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# Characterizations of geometric tripotents in strongly facially symmetric spaces

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**Abstract.** The concept of a geometric tripotent is one of the key concepts in the theory of strongly facially symmetric spaces. This paper studies the properties of geometric tripotents. We establish necessary and sufficient conditions under which a norm-one element of the dual space (real or complex) of a strongly facially symmetric space is a geometric tripotent. We prove that two geometric tripotents in such a space are mutually orthogonal if and only if both their sum and difference have norm one. Furthermore, we show that the set of extreme points of the unit ball coincides with the set of maximal geometric tripotents in the dual of a strongly facially symmetric space. Finally, we examine the relationship between M-orthogonality and ordinary orthogonality in the dual of a complex strongly facially symmetric space, providing a geometric characterization of geometric tripotents.

**Keywords:** geometric tripotent, norm exposed face, Peirce projection, strongly facially symmetric space

**Mathematics Subject Classification:** 46B20, 46E30.

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# Характеризация геометрических трипотентов в сильно гранево симметричных пространствах

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**Аннотация.** Понятие геометрического трипотента является одним из ключевых в теории сильно гранево симметричных пространств. В данной статье исследуются свойства геометрических трипотентов. Определены необходимые и достаточные условия для того, чтобы элемент с единичной нормой сопряженного пространства действительного или комплексного сильно гранево симметричного пространства являлся геометрическим трипотентом. Доказано, что два геометрических трипотента в сильно гранево симметричном пространстве взаимно ортогональны тогда и только тогда, когда и норма их суммы, и норма их разности равны единице. Кроме того, показано, что множества экстремальных точек единичного шара и максимальных геометрических трипотентов сопряженного пространства сильно гранево симметричного пространства совпадают. В заключение, исследованы связи между M-ортогональностью и ортогональностью в сопряженном пространстве комплексного сильно гранево симметричного пространства, а также дана геометрическая характеристика геометрических трипотентов.

**Ключевые слова:** геометрический трипотент, выставленная по норме грань, Пирсовский проектор, сильно гранево симметричные пространства

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## Introduction

In the early 1980s, the development of $JB^*$-triple theory was initiated by Kaup, establishing a framework that parallels the functional-analytic aspects of operator algebra theory [1, 2]. These triples, distinguished by the holomorphic properties of their unit balls, constitute a broad class of Banach spaces based on ternary algebraic structures, encompassing $C^*$-algebras, Hilbert spaces, and spaces of rectangular matrices. The axiomatic approach developed by Alfsen and Schultz suggests the existence of unordered analogs of $JB^*$-triples [3]. A significant advancement in this direction was the introduction of facially symmetric spaces by Friedman and Russo in [4, 5], motivated by the geometric characterization of predual spaces of Banach spaces admitting an algebraic structure. Many of the properties required for such characterizations arise naturally in state spaces of physical systems, making these spaces a compelling geometric model for quantum mechanics.

In [6], it was established that the predual space of a complex von Neumann algebra, as well as that of a general $JB^*$-triple, forms a neutral strongly facially symmetric space. Further developments in [7] demonstrated that the predual of the real part of a von Neumann algebra is a strongly facially symmetric space if and only if the algebra is the direct sum of an Abelian algebra and a type $I_2$ algebra. A similar result was obtained for $JBW$-algebras in [8], where it was shown that the predual space of a $JBW$-algebra is a strongly facially symmetric space if and only if the algebra is the direct sum of an Abelian algebra and an algebra of type $I_2$.

Subsequent studies provided further geometric characterizations: in [9], a characterization of complex Hilbert spaces and spin factors was given, while in [10], a description of atomic facially symmetric spaces was presented, establishing that a neutral strongly facially symmetric space is isometrically isomorphic to the predual of one of the Cartan factors of types 1-6. Neal and Russo [11] identified geometric conditions under which a facially symmetric space is isometric to the predual of a complex $JBW^*$-triple. A complete description of strongly facially symmetric spaces that are isometrically isomorphic to the predual of an atomic commutative von Neumann algebra was obtained in [12]. In [13], a classification of finite dimensional real neutral strongly facially symmetric spaces with property (JP) (joint Peirce decomposition property) was proposed. It was shown that every real neutral strongly facially symmetric space with a unitary tripotent is isometrically isomorphic to $L_1(\Omega, \Sigma, \mu)$, where $(\Omega, \Sigma, \mu)$ is a measure space satisfying the direct sum property.

