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Linear integral operators in spaces of continuous and essentially bounded vector functions

Manuel J. ALVES[1] , **Elena V. ALVES**[2] , **João S.P. MUNEMBE**[1] , **Yury V. NEPOMNYASHCHIKH**[1]

[1] Eduardo Mondlane University

Main Campus, PO. Box 257, Maputo, Republic of Mozambique

[2] Higher Institute of Sciences and Technology of Mozambique

Street 1.194 no 332, Central C, Municipal District KaMpfumu, Maputo, Republic of Mozambique

Abstract. The well-established criterion for the action and boundedness of a linear integral operator K from the space L_∞ of essentially bounded functions to the space C of functions continuous on a compact set is extended to the case of functions taking values in Banach spaces.

The study further shows that if the operator K is active and bounded in the space C, it is also active and bounded in the space L_∞, with the norms of K in C and L_∞ being identical. A precise expression for the general value of the norm of the operator K in these spaces, expressed in terms of its operator kernel, is provided. Addicionally, an example of an integral operator (for scalar functions) is given, active and bounded in each of the spaces C and L_∞, but not acting from L_∞ into C.

Convenient conditions for checking the boundedness of the operator K in C and L_∞ are discussed. In the case of the Banach space Y of the image function values of K being finite-dimensional, these conditions are both necessary and sufficient. In the case of infinite-dimensionality of Y, they are sufficient but not necessary (as proven).

For $\dim Y < \infty$, unimprovable estimates for the norm of the operator K are provided in terms of a 1-absolutely summing constant $\pi_1(Y)$, determined by the geometric properties of the norm in Y. Specifically, it is defined as the supremum over finite sets of nonzero elements of Y of the ratio of the sum of the norms of these elements to the supremum (over functionals with unit norm) of the sums of absolute values of the functional on these elements.

Keywords: Banach space, linear integral operator, norm of linear operator, 1-absolutely summing constant

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НАУЧНАЯ СТАТЬЯ

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Линейные интегральные операторы в пространствах непрерывных и существенно ограниченных вектор-функций

Мануэль Жоаким АЛВЕС[1], **Елена Владимировна АЛВЕС**[2], **Жоао Себастьян Паулу МУНЕМБЕ**[1], **Юрий Витальевич НЕПОМНЯЩИХ**[1]

[1] «Университет Эдуардо Мондлане»
1100, Республика Мозамбик, г. Мапуто, Главный кампус, П.Я. 257
[2] «Высший институт наук и технологий Мозамбика»
1100, Республика Мозамбик, г. Мапуто, улица 1.194 No 332,
центральный С, Муниципальный район КаМпфуму

Аннотация. Известный критерий действия и ограниченности линейного интегрального оператора K из пространства L_∞ существенно ограниченных функций в пространство C непрерывных на компакте функций обобщается на случай функций со значениями в банаховых пространствах.

В работе также доказано, что из действия и ограниченности оператора K в пространстве C вытекает его действие и ограниченность в пространстве L_∞, причем нормы оператора K, рассматриваемого в C и L_∞, совпадают. Приводится точное выражение общего значения нормы оператора K в этих пространствах в терминах ядра оператора. В дополнение к этому, приводится пример интегрального оператора (для скалярных функций), который действует и ограничен в каждом из пространств C и L_∞, но не действует из L_∞ в C.

Также обсуждаются удобные для проверки условия ограниченности оператора K в C и L_∞. В случае конечномерности банахова пространства Y значений функций образа оператора K эти условия являются одновременно необходимыми и достаточными. В случае бесконечномерности Y они являются достаточными, но не являются необходимыми (это доказывается).

В случае $\dim Y < \infty$ приводятся неулучшаемые оценки для нормы оператора K в терминах 1-абсолютно суммирующей константы $\pi_1(Y)$, определяемой геометрическими свойствами нормы в Y, более точно, как супремум по конечным наборам ненулевых элементов Y отношения суммы норм этих элементов и супремума (по функционалам с единичной нормой) сумм абсолютных значений функционала на этих элементах.

Ключевые слова: банахово пространство, линейный интегральный оператор, норма линейного оператора, 1-абсолютно суммирующая константа

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Introduction

The criteria governing the action, continuity, and compactness of the Fredholm linear integral operator K within function spaces underwent comprehensive and thorough examination during the 20th century. These considerations are extensively documented in classical monographs on functional analysis, as exemplified by references such as [1,2]. The monograph [3, p. 100] formulates necessary and sufficient conditions for the action and boundedness of K from the space of essentially bounded functions to the space of continuous functions on a compact set. This work also provides expressions for the norm of the operator K in terms of its kernel.

In this paper, we extend and partially generalize these findings, building upon the characteristics of the integral functional derived in the study [4]. We outline the principal directions of advancement in our work concerning the well-established classical aspects of the operator K within the space of continuous functions.

- We establish necessary and sufficient conditions for the action and boundedness of the operator K from spaces of continuous functions, as well as from the space of essentially bounded functions to the space of bounded functions, when the functions from these spaces take values in Banach spaces.
- We demonstrate that the action and boundedness of the operator K in the space of continuous functions imply the action and boundedness in the space of essentially bounded functions, with the norms of both operators being equal.
- We provide an expression for the norm of the operator K in terms of its kernel.
- In the scenario of finite-dimensionality of the function value space, we establish an optimal estimate for the norm of the operator K in terms of a convenient expression through the kernel of the operator using 1-absolutely summing constant.

1. Key notations and concepts

We will use the following notations:

Ω is closed bounded set in $\mathbb{R}^n$ with the classical Lebesgue measure μ in σ-algebra Σ Lebesgue measurable subsets of Ω.

χ_E is characteristic function of a set $E \subset \Omega$ ($\chi_E(s) = 1$ if $s \in E$ and $\chi_E(s) = 0$ if $s \notin E$).

X and Y are real separable Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, accordingly, moreover Y contains no copy of c_0 ($Y \not\supset c_0$), in particular, Y reflexively.

X^* is the dual space of X with a norm $\|\cdot\|_{X^*}$; value of a functional $f \in X^*$ in the point $x \in X$ we will denote by $f[x]$ or $\langle x, f\rangle$.

$B(X,Y)$ — the space of linear bounded operators from X to Y with the natural norm.

By $B_1(Z)$, we denote the closed unit ball centered at zero in the Banach space Z, that is $B_1(Z) = \{x \in Z \colon \|x\|_Z \le 1\}$.

