_+^n}$ be an n-parameter exponentially equicontinuous C_0-semigroup on X. Then the infinitesimal generator of $(J(u))_{u\in\mathbb{R}_+^n}$ is the linear transformation $L:\mathbb{R}_+^n\to\mathcal{L}(\bigcap_{k=1}^{n}D(A_k),X)$ defined by for any $x\in\bigcap_{k=1}^{n}D(A_k)$ and $(a_1,\ldots,a_n)\in\mathbb{R}_+^n$,*

$$L(a_1,\ldots,a_n)x=(A_1,\ldots,A_n)\begin{pmatrix}a_1\\ \vdots\\ a_n\end{pmatrix}x=\sum_{k=1}^{n}a_kA_kx.$$

Furthermore if $x\in\bigcap_{k=1}^{n}D(A_k)$, we have for all $(a_1,\ldots,a_n)\in\mathbb{R}_+^n$,

$$DJ(s,t)\begin{pmatrix}a_1\\ \vdots\\ a_n\end{pmatrix}x=(A_1,\ldots,A_n)\begin{pmatrix}a_1\\ \vdots\\ a_n\end{pmatrix}J(s,t)x,$$

where $A_1,\ldots,A_n$ are the infinitesimal generators of the one parameter exponentially equicontinuous C_0-semigroups $(J(s_1e_1))_{s_1\in\mathbb{R}_+}$, $\ldots$, $(J(s_ne_n))_{s_n\in\mathbb{R}_+}$ respectively.

Proof. The proof is similar to the proof of Theorem 3.3. □

Definition 4.6. Let $(J(u))_{u\in\mathbb{R}_+^n}$ be an n-parameter semigroup of continuous linear operators on a sequentially complete locally convex Hausdorff space X. Then the directional derivative D_wJ of $J(u)$ at $(0,\ldots,0)$ in the direction of $w=(a_1,\ldots,a_n)\in\mathbb{R}_+^n$ is defined by

$$Domain(D_wJ)=\{x\in X:\lim_{h\to 0^+}\frac{J(hw)x-x}{h}\ \text{exists in}\ X\}$$

and for all $x\in Domain(D_wJ)$,

$$D_wJx=\lim_{h\to 0^+}\frac{J(hw)x-x}{h}.$$

Similarly to the Proposition 3.1, we have

P r o p o s i t i o n 4.1. *Let* $(J(u))_{u\in\mathbb{R}_+^n}$ *be an* n*-parameter exponentially equicontinuous* C_0*-semigroup of continuous linear operators on a sequentially complete locally convex Hausdorff space* X*. Then for* $w=(a_1,\ldots,a_n)\in\mathbb{R}_+^n$, $D_wJx=\sum_{k=1}^{n}a_kA_kx$ *is the infinitesimal generator of the one parameter product semigroup* $\prod_{k=1}^{n}J(a_ks_ke_k)$, *where* A_k *are the infinitesimal generator of the one parameter exponentially equicontinuous* C_0*-semigroup* $(J(s_ke_k))_{s_k\in\mathbb{R}_+}$, $k=1,\ldots,n$.

Similarly to the Theorem 3.4, we have

Theorem 4.2. *Let* X *be a sequentially complete locally convex Hausdorff space, and let* $(J(u))_{u\in\mathbb{R}_+^n}$ *be an* n*-parameter exponentially equicontinuous* C_0*-semigroup of continuous linear operators on* X*. Then the followings hold:*

(i) $DJ(u)\begin{pmatrix}a_1\\ \vdots\\ a_n\end{pmatrix}x=(A_1,\ldots,A_n)\begin{pmatrix}a_1\\ \vdots\\ a_n\end{pmatrix}J(u)x$ *for all* $(a_1,\ldots,a_n)\in\mathbb{R}_+^n$ *and all* $x\in X$, *where* $A_1,\ldots,A_n$ *are the infinitesimal generators of the one parameter exponentially equicontinuous* C_0*-semigroups* $(J(s_1e_1))_{s_1\in\mathbb{R}_+},\ldots,(J(s_ne_n))_{s_n\in\mathbb{R}_+}$ *respectively.*

