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On the problem of normal modes of a waveguide

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Abstract. Various approaches to calculating normal modes of a closed waveguide are considered. A review of the literature was given, a comparison of the two formulations of this problem was made. It is shown that using a self-adjoint formulation of the problem of normal waveguide modes eliminates the occurrence of artifacts associated with the appearance of a small imaginary additive to the eigenvalues. The implementation of this approach for a rectangular waveguide with rectangular inserts in the Sage computer algebra system is presented and tested on hybrid modes of layered waveguides. The tests showed that our program copes well with calculating the points of the dispersion curve corresponding to the hybrid modes of the waveguide.

Key words and phrases: polarized electromagnetic radiation, normal modes of a waveguide, spectral problem of waveguide theory, dispersion curve of a waveguide

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1. Introduction

In classical electrodynamics there are two related spectral problems, the problem of normal modes of a waveguide and the problem of eigenmodes of a resonator [1, 2]. The first of these problems in the vector case turned out to be surprisingly difficult, its solution requiring the use of very subtle theorems from the field of functional analysis.

2. Scalar model

Let S be regular domain in $\mathbb{R}^2$, the cylinder $S \times \mathbb{R}$ will be called a waveguide, and the Oz axis of the Cartesian coordinate system used is directed along the axis of the cylinder. A nontrivial solution

$$u = u(x, y)e^{i\omega t - i\gamma z}$$

of the oscillation equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \Delta u$$

in the cylinder $S \times \mathbb{R}$, satisfying the Dirichlet boundary condition

$$u|_{\partial S\times\mathbb{R}} = 0$$

or Neumann boundary condition

$$\left.\frac{\partial u}{\partial n}\right|_{\partial S\times\mathbb{R}} = 0$$

is called the normal mode of the scalar waveguide, and the corresponding value of the positive parameter ω is called the natural frequency. The parameter γ/c is called the propagation constant. If $\gamma > 0$, then the normal mode runs along the z axis, if $\gamma < 0$, then against it. These modes are called guided modes. If γ contains an imaginary part, then the normal modes either increase exponentially or decrease exponentially with increasing z. Such modes are called evanescent.

The problem of finding normal modes of a scalar waveguide is reduced to a 2D spectral problem of finding a nontrivial solution u of the equation

$$\Delta_2 u + (k^2 - \gamma^2)u = 0$$

with Dirichlet or Neumann boundary conditions.

Let the eigenvalues of the problem

$$\Delta_2\phi + \alpha^2\phi = 0, \quad \phi|_{\partial S} = 0$$

be numbered in ascending order taking into account the multiplicity as $\alpha_1^2, \alpha_2^2, \dots,$ and the corresponding eigenfunctions be denoted as ϕ_n. In this case, functions

$$\phi_n(x,y)e^{i\alpha_n t}$$

describe the natural oscillations of the membrane S. For a given domain S, the numbers $\alpha_1^2, \alpha_2^2, \dots$ are uniquely determined. For some domains, they can be calculated analytically, for all others they are found using the Galerkin method.

For this reason, the parameters ω and γ of the normal modes of a scalar waveguide are related as

$$k^2 - \gamma^2 = \alpha_n^2.$$

Therefore, for a fixed frequency ω, there are at most a finite number of positive values of the parameter γ, for which normal modes exist. These modes describe waves traveling along the waveguide and, as already said, are called guided modes. All other normal modes have an imaginary γ and for this reason they exponentially increase or decrease along the waveguide axis.

By Steklov's theorem, a monochromatic scalar field in a waveguide can always be represented as a sum

$$\sum \left(a_n e^{i\omega t - i\gamma_n z} + b_n e^{i\omega t + i\gamma_n z}\right)\phi_n(x,y),$$

where a_n, b_n are the complex amplitudes. Therefore, the field, say, at large z is a superposition of a finite number of running normal modes, a sum of exponentially decreasing evanescent modes, and a sum of exponentially growing modes. The partial radiation conditions are such that there should be no exponentially growing terms [1].