$T(X)$ is Banach space consisting of bounded functions $u : \Omega \to X$, with the sup-norm

$$\|u\|_\infty = \sup_{t\in\Omega} \|u(t)\|_X.$$

A function $u : \Omega \to X$ is called measurable if the preimage of any Borel set in X is Lebesgue measurable. The set of all measurable functions $u : \Omega \to X$ is denoted by $L_0(X)$.

By $L_\infty(X)$, $L^c_\infty(X)$, $C(X)$ and $P(X)$ we will denote the linear subspaces of the space $T(X)$, consisting of measurable bounded functions, measurable compact-valued functions, continuous functions, and measurable finite-valued functions, respectively, equipped with the sup-norm. It is clear that the first three of these linear subspaces are closed, hence they are Banach spaces. Let's note that $P(X) \cup C(X) \subset L^c_\infty(X) \subset L_\infty(X) \subset T(X)$ and in the case of finite-dimensionality of X we have $L^c_\infty(X) = L_\infty(X)$. In particular $X = \mathbb{R}$ in the notations of the introduced functional spaces, we will omit the notation of the function value space: $L_\infty = L_\infty(\mathbb{R})$, $C = C(\mathbb{R})$ and etc.

By $\mathrm{L}_\infty(X)$ we will denote the factor space of space $L_\infty(X)$, consisting of classes μ-equivalent to essentially bounded functions with the essential supremum norm

$$\|u\|_\infty = \operatorname*{ess\,sup}_{t\in\Omega} \|u(t)\|_X.$$

The closed subspace $\mathrm{L}^c_\infty(X)$ of the Banach space $\mathrm{L}_\infty(X)$ is defined similarly.

If the function $v : \Omega \to Y$ is such, that $\forall g \in Y^*$ (which is equivalent to, $\forall g \in B_1(Y^*)$) the real function $\langle v(\cdot), g\rangle$ is Lebesgue integrable on Ω, then there exists a unique element $I \in Y$ such that

$$\langle I, g\rangle = \int_\Omega \langle v(s), g\rangle\, ds \qquad (\forall g \in Y^*) \tag{1.1}$$

(see, for example, [5, p. 54]). In this case, the function v is called Pettis integrable on Ω, and I is called the Pettis integral of the function v on Ω, denoted by

$$I = (P)\int_\Omega v(s)\, ds.$$

From the separability of Y it follows that any Pettis integrable function is measurable (see [5, p. 42, 53]).

Note that in the case when Y contains a subspace isomorphic to c_0, from Lebesgue integrability on Ω the functions $\langle v(\cdot), g\rangle$ for each $\forall g \in Y^*$, in general, it does not follow that there exists $I \in Y$ satisfying (1.1). For an arbitrary Banach space Y, the concept of the Danford integral (which is an element of the second dual space Y^{**}) is known, which generalizes the concept of the Pettis integral [5, p. 52]. To avoid complicating the results in the direction related to different definitions of integrals, we assume in the paper that $Y \not\supset c_0$.

If the function $v : \Omega \to Y$ is measurable and the real-valued function $\|v(\cdot)\|_Y$ is Lebesgue integrable on Ω, then the function v is called Bochner integrable on Ω. The definition of the Bochner integral is analogous to the definition of the Lebesgue integral for real-valued functions (see [5, p. 44]). For the Bochner integral we will use the notation $(B)\int_\Omega v(s)\, ds$.

If the function $v : \Omega \to Y$ is Bochner integrable, then it is Pettis integrable and the values of the integrals coincide. The reciprocal statement holds true if and only if Y is finite-dimensional [5]. For the Lebesgue integral of real functions, instead of $(P)\int$ and $(B)\int$, we will write $\int$.

2. Some auxiliary results

Lemma 2.1. *The set $P(X)$ is dense in $L^c_\infty(X)$.*

P r o o f. Let's now fix arbitrary $u \in L^c_\infty(X)$ and $\varepsilon > 0$. Let's choose for the relatively compact set $u(\Omega)$ a finite ε-net $z_1, z_2, \dots, z_m \subset u(\Omega)$ and let's define

$$A_i = \{s \in \Omega\colon \|u(s) - z_i\|_X \le \varepsilon\} \quad (i = 1, 2, \dots, m),$$
$$E_1 = A_1, \qquad E_i = A_i \setminus \bigcup_{j=1}^{k-1} A_j \quad (i = 2, 3, \dots, m).$$

Then $u_\varepsilon = \sum_{i=1}^m \chi_{E_i} z_i \in P(X)$ and $\|u - u_\varepsilon\|_\infty \le \varepsilon$. □

The next two lemmas follow directly from [4, Lemma 2.1 and Assertion 2.1].

Lemma 2.2. *Let $D \subset X$ be a convex and closed set, and let $u : \Omega \to Y$ be some measurable function with values in D. Then there exists a sequence of functions $u_n \in C(X)$ with values in D that converges in measure to u.*

Lemma 2.3. *Let the function $f : \Omega \times X \to \mathbb{R}$ satisfy the Carathéodory conditions, which means that the function $f(\cdot, x)$ is measurable for each $x \in X$, and the function $f(s, \cdot)$ is continuous for each $s \in \Omega$. Then the function $\psi : \Omega \to [0, \infty]$ defined by $\psi(s) = \sup\limits_{x \in B_1(X)} |f(s, x)|$ is measurable, and*

$$\sup_{u \in B_1(L_\infty(X))} \int_\Omega |f(s, u(s))|\, ds = \int_\Omega \sup_{x \in B_1(X)} |f(s, x)|\, ds.$$

To prove the main results about the linear integral operator, we will need the criterion for the action and boundedness of the linear integral functional, as well as one of its properties established in the work [4], which we will present here without proofs.

Let $a : \Omega \to X^*$. The functional H will be formally defined by the equation

$$H[u] = \int_\Omega a(s)[u(s)]\, ds. \tag{2.1}$$

If the finite integral (2.1) exists for all functions $u : \Omega \to X$ from a certain linear subspace V of the space $L_0(X)$, then expression (2.1) defines a linear functional $H : V \to \mathbb{R}$.

Function $a(\cdot) : \Omega \to X^*$ is said weak*-measurable (see [5, p. 41]), if $\forall x \in X$ the real function $a(\cdot)[x]$ is measurable.