(ii) *For any* $x\in X$ *and* $t=(t_1,\ldots,t_n)\in\mathbb{R}_+^n$, *we have*

$$\lim_{h\to(0^+,\ldots,0^+)}\frac{1}{h_1\ldots h_n}\int_{t_n}^{t_n+h_n}\ldots\int_{t_1}^{t_1+h_1}J(u)x\,ds_1\ldots ds_n=J(t)x,$$

where $h=(h_1,\ldots,h_n)\in\mathbb{R}_+^n$.

(iii) *For any* $x\in X$ *and* $(h_1,\ldots,h_n)\in\mathbb{R}_+^n$, *we have*

$$\int_0^{h_n}\ldots\int_0^{h_1}J(u)x\,ds_1\ldots ds_n\in D\Big((A_1,\ldots,A_n)\begin{pmatrix}a_1\\ \vdots\\ a_n\end{pmatrix}\Big).$$

As a generalization of Theorem 3.5, we have

Theorem 4.3. *Let* X *be a sequentially complete locally convex Hausdorff space, and let* $(J(u))_{u\in\mathbb{R}_+^n}$ *be an* n*-parameter exponentially equicontinuous* C_0*-semigroup of continuous linear operators on* X *with infinitesimal generator* $(A_1,\ldots,A_n)$*. Then* $(A_1,\ldots,A_n)\begin{pmatrix}a_1\\ \vdots\\ a_n\end{pmatrix}$ *is a closed operator with dense domain in* X *for any* $(a_1,\ldots,a_n)\in\mathbb{R}_+^n$

As a generalization of Theorem 3.6, we obtain.

Theorem 4.4. *Let* X *be a sequentially complete locally convex Hausdorff space, and let* $(T(u))_{u\in\mathbb{R}_+^n}$ *and* $(S(u))_{u\in\mathbb{R}_+^n}$ *be two parameter exponentially equicontinuous* C_0*-semigroups of continuous linear operators on* X*. Suppose that both has the same generator* $(A_1,\ldots,A_n)$*. Then* $T(u)=S(u)$ *for any* $u=(s_1,\ldots,s_n)\in\mathbb{R}_+^n$.

Now, we obtain a generalization of Theorem 3.9.

Theorem 4.5. *Let X be a sequentially complete locally convex Hausdorff space, and let $(J_1(t))_{t\in\mathbb{R}_+}, \ldots, (J_n(t))_{t\in\mathbb{R}_+}$ be exponentially equicontinuous C_0-semigroups with generators $A_1,\ldots,A_n$ respectively. If for all $i,j\in\{1,\ldots,n\}$ and for any $t\in\mathbb{R}_+$, $J_i(t)J_j(t)=J_j(t)J_i(t)$, then the product $U(t)=\prod_{k=1}^n J_k(t)$ is a C_0-semigroup, whose generator is the closure of $\sum_{k=1}^{n} A_k$.*

Remark 4.3. Let X be a sequentially complete locally convex Hausdorff space, and let $(J_1(t))_{t\in\mathbb{R}_+}, \ldots, (J_n(t))_{t\in\mathbb{R}_+}$ be exponentially equicontinuous C_0-semigroups with generators $A_1,\ldots,A_n$ respectively. It is clear that if $A_1,\ldots,A_n$ satisfy (a), (b) and (c), hence the conclusion of Trotter's theorem holds.

Now, we demonstrate one of the main results of this paper (Generalization of the Hille–Yosida Theorem to n-parameter semigroups).