It should also be noted that normal modes in the framework of the scalar model with Dirichlet conditions exist only for those values of k, γ that lie on hyperboles

$$k^2 - \gamma^2 = \alpha_n^2, \quad n = 1, 2, \dots .$$

The set of such points (k, γ), which correspond to normal modes, is called the dispersion curve of the waveguide. The dispersion curve of a scalar waveguide consists of a countable number of hyperbolas.

The case of Neumann boundary conditions does not present any fundamental difficulties. Let us agree that the eigenvalues of the problem

$$\Delta_2\psi + \beta^2\psi = 0, \quad \frac{\partial\psi}{\partial n}|_{\partial S} = 0$$

are numbered in ascending order and taking into account the multiplicity as $\beta_1^2, \beta_2^2, \dots$, and the corresponding eigenfunctions are denoted as ψ_n. Let us add to them the zero eigenvalue $\beta_0 = 0$ and the corresponding eigenfunction $\psi_0 = 1$. The system of functions ψ_n is again complete, and normal modes within the framework of the scalar model with Neumann conditions exist only for those values of k, γ that lie on the hyperbolas

$$k^2 - \gamma^2 = \beta_n^2, \quad n = 0, 1, 2, \dots .$$

Thus, in the scalar case, classical theorems of mathematical physics are sufficient to construct the theory of waveguides [1].

3. Vector model

Let us now turn to the vector model of an electromagnetic waveguide. A nontrivial field of the form

$$\vec{E} = \vec{E}(x, y)e^{i\omega t - i\gamma z}, \quad \vec{H} = \vec{H}(x, y)e^{i\omega t - i\gamma z},$$

satisfying the system of homogeneous Maxwell's equations and the boundary conditions

$$\vec{n} \times \vec{E} = 0, \quad \vec{n} \cdot \vec{H} = 0,$$

is called an eigenmode, and the corresponding value of the positive parameter ω is called an eigenfrequency. The parameter $\beta = \gamma/c$ is called the propagation constant. To find the eigenfrequencies, it is necessary to solve the eigenvalue problem

$$\overline{\mathrm{rot}}\vec{E} = -ik\mu\vec{H}, \quad \overline{\mathrm{rot}}\vec{H} = ik\epsilon\vec{E} \tag{1}$$

with the boundary conditions

$$\vec{n} \times \vec{E} = 0, \quad \vec{n} \cdot \vec{H} = 0.$$

Here $\overline{\mathrm{rot}}$ is a differential operator in which differentiation with respect to z is replaced with multiplication by $-i\gamma$. As in the scalar case, the points of the $k\gamma$ plane at which this problem has a nontrivial solution form a certain curve called the dispersion curve of the waveguide.

In the case where the waveguide filling is uniform, Tikhonov A.N. and Samarskii A.A. [3] proved a field decomposition theorem, from which it follows that the complete system of waveguide modes can be composed of two types of modes: transverse magnetic (TM, $H_z = 0$) and transverse electric (TE, $E_z = 0$). For a TE mode, from the equation

$$\mathrm{div}\, \epsilon\vec{E} = 0$$

it follows that there is such a function u that

$$E_x = \frac{\partial u}{\partial y}, \quad E_y = -\frac{\partial u}{\partial x}, \quad E_z = 0.$$