It is worth noting that the study of the relationship between M-orthogonality and orthogonality in strongly facially symmetric spaces within their dual space was presented in [14], where a geometric characterization of geometric tripotents in reflexive complex strongly facially symmetric spaces was provided. In the present work, we establish necessary and sufficient conditions under which an element of the dual space of a strongly facially symmetric space is geometric tripotent.

## 1. Preliminaries

We present necessary information from the theory of facially symmetric spaces, [4, 5]. Let $Z$ be a real or complex normed space, and let $Z^*$ denote its dual space. We say that elements $f, g \in Z$ are orthogonal and write $f \diamond g$ if $\|f+g\| = \|f-g\| = \|f\| + \|g\|$. We say subsets $S, T \subset Z$ are orthogonal and write $S \diamond T$, if $f \diamond g$ for all $(f, g) \in S \times T$. For a subset $S$ of $Z$, we put $S^\diamond = \{f \in Z : \forall g \in S \ f \diamond g\}$; the set $S^\diamond$ is called the orthogonal complement of $S$. Recall that a face $F$ of a convex set $K$ is a non-empty convex subset of $K$ such that if

$g, h \in K$ satisfy $\lambda g + (1-\lambda)h \in F$ for some $\lambda \in (0,1)$, then $g, h \in F$.

A norm exposed face of the unit ball $Z_1 = \{f \in Z : \|f\| \leq 1\}$ of $Z$ is a non-empty set (necessarily $\neq Z_1$) of the form $F_u = \{f \in Z_1 : u(f) = 1\}$, where $u \in Z^*$ with $\|u\| = 1$. While every norm exposed face is a face, the converse does not hold in general.

E x a m p l e 1.1. Let the set $Z_1$ be the unit ball of the space $Z = \mathbb{R}^2$ with respect to some norm. A point $f$ on this ball is a face, but not a norm exposed face, since there is no hyperplane $H$ such that $H \cap Z_1 = \{f\}$ (see Fig. 1).

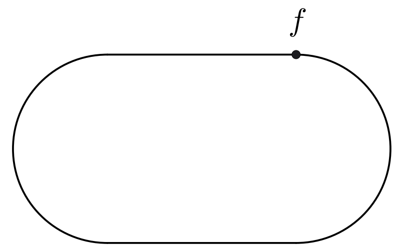


**Fig. 1**

An element $u \in Z^*$ is called a projective unit if $\|u\| = 1$ and $u(g) = 0$ for all $g \in F_u^\diamond$.

D e f i n i t i o n 1.1. A norm exposed face $F_u$ in $Z_1$ is called a symmetric face if there exists a linear isometry $S_u$ from $Z$ to $Z$ such that $S_u^2 = I$ whose fixed point set coincides with the topological direct sum of the closure $\overline{\mathrm{sp}} F_u$ of the linear hull of the face $F_u$ and its orthogonal complement $F_u^\diamond$, i. e., with $\overline{\mathrm{sp}} F_u \oplus F_u^\diamond$.

D e f i n i t i o n 1.2. A space $Z$ is said weakly facially symmetric (WFS) if each norm exposed face in $Z_1$ is symmetric.

For each symmetric face $F_u$, contractive projections (i. e., linear operator $P : Z \to Z$ such that $P^2 = P$ and $\|P\| \leq 1$) $P_k(u), k = 0, 1, 2$ on $Z$ are defined as follows (see [5]). First, $P_1(u) = (I - S_u)/2$ is the projection on the eigenspace corresponding to the eigenvalue $-1$ of the symmetry $S_u$. Next, $P_2(u)$ and $P_0(u)$ are defined as projections of $Z$ onto $\overline{\mathrm{sp}} F_u$ and $F_u^\diamond$, respectively; i. e., $P_2(u) + P_0(u) = (I + S_u)/2$. The projections $P_k(u)$ are called geometric Peirce projections.

E x a m p l e 1.2. Let $A$ be a $C^*$-algebra. If $v$ is a partial isometry from $A$ then the elements $l = vv^*$ and $r = v^*v$ are projections. For each partial isometry $v$, we define projections $E(v)$, $F(v)$ and $G(v)$ on the Banach space $A$. We put

$$E(v)x = lxr, \quad F(v)x = (1-l)x(1-r), \quad G(v)x = lx(1-r) + (1-l)xr.$$

We call $E(v)$, $F(v)$ and $G(v)$ the Peirce projections corresponding to $v$.