Assertion 2.1. (see [4, Theorem 4.1]) *The following conditions* a), b), *and* c) *are equivalent to each other:*

a) *$H \in (C(X))^*$, in other words, the functional H acts from $C(X)$ to $\mathbb{R}$ and is bounded;*

b) *$H \in (\mathrm{L}_\infty(X))^*$;*

c) *Function $a(\cdot)$ weak*-measurable and $\|a\|_1 \overset{\text{def.}}{=} \int_\Omega \|a(s)\|_{X^*}\, ds < \infty$.*

When any of these conditions is satisfied

$$\|H\|_{(C(X))^*} = \|H\|_{(\mathrm{L}_\infty(X))^*} = \|a\|_1.$$

Assertion 2.2. (see [4, Corollary 4.2]) *If $H \in (C(X))^*$, then for any bounded sequence $u_n \subset L_\infty(X)$ converging in measure to some $u \in L_\infty(X)$, it holds that $H[u_n] \to H[u]$.*

3. Definition of an integral operator and conditions on its kernel

Let $k : \Omega^2 \to B(X,Y)$. Let's consider the linear Fredholm integral operator K with the kernel k, defined by equality

$$(Ku)(t) = (P)\int_{\Omega} k(t,s)u(s)\,ds, \qquad t \in \Omega.$$

Under certain conditions on the kernel k, the operator K transforms functions $u : \Omega \to X$ into functions $Ku : \Omega \to Y$.

For fixed $(t,s) \in \Omega^2$, let $k_*(t,s)$ denote the adjoint operator of $k(t,s)$, so that $k_* : \Omega^2 \to B(Y^*, X^*)$ (the asterisk notation as a subscript is used to avoid confusion with the adjoint kernel $k^*(t,s) = k(s,t)$).

Let's introduce certain constants expressed in terms of the kernel k of the operator K and consider certain conditions on the kernel k that will be used in the following sections.

Let's define, for now formally, the quantities $\|k\|_u$ and $\|k\|_w$ by:

$$\|k\|_u \overset{\text{def.}}{=} \sup_{t\in\Omega} \int_{\Omega} \|k(t,s)\|_{B(X,Y)}\,ds, \qquad \|k\|_w \overset{\text{def.}}{=} \sup_{t\in\Omega;\ g\in B_1(Y^*)} \int_{\Omega} \|k_*(t,s)g\|_{X^*}\,ds.$$

and the conditions:

(a_0) For all $t \in \Omega$ and $x \in X$ function $k(t,\cdot)x$ is measurable;

(a_c) For all $A \in \Sigma$ and $x \in X$ holds $\displaystyle\int_A k(\cdot,s)x\,ds \in C(Y)$;

(b_u) Exists and is finite the quantity $\|k\|_u$;

(b_w) Exists and is finite the quantity $\|k\|_w$.

Let's emphasize that we have introduced notations for a series of conditions but do not assume them to be a priori satisfied.

Assertion 3.1. *Under the condition* (a_0), *the following properties hold:*

1) *For any* $t \in \Omega$ *and* $u \in L_0(X)$, *the function* $k(t,\cdot)u(\cdot) : \Omega \to Y$ *is measurable;*

2) *For any* $t \in \Omega$, *the function* $\|k(t,\cdot)\|_{B(X,Y)} : \Omega \to [0,+\infty]$ *is measurable;*

3) *For any* $t \in \Omega$, $u \in L_0(X)$, *and* $g \in Y^*$, *the function* $\langle k(t,\cdot)u(\cdot), g\rangle : \Omega \to \mathbb{R}$ *is measurable;*

4) *For any* $t \in \Omega$ *and* $g \in Y^*$, *the function* $\|k_*(t,\cdot)g\|_{X^*} : \Omega \to [0,+\infty]$ *is measurable.*

P r o o f. Properties 3) and 4) follow from Lemma 2.3, applied at fixed $t \in \Omega$ to the function $f : \Omega \times X \to \mathbb{R}$ defined as $f(s,x) = \langle k(t,s)x, g\rangle$. In particular, for 4), we use the equality

$$\|k_*(t,s)g\|_{X^*} = \sup_{x\in B_1(X)} |\langle x, k_*(t,s)g\rangle| = \sup_{x\in B_1(X)} |\langle k(t,s)x, g\rangle| = \sup_{x\in B_1(X)} |f(s,x)|. \tag{3.1}$$

Property 1) follows from theorems 2 and 3 of the paper [6].

Finally, property 2) follows from Lemma 2.3 applied at fixed $t \in \Omega$ to the function $\widetilde{f} : \Omega \times X \to \mathbb{R}$ defined by $\widetilde{f}(s,x) = \|k(t,s)x\|_Y$, taking into account the equality $\|k(t,s)\|_{B(X,Y)} = \sup_{x\in B_1(X)} |\widetilde{f}(s,x)|$. □

From Assertion 3.1, in particular, it follows that under condition (a_0), the quantities $\|k\|_u$ and $\|k\|_w$ are well-defined, which can take finite non-negative values or the value $+\infty$.

4. The criterion for the action and boundedness of the operator K from $C(X)$ and from $L_\infty(X)$ to $T(Y)$

Theorem 4.1. *The following conditions* A) *to* D) *are equivalent to each other:*
A) $K \in B(C(X), T(Y))$ *(the operator* K *acts from* $C(X)$ *to* $T(Y)$ *and is bounded);*
B) $K \in B(\mathrm{L}^c_\infty(X), T(Y))$ *;*
C) $K \in B(\mathrm{L}_\infty(X), T(Y))$ *;*
D) *The conditions* (a_0) *and* $(\mathrm{b_w})$ *are satisfied.*
Moreover, if $K \in B(C(X), T(Y))$, *then*

$$\|K\|_{C(X)\to T(Y)} = \|K\|_{\mathrm{L}^c_\infty(X)\to T(Y)} = \|K\|_{\mathrm{L}_\infty(X)\to T(Y)} = \|k\|_w. \tag{4.1}$$

P r o o f. 1^0 *step.* Let it be fair D). We will prove that $K \in B(\mathrm{L}_\infty(X), T(Y))$ and

$$\|K\|_{\mathrm{L}_\infty(X)\to T(Y)} \le \|k\|_w. \tag{4.2}$$

Let's fix arbitrary $u \in L_\infty(X)$ and $t \in \Omega$. Due to condition $(\mathrm{b_w})$, taking into account Assertion 3.1, for any $g \in Y^*$ with a norm $\|g\|_{Y^*} \le 1$, we have

$$\int_\Omega |\langle k(t,s)u(s), g\rangle|\, ds = \int_\Omega |\langle u(s), k_*(t,s)g\rangle|\, ds \le \|u\|_\infty \|k\|_w < \infty, \tag{4.3}$$

therefore, there exists an integral $(P)\int_\Omega k(t,s)u(s)\, ds \in Y$.