Theorem 4.6. *Let X be a sequentially complete locally convex Hausdorff space, and let $(J(u))_{u\in\mathbb{R}_+^n}$ be an n-parameter C_0-semigroup on X satisfying for any $p\in\Gamma_X$, there exist $\omega_1,\ldots,\omega_n>0$ and $q\in\Gamma_X$ such that for all $x\in X$, $p(J(u)x)\le e^{\sum_{k=1}^n \omega_k s_k} q(x)$. Then the linear transformation $(A_1,\ldots,A_n)$ on $\mathbb{R}_+^n$ into $\mathcal{L}(X)$ defined by $(A_1,\ldots,A_n)\begin{pmatrix}a_1\\ \vdots\\ a_n\end{pmatrix} x=\sum_{k=1}^n a_k A_k$ is the infinitesimal generator of an n-parameter exponentially equicontinuous C_0-semigroup $(J(u))_{u\in\mathbb{R}_+^n}$ on X if and only if the following assertions hold:*

(i) *$(A_1,\ldots,A_n)\begin{pmatrix}a_1\\ \vdots\\ a_n\end{pmatrix}$ is a closed operator and $\overline{D\Big((A_1,\ldots,A_n)\begin{pmatrix}a_1\\ \vdots\\ a_n\end{pmatrix}\Big)}=X$ for all $(a_1,\ldots,a_n)\in\mathbb{R}_+^n$.*

(ii) *$(\omega_i,\infty)\subseteq\rho(A_i)$ for all $i\in\{1,\ldots,n\}$.*

(iii) *The families $\{(\lambda-\omega_i)^n R(\lambda,A_i)^n;\lambda>\omega_i, n\in\mathbb{N}\}$ are equicontinuous for all $i\in\{1,\ldots,n\}$.*

(iv) *For all $i,j\in\{1,2,\ldots,n\}$ $(i\ne j)$, A_i and A_j satisfy conditions (a), (b) and (c).*

5. Genaration theorem for n-parameter semigroups of operators in Banach spaces

In this section, we put $u=(s_1,\ldots,s_n)$, $v=(t_1,\ldots,t_n)\in\mathbb{R}_+^n$, then we get the following definition.

Definition 5.1. [1] Let X be a Banach space. An n-parameter family $(J(u))_{u\in\mathbb{R}_+^n}$ of continuous linear operators on X is said to be an n-parameter semigroup if

(i) $J(0,\ldots,0)=I$ where $(0,\ldots,0)\in\mathbb{R}_+^n$,

(ii) For all $u,v\in\mathbb{R}_+^n$, $J(u+v)=J(u)J(v)$.

Definition 5.2. Let X be a Banach space. An n-parameter semigroup $(J(u))_{u\in\mathbb{R}_+^n}$ on X is an n-parameter strongly continuous semigroup or an n-parameter C_0-semigroup if $\lim_{u\to(0^+,\ldots,0^+)} J(u)x=x$ for all $x\in X$.

R e m a r k 5.1. [1] Let X be a Banach space. Let $(J(u))_{u\in\mathbb{R}_+^n}$ be an n-parameter semigroup on X, then for all $u=(s_1,\dots,s_n)\in\mathbb{R}_+^n$,

$$J(u)=\prod_{k=1}^{n} J(u_k)\quad u_k=(0,\dots,s_k,0\dots)=s_k e_k,$$

where $(e_k)_{1\leq k\leq n}$ is the canonical basis of $\mathbb{R}^n$.

D e f i n i t i o n 5.3. Let X be a Banach space. Let $(J(u))_{u\in\mathbb{R}_+^n}$ be an n-parameter semigroup on X. The infinitesimal generator of the n-parameter semigroup $(J(u))_{u\in\mathbb{R}_+^n}$ is the derivative of J at $(0,\dots,0)$.

R e m a r k 5.2. Let X be a Banach space. Let $(J(u))_{u\in\mathbb{R}_+^n}$ be an n-parameter C_0-semigroup on X. Then for any $x\in X$, the map $(s_1,\dots,s_n)\mapsto J(s_1,\dots,s_n)x$ is continuous on $\mathbb{R}_+^n$ into X. Let $A_1,\dots,A_n$ be the linear operators defined by for all $k\in\{1,\dots,n\}$,

$$D(A_k)=\{x\in X:\lim_{h\to 0^+}\frac{J(he_k)x-x}{h}\text{ exists in } X\}$$

and

$$A_k x=\lim_{h\to 0^+}\frac{J(he_k)x-x}{h}=\frac{\partial J(u)}{\partial s_k}\Big|_{u=(0,\dots,0)},\quad \text{for each } x\in D(A_k).$$

It is easy to see that $A_1,\dots,A_n$ are the infinitesimal generators of the one parameter semigroups $(J(s_1e_1))_{s_1\in\mathbb{R}_+}$, … $(J(s_ne_n))_{s_n\in\mathbb{R}_+}$, respectively. From the definition of the infinitesimal generator, we have.