Let $v$ is a partial isometry from a von Neumann algebra $A$ and $A_*$ is a predual space of $A$. Then the operator

$$S_v = E(v) - G(v) + F(v)$$

defined in terms of the Peirce projections is a linear isometry from $A_*$ onto $A_*$ such that $S_v^2 = I$ and the set of fixed points coincides with $E(v)A_* \oplus F(v)A_*$ (see [6, Lemma 2.8]). Hence, $A_*$ is a weakly facially symmetric space (see [6]).

D e f i n i t i o n 1.3. A WFS-space $Z$ is said to be strongly facially symmetric (SFS) if for each norm exposed face $F_u$ of $Z_1$ and each $y \in Z^*$ satisfying the conditions $\|y\| = 1$ and $F_u \subset F_y$, we have $S_u^* y = y$, where $S_u$ is the symmetry corresponding to $F_u$.

A projective unit $u \in Z^*$ is called geometric tripotent if $F_u$ is a symmetric face and $S_u^* u = u$ for the symmetry $S_u$ corresponding to $F_u$. It should be noted that some properties of geometric tripotents were established in [15]. By $\mathcal{GT}$ and $\mathcal{SF}$ we denote the sets of all geometric tripotents and symmetric faces, respectively; the correspondence $\mathcal{GT} \ni u \mapsto F_u \in \mathcal{SF}$ is one-to-one [5, Proposition 1.6].

Geometric tripotents $u$ and $v$ are said to be orthogonal if $u \in P_0(v)^* Z^*$ (which implies $v \in P_0(u)^* Z^*$ ) or, equivalently, $u \pm v \in \mathcal{GT}$ (see [4, Lemma 2.5]). More generally, elements $x$ and $y$ of $Z^*$ are said to be orthogonal, denoted $x \diamond y$, if one of them belongs to $P_2(u)^* Z^*$ and the other belongs to $P_0(u)^* Z^*$ for some geometric tripotent $u$. The orthogonal complement $X^\diamond$ of a $X \subset Z^*$ is defined as $X^\diamond = \{y \in Z^* : \forall x \in X \ x \diamond y\}$. For a singleton set $\{x\}$ we write $x^\diamond$ instead of $\{x\}^\diamond$.

We present examples of SFS-spaces.

E x a m p l e 1.3. Endowing $\mathbb{R}^n$ with the norm $\|x\| = |x_1| + \ldots + |x_n|$, where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we obtain a strongly facially symmetric space (see [13]).

E x a m p l e 1.4. Every Hilbert space $H$ is a SFS-space (see [13]). Each element $u \in H$ with $\|u\| = 1$ is a geometric tripotent and $F_u = u$. Moreover, the symmetry $S_u$ corresponding to a face $F_u$ is defined as follows:

$$S_u(\lambda u + x) = \lambda u - x, \quad \lambda u + x \in \mathrm{sp} u \oplus u^\perp = H,$$

where $u^\perp$ is the orthocomplement of $u$ in the Hilbert space $H$.

E x a m p l e 1.5. The predual space of a von Neumann algebra $A$ is a strongly facially symmetric space. Notice that there exists a bijective correspondence between the set of geometric tripotents and the set of nonzero partial isometries, (see [6, Theorem 2.11]). If $v$ is a geometric tripotent then the geometric Peirce projections corresponding to $v$ are defined in terms of the Peirce projections corresponding to $v$, i. e., we have

$$P_2(v) = E(v), \quad P_1(v) = G(u), \quad P_0(v) = F(v).$$

E x a m p l e 1.6. The predual space of a JB*-triple $U$ is a strongly facially symmetric space in which the set of geometric tripotents coincides with the set of tripotents (see [6, Theorem 3.1]).

## 2. Main results

Let $Z$ be a real or complex normed space and $x \in Z^*$, $\|x\| = 1$. For $x$ consider sets $D_1(x)$ and $D_2(x)$ defined as

$$D_1(x) = \{y \in Z^* : \exists \alpha > 0 \ \|x + \alpha y\| = \|x - \alpha y\| = 1\},$$

$$D_2(x) = \big\{y \in Z^* : \forall \beta \in \mathbb{C} \ \|x + \beta y\| = \max\{1, \|\beta y\|\}\big\}.$$

**Theorem 2.1.** *Let $Z$ be a real or complex strongly facially symmetric space and $x \in Z^*$ norm-one element. Then $x$ is a geometric tripotent if and only if $F_x \neq \emptyset$ and $D_1(x) = D_2(x)$.*

P r o o f. *Necessity.* Let $x \in Z^*$ be a geometric tripotent. Then we have already noted that $F_x \neq \emptyset$. Let us assume that $y \in D_2(x)$, $y \neq 0$, and put $\alpha = \|y\|^{-1}$. Then it follows from the definition of the set $D_2(x)$ that

$$\|x + \alpha y\| = \|x - \alpha y\| = 1.$$

This shows that $D_2(x) \subset D_1(x)$.

