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Analytic projective geometry for computer graphics

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Abstract. The motivation of this paper was the development of computer geometry course for students of mathematical specialties. The term "computer geometry" hereafter refers to the mathematical foundations of machine graphics. It is important to emphasize separately that this course should be designed for second-year students and, therefore, they can only be required to have prior knowledge of a standard course in algebra and mathematical analysis. This imposes certain restrictions on the material presented. When studying the thematic literature, it was found out that the de facto standard in modern computer graphics is the use of projective space and homogeneous coordinates. However, the authors faced a methodological problem—the almost complete lack of suitable educational literature in both Russian and English. This paper was written to present the information collected by the authors on this issue.

Key words and phrases: projective geometry, Asymptote system, Plücker coordinates, proper and improper points, lines and planes

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1. Introduction

Here are the main reasons that motivated us to write this paper.

1.1. Synthetic and analytical approaches to geometry

Historically, there have been two approaches to the presentation of geometry in mathematics, *synthetic* and *analytical*. In the synthetic presentation of geometry, sets of geometric elements of various kinds are initially introduced, such as points, lines and planes. Then the relationship between them is defined, formulated in the form of axioms that correspond to visual geometric representations. This approach is used in a simplified form when presenting Euclidean geometry in a school geometry course, therefore it is intuitively understandable to most students.

An alternative approach emerged later, with the development of algebra. Despite the fact that the name analytical geometry has been assigned to it, it would be more correct to call it *linear-algebraic*, "since linear algebra forms the basis and provides its own methods, not analysis" [1, p. 12]. The linear-algebraic approach is much more general, much more powerful, and therefore much simpler in describing complex structures than the synthetic approach. He received a particularly strong impetus for his development within the framework of the ideas formulated by Felix Klein in his Erlangen program [2].

It is quite natural for computer graphics algorithms to use a linear-algebraic approach, since it allows you to write down and use algebraic formulas to calculate the necessary quantities. Consequently, projective geometry in the framework of this subject should be presented in this style. Note that we are talking about the projective space model $\mathbb{R}P^3$.

1.2. Lack of educational literature

The literature on projective geometry in Russian is extensive, not to mention English and other foreign languages. Most textbooks, where the presentation is conducted at a level accessible to undergraduates of physics and mathematics faculties, are essentially based on a synthetic approach [3–7]. This technique is justified, since the task of these authors is to provide an understanding of the essence of projective geometry, and the analytical approach "requires more ink and less thought" [8, p. 89]. However, when presenting the basics of projective geometry in a computer geometry course, this approach is questionable because it is too far from the final software implementation.

There are also a large number of monographs where projective geometry is presented in a linear-algebraic style, for example [9–11] and a list of sources in [1]. However, the style of presentation in them is rather abstract and most of them are textbooks for undergraduates, graduate students and researchers working in the framework of theoretical mathematics.

Looking at textbooks on computer graphics, machine vision and robotics, then a different problem arises. In many textbooks, projective geometry begins and ends with an exposition of the concept of homogeneous coordinates, which are introduced exclusively in the context of projective transformations. For example, let's list the sources with the pages: [12, pp. 101, 115][13, p. 18][14, pp. 192, 220][15, p. 146] [16, p. 85] [17, p. 20] [18, p. 176] [19, p. 211] [20, p. 438] [21, p. 56]. In all these books, homogeneous coordinates are introduced ad hoc and used to represent affine transformations as a linear transformation (3×3 matrices on the plane and 4×4 matrices in space). The representation of a straight line and a plane using homogeneous coordinates is not considered, and projective geometry is not applied to standard problems of analytical geometry.

The list of textbooks in Russian concerning the mathematical foundations of computer graphics is extremely limited [12–14, 17, 19, 20, 22, 23]. None of these manuals consistently use projective geometry as a tool for solving computer geometry problems. Some information, mainly about homogeneous coordinates, is available in books [12–14, 17, 19, 24], however, they do not consistently describe how to represent straight lines and planes in a homogeneous form, and homogeneous coordinates are also used only to represent affine transformations as linear.

As an exception, the textbook should be noted [25], which adopted a non-standard approach to the presentation of analytic geometry using Grassmann algebra using non-standard notation. We will also mention several sources on the theory of screws [26–28], in which, in particular, the moment of a sliding vector is introduced, which is directly related to the representation of straight lines in Plucker coordinates (in a homogeneous form).