Theorem 5.1. *Let X be a Banach space. Let $(J(u))_{u\in\mathbb{R}_+^n}$ be an n-parameter C_0-semigroup on X. Then the infinitesimal generator of the n-parameter C_0-semigroup $(J(u))_{u\in\mathbb{R}_+^n}$ is the linear transformation $L:\mathbb{R}_+^n\to\mathcal{L}(\bigcap_{k=1}^{n} D(A_k),X)$ defined by for any $x\in\bigcap_{k=1}^{n} D(A_k)$ and $(a_1,\dots,a_n)\in\mathbb{R}_+^n$,*

$$L(a_1,\dots,a_n)x=(A_1,\dots,A_n)\begin{pmatrix}a_1\\ \vdots\\ a_n\end{pmatrix}x=\sum_{k=1}^{n}a_kA_kx.$$

Furthermore if $x\in\bigcap_{k=1}^{n} D(A_k)$, we have for all $(a_1,\dots,a_n)\in\mathbb{R}_+^n$,

$$DJ(s,t)\begin{pmatrix}a_1\\ \vdots\\ a_n\end{pmatrix}x=(A_1,\dots,A_n)\begin{pmatrix}a_1\\ \vdots\\ a_n\end{pmatrix}J(s,t)x$$

where A_k are the infinitesimal generator of the one parameter C_0-semigroup $(J(s_ke_k))_{s_k\in\mathbb{R}_+}$, $k=1,\dots,n$.

P r o o f. The proof is similar to the proof of Theorem 3.3. □

D e f i n i t i o n 5.4. Let $(J(u))_{u\in\mathbb{R}_+^n}$ be an n-parameter semigroup of continuous linear operators on a Banach space X. Then the directional derivative D_wJ of $J(u)$ at $(0,\ldots,0)$ in the direction of $w=(a_1,\ldots,a_n)\in\mathbb{R}_+^n$ is defined by

$$Domain(D_wJ)=\{x\in X:\lim_{h\to 0^+}\frac{J(hw)x-x}{h}\text{ exists in } X\}$$

and for all $x\in Domain(D_wJ)$,

$$D_wJx=\lim_{h\to 0^+}\frac{J(hw)x-x}{h}.$$

Similarly to the Proposition 3.1, we have

P r o p o s i t i o n 5.1. *Let $(J(u))_{u\in\mathbb{R}_+^n}$ be an n-parameter C_0-semigroup of continuous linear operators on a Banach space X. Then for $w=(a_1,\ldots,a_n)\in\mathbb{R}_+^n$, $D_wJx=\sum_{k=1}^n a_kA_k$ is the infinitesimal generator of the one parameter product semigroup $\prod_{k=1}^n J(a_ks_ke_k)$ where A_k are the infinitesimal generator of the one parameter C_0-semigroup $(J(s_ke_k))_{s_k\in\mathbb{R}_+}$, $k=1,\ldots,n$.*

Similarly to the Theorem 3.4, we have

Theorem 5.2. *Let X be a Banach space. Let $(J(u))_{u\in\mathbb{R}_+^n}$ be an n-parameter C_0-semigroup of continuous linear operators on X. Then the followings hold:*

(i) $DJ(u)\begin{pmatrix}a_1\\ \vdots\\ a_n\end{pmatrix}x=(A_1,\ldots,A_n)\begin{pmatrix}a_1\\ \vdots\\ a_n\end{pmatrix}J(u)x$ *for all* $(a_1,\ldots,a_n)\in\mathbb{R}_+^n$ *and all* $x\in\bigcap_{k=1}^n D(A_k)$, *where $A_1,\ldots,A_n$ are the infinitesimal generators of the one parameter C_0-semigroups $(J(s_1e_1))_{s_1\in\mathbb{R}_+}$, ..., $(J(s_ne_n))_{s_n\in\mathbb{R}_+}$ respectively.*