Let now $y \in D_1(x)$. Then for every $g \in F_x$ we have

$$|1 \pm \alpha y(g)| = |(x \pm \alpha y)(g)| \leq \|x \pm \alpha y\| = 1.$$

But this inequality is true only when $y(g) = 0$. Therefore, $F_x \subset F_{x+\alpha y}$. Then, by [4, Lemma 2.8], we get

$$x + \alpha y = x + P_0(x)^*(x + \alpha y) = x + \alpha P_0(x)^* y,$$

i. e., $y = P_0(x)^* y$. So, $x \diamondsuit y$. Then it follows from [4, Lemma 2.8(i)] that

$$\|x + \beta y\| = \max\{\|x\|, \|\beta y\|\} = \max\{1, \|\beta y\|\},$$

for every $\beta \in \mathbb{C}$. Therefore, $D_1(x) \subset D_2(x)$. Thus, if $x$ is a geometric tripotent, then $D_1(x) = D_2(x)$.

*Sufficiency.* Suppose that $F_x \neq \emptyset$ and $D_1(x) = D_2(x)$, but $x$ is not a geometric tripotent. Since $Z$ is a strongly facially symmetric space, $F_x$ is a symmetric face. Consequently, by [5, Proposition 1.6] there $u$ is a geometric tripotent such that $F_u = F_x$. Therefore, by [4, Lemma 2.8] we have $x = u + P_0(u)^* x$.

Set $y = \big(\|P_2(u)^* x\| - 1\big) \dfrac{P_2(u)^* x}{\|P_2(u)^* x\|}$. Then by [4, Lemma 2.1(i)] it follows that

$$\|x + y\| = \left\| u + \big(2\|P_0(u)^* x\| - 1\big) \frac{P_0(u)^* x}{\|P_0(u)^* x\|} \right\| = \max\big\{\|u\|, |2\|P_0(u)^* x\| - 1|\big\} = 1,$$

$$\|x - y\| = \left\| u + \frac{P_0(u)^* x}{\|P_0(u)^* x\|} \right\| = \max\big\{\|u\|, 1\big\} = 1.$$

So, $y \in D_1(x)$.

On the other hand, again according to [4, Lemma 2.1(i)], the following equalities hold for every $\beta \in \mathbb{C}$

$$a = \|x + \beta y\| = \left\| u + \big(\|P_0(u)^* x\| + \beta\|P_0(u)^* x\| - \beta\big) \frac{P_0(u)^* x}{\|P_0(u)^* x\|} \right\|$$
$$= \max\big\{\|u\|, \big|\|P_0(u)^* x\| - \beta\|P_0(u)^* x\| - \beta\big|\big\},$$

$$b = \max\big\{\|u\|, \|\beta y\|\big\} = \max\left\{\|u\|, \left\| \|P_0(u)^* x\|\beta - \beta) \frac{P_0(u)^* x}{\|P_0(u)^* x\|} \right\| \right\}$$
$$= \max\big\{\|u\|, \big|\|P_0(u)^* x\|\beta - \beta\big|\big\}.$$

Since $\|P_0(u)^* x\| > 0$, then $a \neq b$. From here

$$\|x + \beta y\| \neq \max\{1, \|\beta y\|\}.$$

This contradicts the assumption that $D_1(x) = D_2(x)$, hence $x$ must be a geometric tripotent. □

**Corollary 2.1.** *Let $Z$ be a reflexive SFS-space, and let $x$ be a norm-one element of $Z^*$. Then $x$ is a geometric tripotent if and only if $D_1(x) = D_2(x)$.*

The following corollary follows directly from the proof of necessity in Theorem 2.1.

**Corollary 2.2.** *Let $Z$ be a SFS-space and $u \in \mathcal{GT}$. Then $P_0(u)^* Z^* = D_1(u)$.*

**Corollary 2.3.** *Elements $u, v \in \mathcal{GT}$ are orthogonal if and only if $\|u \pm v\| = 1$.*

The geometric tripotent $u$ is called maximal if $P_0(u) = 0$.