The situation with books in English is only slightly better. You can specify the book [29], where the presentation is not limited to the introduction of homogeneous coordinates only for points, on the contrary, homogeneous coordinates for straight lines and planes are considered, as well as solutions to some standard problems [29, pp. 25, 65]. As a disadvantage, we note that all formulas are written mainly in component rather than vector form, and we also note the focus on the field of computer vision rather than computer graphics. These are different areas, despite their proximity.

Of particular note is the extremely capacious but extremely informative book [30] by Eric Lengyel, which stands out in several ways.

- The presentation is conducted at a good mathematical level, but with an emphasis on practical application. For almost every formula, an example of its implementation is given in the form of programs in C-like pseudocode.
- Due attention is paid to the application of the principles of projective geometry in computer graphics problems. In the third chapter, the author provides an extremely useful table 3.1 with a summary of basic formulas using homogeneous coordinates for points, lines and planes.
- The fourth chapter is devoted to Grassmann algebra, which is the basis of geometric algebra. The author gives a description of projective spaces in terms of n–vectors.

This concludes the list of textbooks found by the authors, where analytical projective geometry is somehow touched upon.

1.3. Paper structure

In this article and its continuation, we attempt to eliminate this drawback and provide a detailed description of analytic projective geometry, or rather the model of the projective space $\mathbb{RP}^3$. At the same time, we focus on practical applications in the field of computer graphics and implement all proven formulas programmatically in the language Asymptote [31–33]. The article contains a large number of drawings and all of them are created programmatically using Asymptote. Writing points, lines, and planes in a projective form allows you to associate each of these geometric entities with a certain algebraic entity.

In the first article, we describe the theory in detail, based on the notation from the book [30]. To prove the formulas, we use both projective space and three-dimensional Euclidean space with a Cartesian coordinate system. For the sake of completeness, we repeat some things from classical analytic geometry, but they are also interpreted within the framework of projective space. We write down all the proven formulas in the table, which is an expanded and supplemented version of the above-mentioned Table 3.1. This table allows you to solve any problem about the relative position of points, lines and planes.

In the second article, the proven formulas will be translated into the Asymptote programming language. This language is designed to create two-dimensional and three-dimensional vector illustrations, has a C-like syntax, and allows you to set custom data structures by attaching member functions (methods) to them and overloading existing functions for these structures. This made it possible to create structures for a projective point, a straight line, and a plane and to implement standard tasks for studying their relative positions. The results are immediately visualized.

2. Projective space and homogeneous coordinates

In art, the concept of *perspective* (Latin *perspicere* - to look through) has emerged since ancient times as a technique for depicting spatial objects in accordance with the distortion of proportions and shapes of depicted bodies during their visual perception. Visual means for conveying perspective

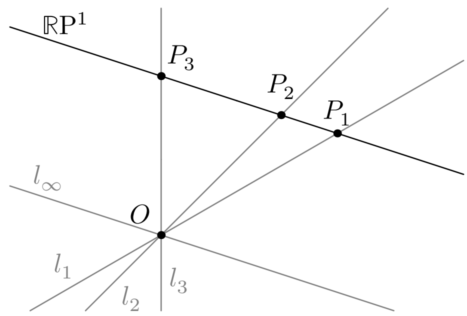

Figure 1. Real projective line $\mathbb{R}P^1$. The points on the projective line correspond to the lines of the space $\mathbb{R}^2$ passing through the origin O. An irregular (ideal) point corresponds to a parallel line l_∞

were initially passed on as practical skills in the fine arts from experienced craftsmen to apprentices. The mathematical foundations of perspective were laid by the French geometer and architect Girard Desargues (1591–1661), under the name *projective geometry*. Later, one of the main theorems of which bears his name (Desargues' theorem) was named after him. Further contributions to the development of projective geometry were made by Jean-Victor Poncelet, M. Chasles, K. G. H. von Staudt, A. F. Möbius, J. Plücker, F. H. Klein. For more information about the history of the development of projective geometry, see the popular book [34].

A projective space or, more precisely, a model of the projective space $\mathbb{R}P^n$ of dimension n over the field of real numbers $\mathbb{R}$ is a set of straight lines in an ordinary Euclidean affine space $\mathbb{R}^{n+1}$ passing through the origin point O. Three-dimensional computer graphics uses the projective space model $\mathbb{R}P^3$. This model is based on the four-dimensional Euclidean space $\mathbb{R}^4$, which greatly limits visibility due to the natural complexity of illustrating four-dimensional spaces. Therefore, for greater clarity, we will give examples of the spaces $\mathbb{R}P^1$ and $\mathbb{R}P^2$ for which illustrations can be given on a plane and in three-dimensional space, respectively.