(ii) *For any $x\in X$ and $t=(t_1,\ldots,t_n)\in\mathbb{R}_+^n$, we have*

$$\lim_{h\to(0^+,\ldots,0^+)}\frac{1}{h_1\ldots h_n}\int_{t_n}^{t_n+h_n}\ldots\int_{t_1}^{t_1+h_1}J(u)x\,ds_1\ldots ds_n=J(t)x,$$

where $h=(h_1,\ldots,h_n)\in\mathbb{R}_+^n$.

(iii) *For any $x\in X$ and $(h_1,\ldots,h_n)\in\mathbb{R}_+^n$, we have*

$$\int_0^{h_n}\ldots\int_0^{h_1}J(u)x\,ds_1\ldots ds_n\in D\Big((A_1,\ldots,A_n)\begin{pmatrix}a_1\\ \vdots\\ a_n\end{pmatrix}\Big).$$

As a generalization of Theorem 3.5, we have

Theorem 5.3. *Let X be a Banach space. Let $(J(u))_{u\in\mathbb{R}_+^n}$ be an n-parameter C_0-semigroup of continuous linear operators on X with infinitesimal generator $(A_1,\ldots,A_n)$. Then*

$(A_1,\ldots,A_n)\begin{pmatrix}a_1\\ \vdots\\ a_n\end{pmatrix}$ *is a closed operator with dense domain in X for any $(a_1,\ldots,a_n)\in\mathbb{R}_+^n$.*

As a generalization of Theorem 3.6, we obtain.

Theorem 5.4. *Let X be a Banach space. Let $(T(u))_{u\in\mathbb{R}_+^n}$ and $(S(u))_{u\in\mathbb{R}_+^n}$ be two parameter C_0-semigroups of continuous linear operators on X. Suppose that both has the same generator $(A_1,\dots,A_n)$. Then $T(u)=S(u)$ for any $u=(s_1,\dots,s_n)\in\mathbb{R}_+^n$.*

Now, we obtain a generalization of Theorem 3.9.

Theorem 5.5. *Let X be a Banach space. Let $(T_1(t))_{t\in\mathbb{R}_+}, \dots, (T_n(t))_{t\in\mathbb{R}_+}$ be C_0-semigroups with generators $A_1,\dots,A_n$ respectively. If for all $i,j\in\{1,\dots,n\}$ and for any $t\in\mathbb{R}_+$, $T_i(t)T_j(t)=T_j(t)T_i(t)$, then the product $U(t)=\prod_{k=1}^n T_k(t)$ is a C_0-semigroup, whose generator is the closure of $\sum\limits_{k=1}^n A_k$.*

R e m a r k 5.3. Let X be a Banach space. Let $(T_1(t))_{t\in\mathbb{R}_+}, \dots, (T_n(t))_{t\in\mathbb{R}_+}$ be C_0-semigroups with generators $A_1,\dots,A_n$ respectively. It is clear that if $A_1,\dots,A_n$ satisfy (a), (b) and (c), hence the conclusion of Trotter's theorem holds.

There exists a generalization of the Hille–Yosida Theorem for n-parameter semigroups in Banach spaces due to Abdelaziz see [1, Theorem 2]. Now, we get a another characterization used the definition of the infinitesimal generator for n-parameter semigroups in Banach spaces.