Furthermore, for any $u \in L_\infty(X)$ and $t \in \Omega$, we have, due to $(\mathrm{b_w})$, taking into account (4.3):

$$\|Ku(t)\|_Y = \sup_{g\in B_1(Y^*)} |\langle Ku(t), g\rangle| = \sup_{g\in B_1(Y^*)} \left| \int_\Omega \langle u(s), k_*(t,s)g\rangle\, ds \right| \le \|k\|_w\, \|u\|_\infty.$$

Therefore, the operator K acts from $\mathrm{L}_\infty(X)$ to $T(Y)$, is bounded and holds (4.2).

2^0 *step.* Let $K \in B(C(X), T(Y))$. We will prove that property D) holds and the equality

$$\|K\|_{C(X)\to T(Y)} = \|k\|_w. \tag{4.4}$$

From the condition $K : C(X) \to T(Y)$ and the fact that constant functions are continuous, condition (a_0) follows.

Let us fix arbitrary $t \in \Omega$ and $g \in Y^*$. We define the function $a : \Omega \to X^*$ as follows by

$$a(s)[x] = \langle k(t,s)x, g\rangle, \qquad s \in \Omega,\ x \in X \tag{4.5}$$

and let us consider the functional H, defined by equation (2.1). From the condition $K \in B(C(X), T(Y))$, it follows that $H \in (C(X))^*$. According to Assertion 3.1, taking into account equation (3.1), we have, using the notation $C_1 = B_1(C(X))$,

$$\int_\Omega \|k_*(t,s)g\|_{X^*} ds = \int_\Omega \|a(s)\|_{X^*}\, ds = \|F\|_{(C(X))^*} = \sup_{u\in C_1} \left| \int_\Omega a(s)[u(s)]\, ds \right|. \tag{4.6}$$

By the definition of the Pettis integral

$$\sup_{u\in C_1} \left| \int_\Omega a(s)u(s)\, ds \right| = \sup_{u\in C_1} \left| \int_\Omega \langle k(t,s)u(s), g\rangle\, ds \right| = \sup_{u\in C_1} \left| \left\langle \int_\Omega k(t,s)u(s)\, ds,\ g \right\rangle \right|. \tag{4.7}$$

From equations (4.6) and (4.7), it follows that

$$\sup_{u \in C_1} \left| \left\langle \int_\Omega k(t,s)u(s)\,ds,\ g \right\rangle \right| = \int_\Omega \|k_*(t,s)g\|_{X^*} ds.$$

Taking the supremum over all $t \in \Omega$ and $g \in B_1(Y^*)$ in this equality, we obtain equation (4.4). From this equation and the inequality $\|K\|_{C(X)\to T(Y)} < \infty$, condition $(\mathrm{b_w})$ follows.

From the properties established in steps 1^0 and 2^0, the statement of the theorem follows. □

5. Integral operator with values in the space of continuous functions

The following theorem provides necessary and sufficient conditions for the action and boundedness of the operator K from $\mathrm{L}^c_\infty(X)$ to $C(Y)$ in terms of the norm expression of K using the kernel k. It also establishes the equality of norms of the operator considered from $\mathrm{L}^c_\infty(X)$ to $C(Y)$ and from $C(X)$ to $C(Y)$. This theorem partially generalizes Theorem 1.1 in [3, p. 100], for the case of $p = \infty$.

Theorem 5.1. *$K \in B(\mathrm{L}^c_\infty(X), C(Y))$ if and only if the conditions $(\mathrm{a_c})$ and $(\mathrm{b_w})$ hold. Moreover, if $K \in B(\mathrm{L}^c_\infty(X), C(Y))$, then*

$$\|K\|_{C(X)\to C(Y)} = \|K\|_{\mathrm{L}^c_\infty(X)\to C(Y)} = \|k\|_w. \tag{5.1}$$

P r o o f. 1) Let the conditions $(\mathrm{a_c})$ and $(\mathrm{b_w})$ be satisfied.

Its clear that condition $(\mathrm{a_0})$ is fair, and by virtue of Theorem 4.1 $K \in B(\mathrm{L}^c_\infty(X), T(Y))$.

Each function $v \in P(X)$ has a representation $v(s) = \sum_{i=1}^n \chi_{A_i}(s)x_k$ for some positive integer n, some $x_i \in X$, and pairwise disjoint sets $A_i \in \Sigma$. The linearity and additivity properties of the Pettis integral [5] and the condition $(\mathrm{a_c})$ imply

$$(P)\int_\Omega k(t,s)u(s)\,ds = \sum_{i=1}^n (P)\int_{A_i} k(t,s)x_i\,ds$$

moreover, each of the integrals in the right-hand side exists and is a continuous function of t. Therefore, the integral on the left-hand side and the equality itself will be valid. Thus, it is proven that $K(P(X)) \subset C(Y)$.

From the continuity of the operator $K : \mathrm{L}^c_\infty(X) \to T(Y)$, the inclusion $K(P(X)) \subset C(Y)$, and Lemma 2.1, it follows straightforwardly that $K \in B(\mathrm{L}^c_\infty(X), C(Y))$.

2) Let $K \in B(\mathrm{L}^c_\infty(X), C(Y))$. For any $A \in \Sigma$ and $x \in X$, we have $v = \chi_A x \in L^c_\infty(X)$, thus $(Kv)(\cdot) = (P)\int_A k(\cdot,s)x, ds \in C(Y)$. Thus, condition $(\mathrm{a_c})$ is satisfied. Condition $(\mathrm{b_w})$ and equality (5.1) follow from Theorem 4.1. □

The following theorem provides necessary and sufficient conditions for the boundedness of the operator K when it operates from $C(X)$ to $C(Y)$, expressing its norm in terms of the kernel k. This theorem generalizes the equality for the norm when $p = \infty$ in Theorem 1.2 from [3, p. 100].

Theorem 5.2. *Let the operator K acts from $C(X)$ to $C(Y)$. In order for K to be bounded, it is necessary and sufficient to satisfy condition $(\mathrm{b_w})$.*

Moreover, if $K \in B(C(X), C(Y))$, then we have $\|K\|_{C(X)\to C(Y)} = \|k\|_w$.

P r o o f. From the condition $K : C(X) \to C(Y)$, condition (a_0) of Theorem 4.1 follows. Taking this into account, the statement of the theorem straightforwardly follows from Theorem 4.1. □

As noted in [3, p. 101], a linear integral operator acting from C to C can also be considered as acting from L_∞ to L_∞. The following theorem asserts this fact in the case of function spaces with values in Banach spaces.

Theorem 5.3. *If* $K \in B(C(X), C(Y))$, *then* $K \in B(\mathrm{L}_\infty(X), L_\infty(Y))$ *and*

$$\|K\|_{C(X)\to C(Y)} = \|K\|_{\mathrm{L}_\infty(X)\to L_\infty(Y)} = \|k\|_w.$$

P r o o f. According to Theorem 4.1, from the condition $K \in B(C(X), C(Y))$, it follows that $K \in B(\mathrm{L}_\infty(X), T(Y))$ and equality (4.1) holds. Thus, it sufficient to prove that for every $u \in L_\infty(X)$, the function Ku is measurable.