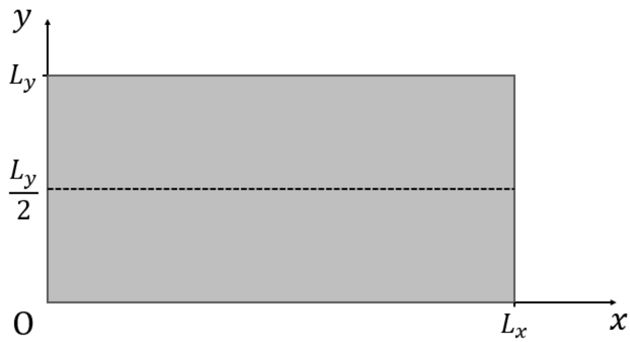

Figure 1. Waveguide filled with layers

This function is called the Borgnis function [1]. By direct substitution of these expressions into Maxwell's equations, it is possible to express all the components of the field through the derivatives of the Borgnis function, and for the Borgnis function itself to obtain a scalar eigenvalue problem with the Dirichlet condition. Similarly, the TM mode can be expressed through the derivatives of the Borgnis function, for which it is possible to obtain a scalar eigenvalue problem with the Neumann condition. It is only necessary to discard the zero eigenvalue, which will correspond to the trivial electromagnetic field.

Thus, the theory of Borgnis functions allows reducing the study of the modes of a waveguide filled with an optically homogeneous substance to the study of the spectrum of the Laplace operator. In this case, the dispersion curve turns out to be the union of a countable number of hyperbolas, which are dispersion curves for a scalar waveguide with the Dirichlet and Neumann conditions.

However, in practice, waveguides with optically inhomogeneous filling are quite common. Such waveguides include waveguides with a core, which are obtained by coating a dielectric cylinder with another dielectric and then with a conducting layer, and multicore waveguides, which are obtained by adding several dielectric cylinders into a bundle covered with a conducting layer on the outside. In this case, it is impossible to decompose the field into TE and TM modes.

4. Rectangular waveguide with two layers

As an example, consider a waveguide with rectangular cross-section $L_x \times L_y$ (see Fig. 1), the filling of which is piecewise constant and depends only on y. In other words, the waveguide consists of several layers, Fig. 1 shows two such layers of equal thickness $L_y/2$. When one of the layers is air, we say that the waveguide is considered half-filled.

There are two families of normal modes in such a waveguide, the SLE and SLH modes, the former have $E_y = 0$, and the latter have $H_y = 0$ [4]. The theory of these modes is in many ways similar to the theory of TM and TE modes developed by Tikhonov and Samarskii [5].

We will search for SLE modes using the method of separation of variables:

$$\vec{E} = \begin{pmatrix} A_x(y)\cos k_x x \\ 0 \\ A_z(y)\sin k_x x \end{pmatrix} e^{ik_z z - i\omega t} \tag{2}$$

and

$$\vec{H} = \begin{pmatrix} B_x(y)\sin k_x x \\ B_y(y)\cos k_x x \\ B_z(y)\cos k_x x \end{pmatrix} e^{ik_z z - i\omega t}.$$

Here the choice of sines and cosines is determined in such a way as to satisfy the boundary conditions.

First of all, let us consider what Maxwell's equations lead to in a layer, where ϵ and μ are constant. From Maxwell's equations we have:

$$\begin{cases} ik\mu B_x + \dfrac{dA_z}{dy} = 0, \\ ik\mu B_y + ik_z A_x - k_x A_z = 0, \\ -ik\mu B_z + \dfrac{dA_x}{dy} = 0, \\ -i\varepsilon k A_x - ik_z B_y + \dfrac{dB_z}{dy} = 0, \\ ik_z B_x + k_x B_z = 0, \\ i\varepsilon k A_z + k_x By + \dfrac{dB_x}{dy} = 0. \end{cases} \tag{3}$$

Three equations from this system allow us to express $\vec{B}$ in terms of A_x and A_z and their derivatives.