**Corollary 2.4.** *Let $Z$ be a strongly facially symmetric space and $x \in Z^*$, $\|x\| = 1$, $F_x \neq \emptyset$. Then the following statements are equivalent:*
1) *$x$ is a maximal geometric tripotent,*
2) *$x$ is an extreme point in $Z_1^*$,*
3) *$D_1(x) = \{0\}$.*

Two elements $x$ and $y$ of $Z^*$ are said to be M-orthogonal (see [16]) and denoted as $x \square y$ if $\|x \pm y\| = \max\{\|x\|, \|y\|\}$.

The M-orthogonal complement (M-complement) $H^\square$ of a subset $H$ of $Z^*$ is defined as $H^\square = \{y \in Z^* : \forall x \in H \ x \square y\}$. For a singleton set $\{x\}$ we write $x^\square$ instead of $\{x\}^\square$.

For each element $x \in Z^*$ with unit norm, the tangent disc $S_x$ is defined as

$$S_x = \big\{y \in Z^* : \forall \alpha \in \mathbb{C} \ |\alpha| \leq 1 \Rightarrow \|x + \alpha y\| = 1\big\}.$$

**Theorem 2.2.** *Let $Z$ is a complex strongly facially symmetric space and $x \in Z^*$, $\|x\| = 1$, $F_x \neq \emptyset$. Then the following conditions are equivalent:*
1) *$x \in \mathcal{GT}$,*
2) *$x^\square \cap Z_1^* = x^\diamond \cap Z_1^*$,*
3) *$x^\square \cap Z_1^* = i x^\square \cap Z_1^*$,*
4) *$S_x = x^\diamond \cap Z_1^*$.*

P r o o f. The implication 1) $\Rightarrow$ 2) follows from [14, Lemma 3].

1) $\Rightarrow$ 3). Suppose $x \in \mathcal{GT}$ and $y \in x^\square \cap Z_1^*$. Then, from [16, Lemma 2.2(i)], it follows that

$$x^\square \cap Z_1^* = \{y \in Z^* : \forall t \in [-1;1] \ \|x + ty\| = 1\} \subset D_1(x).$$

Therefore, from Theorem 2.1, we have $y \in D_2(x)$. Specifically,

$$\|iu \pm y\| = \|u \mp iy\| = \max\{\|u\|, \|iy\|\} = \max\{\|iu\|, \|y\|\}.$$

Thus, $y \in i x^\square \cap Z_1^*$, i. e., $x^\square \cap Z_1^* \subset i x^\square \cap Z_1^*$. The reverse implication follows from similar reasoning. Hence, $x^\square \cap Z_1^* = i x^\square \cap Z_1^*$.

1) $\Rightarrow$ 4). Assume $x \in \mathcal{GT}$ and $y \in x^\diamond \cap Z_1^*$. From Corollary 2.2, $y \in D_1(x)$, and from Theorem 2.1, $y \in D_2(x)$. Therefore, for all $\alpha \in \mathbb{C}, |\alpha| \leq 1$, we have

$$\|x + \alpha y\| = \max\{\|x\|, \|\alpha y\|\} = 1,$$

i. e., $y \in S_x$.

Let $x \in \mathcal{GT}$ and $y \in S_x$. Then $y \in D_1(x)$, and from Corollary 2.2, $y \in P_0(x)^* Z^*$, i. e., $y \in x^\diamond \cap Z_1^*$. Thus, $S_x = x^\diamond \cap Z_1^*$.

To prove the reverse implications, assume $x$ is not a geometric tripotent, leading to a contradiction with the problem's conditions in each case. Since $Z$ is strongly facially symmetric, $F_x$ is a symmetric face. Therefore, by [5, Proposition 1.6], there exists a geometric tripotent $u$ such that $F_u = F_x$. From [4, Lemma 2.8], we have $x = u + P_0(u)^*x$.