Let $u \in L_\infty(X)$. We choose a closed ball $D \supset u(\Omega)$, and according to Lemma 2.2, we find a sequence of functions $u_n \in C(X)$ with values in D that converges to u in measure. Fix arbitrary $t \in \Omega$ and $g \in Y^*$, and define the function $a : \Omega \to X^*$ by (4.5). We consider the functional H defined by equality (2.1). From the condition $K \in B(C(X), C(Y))$, it follows that $H \in (C(X))^*$. By Assertion 2.2, $H[u_n] \to H[u]$, which means that $\langle Ku_n(t), g\rangle \to \langle Ku(t), g\rangle$.

Since $K : C(X) \to C(Y)$ and $C(Y) \subset L_0(Y)$, the real-valued functions $\langle Ku_n(\cdot), g\rangle$ are measurable. Then, the function $\langle Ku(\cdot), g\rangle$ is also measurable as the pointwise limit of measurable functions. Thus, we have shown that for any $g \in Y^*$, the function $\langle Ku(\cdot), g\rangle$ is measurable (this property is commonly referred to as weak μ-measurability of the function Ku, see, for example, [5, p. 41]). Then, by Theorem 2 in [5, p. 42], combined with the separability of Y, it follows that the function Ku is measurable. □

R e m a r k 5.1. Among the theorems in this section, there are no simultaneously necessary and sufficient conditions in terms of the kernel for the action and continuity of the operator K from $C(X)$ to $C(Y)$ (Theorem 5.1 provides a close result by replacing $C(X)$ with $L^c_\infty(X)$, and Theorem 5.2 gives a close result about the boundedness of K under the prior assumption of its action). Currently, we are unaware of a corresponding result even for the case $X = Y = \mathbb{R}$.

As for Theorem 5.3, it can be accurately stated that the condition $K \in B(C(X), C(Y))$ implies $K \in B(\mathrm{L}_\infty(X), L_\infty(Y))$ (with equality of norms). However, it does not generally imply either the action of the operator K from $\mathrm{L}^c_\infty(X)$ to $C(Y)$ or the validity of condition $(\mathrm{a_c})$ from Theorem 5.1, even in the case of $X = Y = \mathbb{R}$. We provide a corresponding counterexample obtained in the works [7,8].

E x a m p l e 5.1. Let $X = Y = \mathbb{R}$ and $\Omega = [0;1]$. We define the sets

$$E(t) = \bigcup_{n=1}^{\infty} \left[1-(1-t)^{n-1} - t^2(1-t)^{n-1};\ 1-(1-t)^{n-1}\right] \quad (0 < t < 1)$$

and let's consider a linear integral operator K with a kernel $k : [0;1]^2 \to \mathbb{R}$ defined by

$$k(t,s) = \begin{cases} t^{-1}\chi_{E(t)}(s) & \text{if } \ 0 < t < 1 \\ 1 & \text{if } \ t \in \{0;1\}. \end{cases}$$

Then, we have $K \in B(C, C)$ and $K \in B(\mathrm{L}_\infty, \mathrm{L}_\infty)$. Furthermore, for any $u \in \mathrm{L}_\infty$, the function Ku is continuous on the interval $(0;1]$. However, the operator K does not act from L_∞ to C, and there exists a Lebesgue measurable set $A \subset [0;1]$ such that $\int_A k(\cdot, s)\, ds \notin C$.

6. A convenient sufficient condition for the boundedness of an integral operator

The main results of the work (Theorems 4.1 and 5.1–5.3) utilize a constant $\|k\|_w$ expressed in terms of the kernel k and represents the exact value of the norm of the operator K in a series of pairs of functional spaces. However, the constant $\|k\|_w$ the constant uses the supremum over all functionals in the unit sphere of the space Y^*, which is not very convenient for application. In this regard, it makes sense to analyze the possibility of replacing the constant in the main results $\|k\|_w$ with a more convenient constant $\|k\|_u$, whose expression is a direct formal generalization of a well-known expression $\sup_{t\in\Omega}\int_\Omega |k(t,s)|\,ds$ for the norm of a linear integral operator in the space C (see, for example, [2, p. 183] and [3, p. 100]).

Theorem 6.1. *The following statements are true:*

1) *If the conditions are satisfied* (a_0) *and* (b_u), *then* $K \in B(\mathrm{L}_\infty(X), T(Y))$.

2) *If the conditions are satisfied* (a_c) *and* (b_u), *then* $K \in B(\mathrm{L}^\mathrm{c}_\infty(X), C(Y))$.

3) *If* $K : C(X) \to C(Y)$ *and if the condition* (b_u) *is satisfied, then* $K \in B(C(X), C(Y))$ *and* $K \in B(\mathrm{L}_\infty(X), L_\infty(Y))$.

4) *The norms of the operator* K *in all pairs of spaces considered in statements* 1)–3) *are equal* $\|k\|_w$, *and the estimation is valid*

$$\|K\| = \|k\|_w \le \|k\|_u.$$

P r o o f. In the conditions of any of statements 1)–3), for any $t \in \Omega$ and $g \in Y^*$, we obtain, taking into account Assertion 3.1, the estimation

$$\int_\Omega \|k_*(t,s)g\|_{X^*}\,ds \le \int_\Omega \|k_*(t,s)\|_{B(Y^*,X^*)}\,\|g\|_{Y^*}ds = \|g\|_{Y^*}\int_\Omega \|k(t,s)\|_{B(X,Y)}\,ds.$$

From this, it follows that

$$\|k\|_w \le \|k\|_u. \tag{6.1}$$

From this inequality and Theorems 4.1, 5.1–5.3, all statements of the proven theorem follow in an obvious manner. □

R e m a r k 6.1. For any infinite-dimensional space Y, the reciprocal propositions of 1)–3) in Theorem 6.1 do not hold. Specifically, the condition (b_u), unlike (b_w), is not necessary for the action and boundedness of the operator K in pairs of functional spaces as stated in the theorem. Let's demonstrate this.

It is known (see, for example, [9, p. 91]) that in every infinite-dimensional Banach space Y there exists a weakly summable sequence that is not strongly summable. In other words, there exists a sequence $(y_n)_n$ of elements in Y such that for every $g \in Y^*$, the series $\sum_{n=1}^\infty |\langle y_n, g\rangle|$ converges, while simultaneously $\sum_{n=1}^\infty \|y_n\|_Y = \infty$.