After this substitution, out of 6 equations, 3 non-trivial ones remain:

$$\begin{cases} -i\varepsilon k A_x + i(k_z A_x + ik_x A_z)\dfrac{k_z}{k\mu} - i\dfrac{d^2 A_x}{dy^2}\dfrac{1}{k\mu} = 0, \\ -ik_x\dfrac{dA_x}{dy}\dfrac{1}{k\mu} - k_z\dfrac{dAz}{dy}\dfrac{1}{k\mu} = 0, \\ i\varepsilon k A_z - (k_z A_x + ik_x A_z)\dfrac{k_x}{k\mu} + i\dfrac{d^2 A_z}{dy^2}\dfrac{1}{k\mu} = 0. \end{cases}$$

The second equation, up to an insignificant constant, allows finding a linear relationship between A_x and A_z:

$$A_z = -\frac{ik_x}{k_z}A_x. \tag{4}$$

As a result, two equations that differ only by a constant factor turn out to be nontrivial. Therefore, in the layer, Maxwell's equations are reduced to the equation

$$\frac{d^2 A_x}{dy^2} + (\varepsilon\mu k^2 - k_x^2 - k_z^2)A_x = 0 \tag{5}$$

and equations that allow calculating A_z and $\vec{B}$ from the known A_x.

At the waveguide boundary, the modes must satisfy the condition of a wall with ideal conductivity: $\vec{n} \times \vec{E} = 0$.

At the boundary $x = 0, L_x$, these conditions yield:

$$E_y = E_z = 0.$$

For SLE modes, the component E_y is identically zero, so the condition $E_z = 0$ remains valid. We took the sine in (2) for E_z so that this condition is always satisfied at $x = 0$. At $x = L_x$, we obtain the condition $\sin(k_x L_x) = 0$, from which suitable values of k_x are determined as

$$k_x = \frac{\pi n}{L_x}, \quad n \in \mathbb{N}. \tag{6}$$

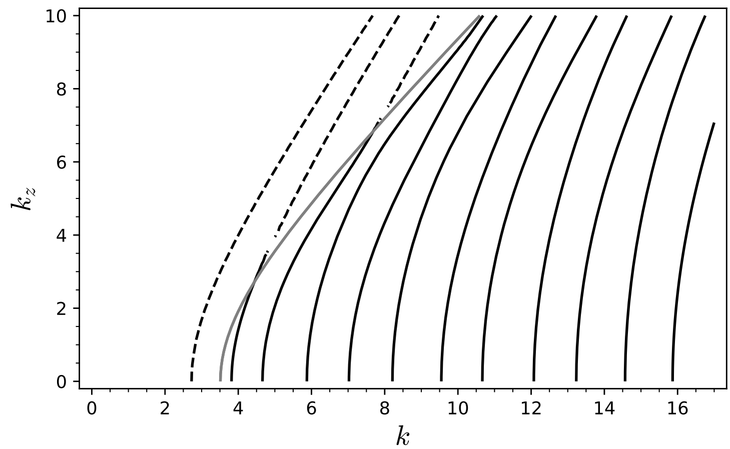

Figure 2. Dispersion curve for a test waveguide with two layers ($\epsilon_0 = 1$, $\epsilon_2 = 1$, $\mu = 1$, $L_x = 1$, $L_y = 2$).

At the boundary $y = 0, L_y$, the conditions of ideal conductivity yield:

$$E_x = E_z = 0.$$

Since A_x and A_y are linearly related in layers, this condition reduces to the Dirichlet condition on A_x:

$$A_x(0) = A_x(L_y) = 0.$$

At the boundary of two layers $y = M$, the requirement of continuity of E_x, E_z and H_x, H_z is to be satisfied.

The continuity of E_x indicates the continuity of A_x. The coefficient in Eq. (4), relating A_z and A_x, does not depend on the filling, so the continuity of A_x implies the continuity of A_z, and, consequently, of E_z.

Maxwell's equations (3) yield

$$ik\mu B_x = -\frac{dA_z}{dy}, \quad ik\mu B_z = \frac{dA_x}{dy}.$$

From this it is clear that the continuity of H_x, H_z is equivalent to the continuity of $\frac{1}{\mu}A'_x$.

As a test example, we consider a waveguide of rectangular cross-section $L_x \times L_y$ with two layers: for $y < L_y/2$ let $\epsilon = \epsilon_1$, and for $y > L_y/2$ let $\epsilon = \epsilon_0$ (see Fig. 1). We consider μ to have a constant value.