Theorem 5.6. *Let X be a Banach space. Let $(J(u))_{u\in\mathbb{R}_+^n}$ be an n-parameter C_0-semigroup $(J(u))_{u\in\mathbb{R}_+^n}$ on X satisfying for all $u\in\mathbb{R}_+^n$ and for each $x\in X$, $\|J(u)x\|\le Me^{\sum_{k=1}^n \omega_k s_k}$ for some $\omega_1,\dots,\omega_n,M>0$. Then the linear transformation $(A_1,\dots,A_n)$ on $\mathbb{R}_+^n$ into $\mathcal{L}(X)$ defined by*

$$(A_1,\dots,A_n)\begin{pmatrix}a_1\\ \vdots\\ a_n\end{pmatrix}x=\sum_{k=1}^n a_kA_kx$$

is the infinitesimal generator of an n-parameter C_0-semigroup $(J(u))_{u\in\mathbb{R}_+^n}$ on X if and only if the following hold:

(i) $(A_1,\dots,A_n)\begin{pmatrix}a_1\\ \vdots\\ a_n\end{pmatrix}$ *is a closed operator and* $\overline{D\Big((A_1,\dots,A_n)\begin{pmatrix}a_1\\ \vdots\\ a_n\end{pmatrix}\Big)}=X$ *for all* $(a_1,\dots,a_n)\in\mathbb{R}_+^n$.

(ii) $(\omega_i,\infty)\subseteq\rho(A_i)$ *for* $i=1,\dots,n$.

(iii) $\|R(\lambda,A_i)^n\|\le\frac{M_i}{(\lambda-\omega_i)^n}$ *for* $\lambda>\omega_i$ *and* $i=1,\dots,n$.

(iv) *For all* $i,j\in\{1,2,\dots,n\}$ $(i\ne j)$, A_i *and* A_j *satisfy conditions* (a), (b) *and* (c).

D e f i n i t i o n 5.5. Let X be a Banach space. An n-parameter semigroup $(J(u))_{u\in\mathbb{R}_+^n}$ on X is an n-parameter uniformly semigroup if

$$\lim_{u\to(0^+,\dots,0^+)}\|J(u)-I\|=0.$$

As a generalization of [2, Theorem 2.9], we have

Theorem 5.7. *Let X be a Banach space. Then the following are equivalent:*

(i) *The linear transformation* $A = (A_1, \ldots, A_n)$ *on* $\mathbb{R}_+^n$ *into* $\mathcal{L}(X)$ *defined by*

$$(A_1, \ldots, A_n)\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} x = \sum_{k=1}^{n} a_k A_k x$$

is the infinitesimal generator of an uniformly continuous n*-parameter semigroup* $(J(u))_{u\in\mathbb{R}_+^n}$ *of continuous linear operators on* X.

(ii) A_i *and* A_j *are bounded linear operators on* X, *and* A_i *and* A_j *commute for all* $i, j \in \{1, \ldots, n\}$ $(i \neq j)$.

As a generalization of [2, Theorem 2.10], we have

Theorem 5.8. *Let* $(J(u))_{u\in\mathbb{R}_+^n}$ *be an uniformly continuous* n*-parameter semigroup of continuous linear operators on a Banach space* X. *Then*

(i) *There exist constants* $\omega_1, \ldots, \omega_n > 0$ *such that* $\|J(u)x\| \leq e^{\sum_{k=1}^{n} \omega_k s_k}$ *for all* $u = (s_1, \ldots, s_n) \in \mathbb{R}_+^n$.

(ii) *There exist bounded linear operators* $A_1, \ldots, A_n$ *such that* $J(u) = \prod_{k=1}^{n} e^{s_k A_k}$ *for all* $u = (s_1, \ldots, s_n) \in \mathbb{R}_+^n$.

(iii) *The linear transformation* $A = (A_1, \ldots, A_n)$ *on* $\mathbb{R}_+^n$ *into* $\mathcal{L}(X)$ *defined by*

$$(A_1, \ldots, A_n)\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} x = \sum_{k=1}^{n} a_k A_k x$$

is the infinitesimal generator of the $(J(u))_{u\in\mathbb{R}_+^n}$.

(iv) *The map* $u \mapsto J(u)$ *is differentiable in the norm and*

$$DJ(u)\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} x = (A_1, \ldots, A_n)\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} x.$$

R e m a r k 5.4. For $n = 2$, the results of this section due to Al-Sharif and Khalil [2].

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