Assuming $\dim Y = \infty$, let us fix some sequence $(y_n)_n$ satisfying the aforementioned property. Take an arbitrary countable measurable partition $\{E_n : n = 1, 2, \ldots\}$ of the set Ω into sets E_n of positive measure, and define $v(s) = \sum_{n=1}^\infty \frac{1}{\mu E_n}\chi_{E_n}(s)y_n$. The constructed function $v : \Omega \to Y$ is clearly measurable. Moreover, it satisfies the following conditions:

$$\int_\Omega |\langle v(s), g\rangle|\,ds = \sum_{n=1}^\infty |\langle y_n, g\rangle| < \infty \quad (\forall g \in Y^*), \qquad \int_\Omega \|v(s)\|_Y\,ds = \sum_{n=1}^\infty \|y_n\|_Y = \infty.$$

Thus, the function v is integrable in the sense of Pettis, but not integrable in the sense of Bochner. Moreover, this follows from [5, p. 224],

$$M \overset{\text{def.}}{=} \sup_{g\in B_1(Y^*)} \int_\Omega |\langle v(s), g\rangle|\, ds < \infty.$$

Let's now consider $X = \mathbb{R}$ and define a function $k : \Omega^2 \to B(\mathbb{R}, Y)$ by the equation $k(t,s)[x] = xv(s)$ $(x \in \mathbb{R})$. We then examine the linear operator K with this kernel k. It is evident that condition (a_0) is satisfied, and

$$\|k\|_w = \sup_{g\in B_1(Y^*)} \int_\Omega |\langle k(t,s), g\rangle|\, ds = M < \infty.$$

So, condition $(\mathrm{b_w})$ holds. According to Theorem 4.1, we have $K \in B(\mathrm{L}_\infty, T(Y))$. Moreover, it is evident that for any $u \in L_\infty$, the function $Ku(\cdot)$ is constant. Hence, $K \in B(\mathrm{L}_\infty, C(Y))$. By Theorem 4.1, we obtain $\|K\|_{\mathrm{L}_\infty\to C(Y)} = \|K\|_{C\to C(Y)} = \|k\|_w = M$. On the other hand,

$$\|k\|_u = \int_\Omega \|v(s)\|_Y\, ds = \sum_{n=1}^{\infty} \|y_n\|_Y = \infty.$$

Therefore, condition $(\mathrm{b_u})$ is not satisfied.

R e m a r k 6.2. In contrast to the property established in Remark 6.1 for any infinite-dimensional Y, we note that in the case of $\dim Y < \infty$, on the contrary, conditions $(\mathrm{b_u})$ and $(\mathrm{b_w})$ are equivalent. Therefore, all the necessary and sufficient conditions from Theorems 4.1 and 5.1–5.3 will remain valid if we replace the condition $(\mathrm{b_w})$ with the condition $(\mathrm{b_u})$ (but without replacing the constant $\|k\|_u$ in these theorems!).

In the case of $\dim Y < \infty$, not only does the estimate (6.1) hold, but there is also a two-sided estimate that can be expressed using a special constant of the finite-dimensional space Y, which depends on the choice of norm in Y.

Dedicating the following section of the work to establishing these properties of the operator K in the case of finite dimension Y.

7. Action and boundedness criteria of the integral operator and norm estimation in the case of $\dim Y < \infty$

D e f i n i t i o n 7.1. (see [10–12]) The quantity

$$\pi_1(Y) \overset{\text{def.}}{=} \sup\Big\{ \sum_{k=1}^{n} \|y_k\|_Y \Big/ \sup_{g\in B_1(Y^*)} \sum_{k=1}^{n} |\langle y_k, g\rangle| \,:\, n\in\{1,2,\dots\},\ y_1,\dots,y_n \in Y\setminus\{0\}\Big\} \tag{7.1}$$

is called the 1-absolutely summing constant of the norm space Y of nonzero dimension.

R e m a r k 7.1. 1) Equality (7.1) correctly defines the constant $\pi_1(Y)$ (taking a finite positive value or the value $+\infty$) for any norm space Y of nonzero dimension. Moreover, $\pi_1(Y) < \infty$ if and only $\dim Y < \infty$. Note that $\pi_1(Y) = \infty$ in every infinite-dimensional Banach space Y, a consequence of the existence of a weakly summable sequence that is not strongly summable (see Remark 6.1).

2) In finite-dimensional spaces of the same dimension equipped with different norms (which, as known, are equivalent), the values of 1-absolutely summing constants, in general, are different and are related to the "geometric properties" of the space that depend on the norm.

3) In [10–12], the p-absolutely summing constant $\pi_p(Y)$ was introduced any $p \in [1;\infty)$, but in our work, it will be needed only for the case $p = 1$.

Throughout this section we assume that the $0 < \dim Y < \infty$.

We will denote the linear space $\mathbb{R}^n$ equipped with the norm $\|\cdot\|_p$ defined by

$$\|x\|_p = \Big(\sum_{i=1}^{n} |x_i|^p\Big)^{1/p} \quad (1 \le p < \infty), \qquad \|x\|_\infty = \max_{i=1,2,\ldots,n} |x_i|$$

as $\mathbb{R}^n_p$ (for any $p \in [1;\infty]$).

We will present, without proof, some properties of the constant $\pi_1(Y)$ established in [11], [12]. Additionally, we will provide values of $\pi_1(Y)$ for certain specific spaces in the table.

Assertion 7.1. [properties of the constant $\pi_1(Y)$]

1. $[\,\dim Y = n\,] \Rightarrow [\,\sqrt{n} \le \pi(Y) \le n\,]$*;*

2. $\pi(\mathbb{R}^n_1) = \dfrac{2^n n}{\sum_{k=0}^{n} C_k^n |n-2k|}$, $\pi(\mathbb{R}^n_2) = \dfrac{\sqrt{\pi}\,\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}$, $\pi(\mathbb{R}^n_\infty) = n$, *in particular,* $\pi(\mathbb{R}^{2n-1}_1) = \pi(\mathbb{R}^{2n-1}_2)$ $(n = 1, 2, \ldots)$ *and*

n	1	2	3	4	5	6	7	8	9	10
$\pi_1(\mathbb{R}^n_1)$	1	2	2	$\frac{8}{3}$	$\frac{8}{3}$	$\frac{16}{5}$	$\frac{16}{5}$	$\frac{128}{35}$	$\frac{128}{35}$	$\frac{256}{63}$
$\pi_1(\mathbb{R}^n_2)$	1	$\frac{\pi}{2}$	2	$\frac{3\pi}{4}$	$\frac{8}{3}$	$\frac{15\pi}{16}$	$\frac{16}{5}$	$\frac{35\pi}{32}$	$\frac{128}{35}$	$\frac{315\pi}{256}$