3) $\Rightarrow$ 1). Suppose $x^{\square} \cap Z_1^* = ix^{\square} \cap Z_1^*$ and

$$y = i\sqrt{1 - \|P_0(u)^*x\|^2}\,\frac{P_0(u)^*x}{\|P_0(u)^*x\|}.$$

Since $\|P_0(u)^*x\| \le 1$, $\|y\| \le 1$. According to [4, Lemma 2.1(i)],

$$\|x \pm y\| = \left\|u + \left(\|P_0(u)^*x\| \pm i\sqrt{1 - \|P_0(u)^*x\|^2}\right)\frac{P_0(u)^*x}{\|P_0(u)^*x\|}\right\| = \max\left\{\|u\|, \left|\|P_0(u)^*x\| \pm i\sqrt{1 - \|P_0(u)^*x\|^2}\right|\right\} = 1.$$

Thus,

$$\max\{\|x\|, \|y\|\} = \max\left\{1, \sqrt{1 - \|P_0(u)^*x\|^2}\right\} = 1 = \|x \pm y\|.$$

This means $x$ and $y$ are M-orthogonal. On the other hand, by [4, Lemma 2.1(i)],

$$\max\{\|x\|, \|iy\|\} = \max\{\|x\|, \|y\|\} = \max\left\{1, \sqrt{1 - \|P_0(u)^*x\|^2}\right\} = 1,$$

$$\|x - iy\| = \left\|u + \left(\|P_0(u)^*x\| + \sqrt{1 - \|P_0(u)^*x\|^2}\right)\right\| = \max\left\{1, \|P_0(u)^*x\| + \sqrt{1 - \|P_0(u)^*x\|^2}\right\} > 1.$$

Thus, $x$ and $iy$ are not M-orthogonal, and $y$ is not in $ix^{\square}$.

2) $\Rightarrow$ 1). Assume $x^{\square} \cap Z_1^* = x^{\diamond} \cap Z_1^*$ and $y = (1 - \|P_0(u)^*x\|)\dfrac{P_0(u)^*x}{\|P_0(u)^*x\|}$. Then $\|y\| = 1 - \|P_0(u)^*x\| < 1$. From [4, Lemma 2.1(i)],

$$\|x - y\| = \left\|u + (2\|P_0(u)^*x\| - 1)\frac{P_0(u)^*x}{\|P_0(u)^*x\|}\right\| = \max\{\|u\|, |2\|P_0(u)^*x\| - 1|\} = 1,$$

$$\|x + y\| = \left\|u + \frac{P_0(u)^*x}{\|P_0(u)^*x\|}\right\| = \max\{\|u\|, 1\} = 1.$$

Thus,

$$\max\{\|x\|, \|y\|\} = \max\{1, 1 - \|P_0(u)^*x\|\} = 1 = \|x \pm y\|.$$

This shows $x$ and $y$ are M-orthogonal, i. e., $y \in x^{\square} \cap Z_1^*$. Assume $x \diamond y$. Then $x \diamond \alpha y$ for every $\alpha \in \mathbb{C}$. By [4, Lemma 2.1(i)],

$$\|x + \alpha y\| = \max\{1, \|\alpha y\|\}. \tag{2.1}$$

On the other hand, from [4, Lemma 2.1(i)], for every $\alpha \in \mathbb{C}$, we have

$$\|x + \alpha y\| = \left\|u + (\|P_0(u)^*x\| + \alpha - \alpha\|P_0(u)^*x\|)\frac{P_0(u)^*x}{\|P_0(u)^*x\|}\right\| = \max\{1, |\|P_0(u)^*x\| + \alpha - \alpha\|P_0(u)^*x\||\},$$

$$\max\{\|x\|, \|\alpha y\|\} = \max\left\{1, \|(\alpha - \alpha\|P_0(u)^*x\|)\frac{P_0(u)^*x}{\|P_0(u)^*x\|}\right\}$$
$$= \max\{1, |\alpha - \alpha\|P_0(u)^*x\||\}.$$

Hence, $\|x+\alpha y\| \neq \max\{1, \|\alpha y\|\}$, contradicting equality (2.1). Therefore, $y$ is not in $x^\diamond$.

4) $\Rightarrow$ 1). Suppose $S_x = x^\diamond \cap Z_1^*$ and $y = (1 - \|P_0(u)^*x\|)\frac{P_0(u)^*x}{\|P_0(u)^*x\|}$. By [4, Lemma 2.1(i)], for each $\alpha \in \mathbb{C}$, $|\alpha| \leq 1$,

$$\|x+\alpha y\| = \left\|u + \big(\|P_0(u)^*x\| + \alpha - \alpha\|P_0(u)^*x\|\big)\frac{P_0(u)^*x}{\|P_0(u)^*x\|}\right\|$$
$$= \max\{1, |\|P_0(u)^*x\| + \alpha - \alpha\|P_0(u)^*x\||\} = 1.$$

Thus, $y \in S_x$. However, $y$ is not in $x^\diamond$ as in case 2) $\Rightarrow$ 1). □

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