According to the discussed above, such a waveguide has a family of SLE modes (2), the parameters of which at a given frequency ω are determined as follows. The number k_x is given by Eq. (6). The number k_z is the eigenvalue of the problem:

$$\begin{cases} \dfrac{d^2A_x}{dy^2} + (\varepsilon\mu k^2 - k_x^2 - k_z^2)A_x = 0, \\ [A_x] = \left[\dfrac{dA_x}{dy}\right] = 0, \quad y = \dfrac{L_y}{2}, \\ A_x(0) = A_x(L_y) = 0. \end{cases} \tag{7}$$

This problem comprises Eq. (5), the boundary conditions and the matching conditions found above. From the eigenfunction A_x we can calculate A_z using Eq. (4), from them we determine $\vec{B}$, and thus determine all the quantities involved in Eq. (2).

For further tests, we are interested in the dependence of the eigenvalues k_z of the problem (7) on $k = \omega/c$.

The solution of (7) for $y < L_y/2$ is

$$A_x = a\sin\left(\sqrt{\varepsilon_1\mu k^2 - k_x^2 - k_z^2}y\right).$$

The solution of (7) for $y > L_y/2$ is

$$A_x = b\sin\left(\sqrt{\varepsilon_0\mu k^2 - k_x^2 - k_z^2}(L_y - y)\right). \tag{8}$$

The choice of sines ensures that the boundary conditions are satisfied. For $y = L_y/2$ we have

$$a\sin\left(\sqrt{\varepsilon_1\mu k^2 - k_x^2 - k_z^2}L_y/2\right) = b\sin\left(\sqrt{\varepsilon_0\mu k^2 - k_x^2 - k_z^2}L_y/2\right)$$

and

$$a\sqrt{\varepsilon_1\mu k^2 - k_x^2 - k_z^2}\cos\left(\sqrt{\varepsilon_1\mu k^2 - k_x^2 - k_z^2}L_y/2\right) =$$
$$-b\sqrt{\varepsilon_0\mu k^2 - k_x^2 - k_z^2}\cos\left(\sqrt{\varepsilon_0\mu k^2 - k_x^2 - k_z^2}L_y/2\right).$$

Thus, k_z is the root of the determinant of this homogeneous system of linear equations.

5. Results

The calculation result is shown in Fig. 2 as solid lines. Their unexpected discontinuity occurs because in Eq. (8) the sine transforms into the hyperbolic sine. We added the second piece of the program, in which the sine is replaced with the hyperbolic sine. The resulting continuation of the dispersion curves is shown in Fig. 2 with dotted line.

Thus, the conjugation method allows finding a family of normal modes of the test waveguide. These modes are neither TE nor TM modes, so they are often called hybrid modes. The considered example proves that hybrid modes exist. This circumstance makes it a very important test for all kinds of calculations of dispersion curves of waveguides, since it is the hybridization of modes that introduces non-self-conjugation into the known approaches to calculating modes.

6. Discussion

Without the decomposition theorem, the normal waveguide mode problem (1) does not decompose into two scalar problems and, thus, does not reduce to any type of problem studied above. It should be noted that the problem (1) contains two parameters, $k = \omega/c$ and γ, and we must choose one of them as the spectral parameter.

In the early 1990s, in the first works on calculating normal waveguide modes [6], the frequency was used as the spectral parameter. This resulted in a self-adjoint spectral problem with respect to k, which could be solved relatively successfully using the software that was available in the early 1990s. The key difficulty at that time was constructing a basis for the Galerkin method that satisfies the condition $\operatorname{div}\vec{H} = 0$.