For any measurable function $v : \Omega \to Y$, let us define

$$\|v\|_1 = \int_\Omega \|v(s)\|_Y \, ds, \qquad \|v\|_* = \sup_{g \in B_1(Y^*)} \int_\Omega |\langle v(s), g\rangle| \, ds.$$

In this case, if the function v is non-integrable (recall that integrability in terms of Bochner and Pettis are equivalent due to the finite-dimensionality of Y), then $\|v\|_1 = \|v\|_* = \infty$, and if it is integrable, both quantities are finite. Moreover (see, for example, [5, p. 50, 224]), on the linear space $L_1(Y)$ consisting of integrable functions $u : \Omega \to Y$ (or more precisely, their classes of μ-equivalence), the quantities $\|\cdot\|_1$ and $\|\cdot\|_*$ are norms.

Assertion 7.2. *The following inequality fulfilled:*

$$\|v\|_* \le \|v\|_1 \le \pi_1(Y)\|v\|_*, \quad v \in L_1(Y), \tag{7.2}$$

in particular, in $L_1(Y)$ the norms $\|\cdot\|_1$ and $\|\cdot\|_*$ are equivalent.

Moreover, the inequality (7.2) is unimprovable, that is

$$\inf_{v \in L_1(Y),\, \|v\|_* \ne 0} \frac{\|v\|_1}{\|v\|_*} = 1, \qquad \sup_{v \in L_1(Y),\, \|v\|_* \ne 0} \frac{\|v\|_1}{\|v\|_*} = \pi_1(Y).$$

P r o o f. Clearly, $\|v\|_* \le \|v\|_1$, and this bound is unimprovable, as for any constant function v we have $\|v\|_* = \|v\|_1$.

Let $v \in L_1(Y)$ and $\varepsilon > 0$ be arbitrary. Let us find, by definition of the Bochner integral [5, p. 44], a function $v_\varepsilon(\cdot) = \sum_{k=1}^{m} \chi_{_k}(\cdot) y_k \in P(Y)$ (where sets $E_k \in \Sigma$ are pairwise disjoint and $y_k \in Y$), such that $\|v - v_\varepsilon\|_1 \le \varepsilon$. Then, by the definition of the constant $\pi_1(Y)$, we have

$$\begin{aligned}\|v\|_1 \le \|v_\varepsilon\|_1 + \varepsilon = \sum_{k=1}^{m} \|y_k\|_Y \cdot \mu(E_k) + \varepsilon \le \pi_1(Y) \sup_{g \in B_1(Y^*)} \sum_{k=1}^{m} |\langle y_k, g\rangle| \mu(E_k) + \varepsilon \\ = \pi_1(Y)\|v_\varepsilon\|_* + \varepsilon \le \pi_1(Y)(\|v\|_* + \varepsilon) + \varepsilon.\end{aligned}$$

Due to the arbitrariness of $\varepsilon > 0$ and $v \in L_1(Y)$, this inequality implies the estimation

$$\|v\|_1 \leq \pi_1(Y)\|v\|_*, \qquad v \in L_1(Y). \tag{7.3}$$

Let's prove the unimprovability of the estimation (7.3). Fix an arbitrary $\delta > 0$ and find, according to the definition of $\pi_1(Y)$, such $n \in \mathbb{N}$ and elements $z_1, z_2, \dots, z_n \in Y$, that

$$\sum_{k=1}^n \|z_k\|_Y \Big/ \sup_{g\in B_1(Y^*)} \sum_{k=1}^n |\langle z_k, g\rangle| > \pi_1(Y) - \delta.$$

Let's take arbitrary measurable sets $A_1, A_2, \dots, A_n \in \Sigma$ with positive measure that are pairwise disjoint, and consider the function $w(s) = \sum_{k=1}^n \frac{1}{\mu(A_k)}\chi_{A_k}(s)z_k$. By construction,

$$\|w\|_1 = \sum_{k=1}^n \|z_k\|_Y > (\pi_1(Y) - \delta) \sup_{g\in B_1(Y^*)} \sum_{k=1}^n |\langle z_k, g\rangle| = (\pi_1(Y) - \delta)\|w\|_*.$$

The unimprovability of the estimation (7.3) is proven. □

Note that inequality (7.2) is known and follows, for example, from Proposition 2.4 in [9, p. 96], formulated in terms of a random element and the p-absolutely summing operator induced by it. However, we preferred a direct proof.

Assertion 7.3. *If condition* (a_0) *is satisfied, then*

$$\pi(Y)^{-1}\|k\|_u \leq \|k\|_w \leq \|k\|_u, \tag{7.4}$$

in particular, conditions $(\mathrm{b_u})$ *and* $(\mathrm{b_w})$ *are equivalent.*

The inequality (7.4) *is unimprovable (in both directions) for* $X = \mathbb{R}$ *in the class of all functions* $k : \Omega^2 \to B(\mathbb{R}, Y)$ *satisfying condition* (a_0).

P r o o f. The inequality $\|k\|_w \leq \|k\|_u$ follows from Theorem 6.1. To prove its unimprovability, it is sufficient to consider the case $X = \mathbb{R}$ and take an arbitrary nonzero element $y_0 \in Y$ and define the kernel $k : \Omega^2 \to B(\mathbb{R}, Y)$ by the equation $k(t,s)[x] = xy_0$. In this case, it is obvious that $\|k\|_w = \|k\|_u = \|y_0\|\mu\Omega$.

Proof of the inequality

$$\pi(Y)^{-1}\|k\|_u \leq \|k\|_w \tag{7.5}$$

we proceed separately for two cases.

1^0 *step.* Let $\|k\|_u < \infty$. Fix arbitrary $t \in \Omega$, $\varepsilon > 0$, and according to Lemma 2.3, let's find a function $u \in L_\infty(X)$ with values in $B_1(X)$ such that

$$\int_\Omega \|k(t,s)\|_{B(X,Y)}\, ds \leq \int_\Omega \|k(t,s)u(s)\|_Y\, ds + \varepsilon = \|v\|_1 + \varepsilon \tag{7.6}$$

where $v : \Omega \to Y$ defined by $v(s) = k(t,s)u(s)$. According to Assertion 7.2,

$$\begin{aligned} \pi(Y)^{-1}\|v\|_1 \leq \|v\|_* &= \sup_{g\in B_1(Y^*)} \int_\Omega |\langle k(t,s)u(s), g\rangle|\, ds \\ \leq \sup_{g\in B_1(Y^*)} \int_\Omega \sup_{x\in B_1(X)} |\langle k(t,s)x, g\rangle|\, ds &= \sup_{g\in B_1(Y^*)} \int_\Omega \|k_*(t,s)g\|_{Y^*}\, ds \leq \|k\|_w. \end{aligned} \tag{7.7}$$

From (7.6) and (7.7) follows

$$\pi_1(Y)^{-1} \int_\Omega \|k(t,s)\|_{B(X,Y)}\, ds \leq \|k\|_w + \pi_1(Y)^{-1}\varepsilon.$$

By taking the supremum over $t \in \Omega$ and the infimum over $\varepsilon > 0$ in this inequality, we obtain the estimate (7.5).