This approach was soon abandoned in favor of an approach in which γ is considered as a spectral parameter, and the frequency ω is considered given [7–11]. With this approach, the problem (1) is reduced to the study of the spectrum of a non-self-adjoint operator pencil that is quadratic with respect to the spectral parameter γ. By analogy with the scalar case, it is necessary to prove the completeness of the system of normal modes. However, the conditions under which Keldysh [12–15] proved the completeness of the root vectors of the quadratic operator pencil are not satisfied in all possible notations of the problem of waveguide normal modes. For the first time, the completeness of the system of root vectors of a waveguide with piecewise constant filling was substantiated in the papers by Yu.G. Smirnov [16–19], for an arbitrary filling in [9, 20–22]. This in turn made it possible to substantiate the formulation of partial radiation conditions. The basis property of the system of root vectors of a waveguide could be substantiated only for the axially symmetric case [23, 24].

Even greater difficulties are offered by the numerical calculation of normal modes. The application of the Galerkin method, as well as any other truncation method, leads to the study of a non-self-adjoint matrix pencil. Numerical methods for calculating its spectrum are very whimsical. In a number of works [8, 10, 11], algorithms built into, for example, MatLab were used according to the "black box" principle. In such computer experiments, the dispersion curve turned out to be non-monotonic, real eigenvalues suddenly went into the complex domain, etc. Generally speaking, all these phenomena are inherent in the spectral theory of non-self-adjoint matrices. However, the physical meaning of these phenomena raises many questions.

Over the past 30 years since the publication [6], the situation has changed radically. It seemed then that new methods for approximate calculation of the spectrum of non-self-adjoint matrices would appear in the near future, which would solve the difficulties noted above. However, instead, computer algebra methods came into use, which allow constructing Galerkin method bases that satisfy certain properties. This renewed the interest in the idea of Ref. [6]. The choice of frequency as a spectral parameter has a simple physical background. There is an obvious connection between the modes travelling along the waveguide axis and the standing modes of a cylindrical resonator. Having studied it, we obtained a method for constructing a dispersion curve, which requires solving the spectral problem in a cylindrical resonator, i.e., a classical self-adjoint problem. This approach was implemented as a program for the Sage computer algebra system and presented in Ref. [25].

To test this program, we considered a waveguide in which the insert occupies the lower half (see Fig. 1, $\epsilon_0 = 1$, $\epsilon_1 = 0.1$, $L_x = 1$, $L_y = 2$). It turned out that for the lower modes, the points found in our program lie on the analytical curve with graphical accuracy even with a very small number of basis elements taken into account (three for each direction).

7. Conclusion

Using a self-adjoint formulation of the problem of normal waveguide modes eliminates the occurrence of artifacts associated with the appearance of a small imaginary additive to the eigenvalues. We implemented this approach for a rectangular waveguide with rectangular inserts in the Sage computer algebra system. Tests on SLE modes of layered waveguides showed that our program copes well with calculating the points of the dispersion curve corresponding to the hybrid modes of the waveguide.

On the other hand, the approach based on a non-self-adjoint formulation gives important results from a theoretical point of view on the completeness of the system of normal modes and, therefore, allows us to justify the partial conditions of Sveshnikov radiation. At the moment, only a combination of two approaches allows us to bring our knowledge of the vector model of the waveguide closer to the well-studied scalar one.

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Задача о нормальных модах волновода

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Аннотация. Рассмотрены различные подходы к вычислению нормальных мод закрытого волновода. Дан обзор литературы, проведено сравнение двух формулировок этой задачи. Показано, что использование самосопряжённой постановки задачи о нормальных модах волновода исключает возникновение артефактов, связанных с появлением малой мнимой добавки у собственных значений. Представлена реализация этого подхода для волновода прямоугольного сечения с прямоугольными вставками в системе компьютерной алгебры Sage и протестирована на гибридных модах слоистых волноводов. Тесты показали, что наша программа прекрасно справляется с вычислением точек дисперсионной кривой, отвечающих гибридным модам волновода.

Ключевые слова: поляризованное электромагнитное излучение, нормальные моды волновода, спектральная задача теории волновода, дисперсионная кривая волновода