2^0 *step.* Let $\|k\|_u = \infty$. Fix arbitrary $R > 0$ and according to Lemma 2.3, let's find $t \in \Omega$ and a function $u \in L_\infty(X)$ with values in $B_1(X)$ such that $\int_\Omega \|k(t,s)u(s)\|_Y ds > R$. By Assertion 7.2, we obtain similarly (7.7):

$$R < \int_\Omega \|k(t,s)u(s)\|_Y \, ds \le \pi_1(Y) \sup_{g \in B_1(Y^*)} \int_\Omega |\langle k(t,s)u(s), g\rangle| \, ds \le \pi_1(Y)\|k\|_w.$$

By taking the supremum over all $t \in \Omega$ and $R > 0$ in this inequality, we obtain $\|k\|_w = \infty$. Thus, inequality (7.5) is proven.

To prove the unimprovability of the estimate (7.5), it sufficient to consider the case when $\|k\|_u < \infty$. Fix an arbitrary $\delta > 0$ and according to Assertion 7.2, let's find a function $v \in L_1(Y)$ such that $\|v\|_1 > (\pi_1(Y) - \delta)\|v\|_*$. We define the kernel $k : \Omega^2 \to B(\mathbb{R}, Y)$ by the equation $k(t,s)[x] = xv(s)$. In this case, it is obvious that

$$\|k\|_u = \|v\|_1 > (\pi_1(Y) - \delta)\|v\|_* = (\pi_1(Y) - \delta)\|k\|_w.$$

Thus, the assertion is proven. □

The main result concerning a linear integral operator in the case of a finite-dimensional Y follows directly from Theorems 4.1, 5.1–5.3 and Assertion 7.3.

Theorem 7.1. *Let* $0 < \dim Y < \infty$. *The following statements are true:*

1) $(\mathrm{a_0}) \wedge (\mathrm{b_u}) \;\Leftrightarrow\; (\mathrm{a_0}) \wedge (\mathrm{b_w}) \;\Leftrightarrow\; K \in B(C(X), T(Y)) \;\Leftrightarrow\; K \in B(\mathrm{L}_\infty(X), T(Y))$.

2) $(\mathrm{a_c}) \wedge (\mathrm{b_u}) \;\Leftrightarrow\; (\mathrm{a_c}) \wedge (\mathrm{b_w}) \;\Leftrightarrow\; K \in B(\mathrm{L}^c_\infty(X), C(Y))$.

3) *If* K *acts from* $C(X)$ *to* $C(Y)$, *then*
$(\mathrm{b_u}) \;\Leftrightarrow\; (\mathrm{b_w}) \;\Leftrightarrow\; K \in B(C(X), C(Y)) \;\Rightarrow\; K \in B(\mathrm{L}_\infty(X), L_\infty(Y))$.

4) *The norms of the operator* K *in all the pairs of spaces considered in statements* 1)–3) *are equal to* $\|k\|_w$, *and additionally, the following estimate holds*

$$\pi_1(Y)^{-1}\|k\|_u \le \|K\| = \|k\|_w \le \|k\|_u,$$

which is unimprovable in the class of bounded operators acting from $C(X)$ *to* $T(Y)$.

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Information about the authors

Manuel J. Alves, PhD of Physics and Mathematics, Full Professor of the Department of Mathematics and Informatics, Eduardo Mondlane University, Maputo, Republic of Mozambique.
E-mail: mjalves.moz@gmail.com
ORCID: https://orcid.org/0000-0003-3713-155X

Elena V. Alves, PhD of Physics and Mathematics, Associate Professor of the School of Economy and Business Administration, Higher Institute of Sciences and Technology of Mozambique, Maputo, Republic of Mozambique.
E-mail: ealves@isctem.ac.mz
ORCID: https://orcid.org/0009-0000-1452-2553

João S. P. Munembe, PhD of Physics and Mathematics, Full Professor of the Department of Mathematics and Informatics, Eduardo Mondlane University, Maputo, Republic of Mozambique.
E-mail: jmunembe3@gmail.com
ORCID: https://orcid.org/0000-0002-0380-6734

Yury V. Nepomnyashchikh, PhD of Physics and Mathematics, Associate Professor of the Department of Mathematics and Informatics, Eduardo Mondlane University, Maputo, Republic of Mozambique. E-mail: yuriy.nepomnyashchikh@uem.ac.mz
ORCID: https://orcid.org/0009-0008-1374-4283

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Corresponding author:
Yury V. Nepomnyashchikh
E-mail: yuvn2@yandex.ru

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Информация об авторах

Алвес Мануэль Жоаким, кандидат физико-математических наук, профессор кафедры математики и информатики, Университет Эдуардо Мондлане, г. Мапуто, Республика Мозамбик. E-mail: mjalves.moz@gmail.com
ORCID: https://orcid.org/0000-0003-3713-155X

Алвес Елена Владимировна, кандидат физико-математических наук, доцент Школы экономики и делового администрирования, Высший институт наук и технологий Мозамбика, г. Мапуто, Республика Мозамбик. E-mail: ealves@isctem.ac.mz
ORCID: https://orcid.org/0009-0000-1452-2553

Мунембе Жоао Себастьян Паулу, кандидат физико-математических наук, профессор кафедры математики и информатики, Университет Эдуардо Мондлане, г. Мапуто, Республика Мозамбик. E-mail: jmunembe3@gmail.com
ORCID: https://orcid.org/0000-0002-0380-6734

Непомнящих Юрий Витальевич, кандидат физико-математических наук, доцент кафедры математики и информатики, Университет Эдуардо Мондлане, г. Мапуто, Республика Мозамбик. E-mail: yuriy.nepomnyashchikh@uem.ac.mz
ORCID: https://orcid.org/0009-0008-1374-4283

Конфликт интересов отсутствует.

Для контактов:
Непомнящих Юрий Витальевич
E-mail: yuvn2@yandex.ru

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