2.1. The real projective line

A model of a projective line, that is, a space of dimension 1, can be constructed if we define a certain line $\mathbb{R}P^1$ on the plane $\mathbb{R}^2$ that does not pass through the origin. Each point P of this straight line will correspond to a straight line in the plane that passes through the origin and intersects $\mathbb{R}P^1$ at the point P, as shown in the figure 1. Such points are called *endpoints* or *proper* points. The only straight line l_∞ passing through O and parallel to $\mathbb{R}P^1$ will correspond to a point of a special kind called *improper* or *ideal point*.

Coordinates can be entered on the projective line. The most convenient way to do this is to draw a projective line $\mathbb{R}P^1$ parallel to the Ox axis, through the point $(0, w)$ as shown in the figure 2. Since each projective point is defined by a straight line passing through the origin, the components of the non-normalized guide vector of this line can be taken as the projective coordinates, for example, $P = (x, w)$. When multiplying the components of the guide vector by the same number $\lambda \neq 0$, it will still set the same straight line, so you can write $P = (x, w) = (\lambda x, \lambda w)$. Therefore, the coordinates are a pair of numbers considered up to the proportionality of x/w. To emphasize this fact, the coordinates are written using a colon as a separator — the division symbol $(x : w)$. You can choose a fixed number as w. Traditionally, $w = 1$ is taken and such coordinates are called *homogeneous* or *homogeneous*. Points with coordinates like $(x : 1)$ correspond to proper points, and points of the form $(x : 0)$ are non-proper (in the one-dimensional case, there is one such point corresponding to the line Ox).

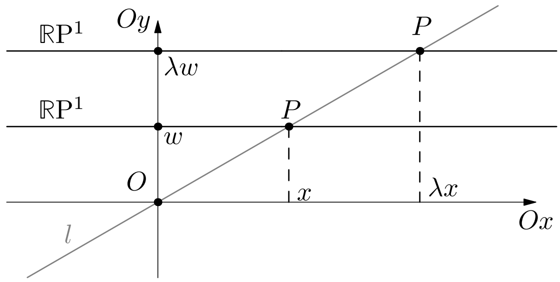

Figure 2. A homogeneous coordinate system for a one-dimensional projective space. The coordinates of the point are determined by the guide vector $\boldsymbol{v} = (x, w)^T = (\lambda x, \lambda)^T$ of the straight line l. In homogeneous coordinates, they are written as $\vec{\boldsymbol{v}} = (x : w)$

2.2. The real projective plane

The projective plane can be defined axiomatically [5, 35]. Thus, a set is called a projective plane, the elements of which are called points. It identifies subsets called straight lines and requires the following properties to be fulfilled.

1. For any two different points, there is a single line containing them.
2. Any two different lines have a single point in common.
3. There are four points, none of which lie on the same straight line.

These axioms define the projective plane in an non-constructive way, since they say nothing about how to construct a specific set of objects that satisfy these axioms — *model* of an abstract object.

Consider *real projective plane* $\mathbb{R}\mathrm{P}^2$ and introduce homogeneous coordinates on the projective plane. Such a space can be represented as a plane in a three-dimensional Euclidean affine space with a Cartesian coordinate system. The plane $\mathbb{R}\mathrm{P}^2$ passes through a point $(0, 0, 1)$, parallel to the Ox and Oy axes. As before, each endpoint of the plane corresponds to a straight line passing through the origin O and intersecting the plane at this point, as shown in the figure 3. The coordinates of all points on the plane have the form $(x, y, 1)$ and correspond to the homogeneous coordinates $(x : y : 1)$ of the point P.

Unlike a projective line, there are an infinite number of ideal points on the projective plane. Each such point corresponds to a straight line lying in the Oxy plane. These include, for example, the Ox and Oy axes themselves. The guide vectors of such lines have components of the form $(x, y, 0)$ and define the *direction* on the projective plane.

Due to this, two types of vectors are distinguished in the projective space $\mathbb{R}\mathrm{P}^2$ with component notation.

- Radius vectors or *point vectors* — vectors anchored to the origin, having homogeneous coordinates of the form $(x : y : 1)$ and defining endpoints on the plane.
- Direction vectors — loose, free vectors having homogeneous coordinates of the form $(x : y : 0)$ and defining ideal points lying at infinity.

In addition to points, straight lines are also present on the projective plane. Each straight line corresponds to a plane passing through the origin and intersecting $\mathbb{R}\mathrm{P}^2$ along a certain straight line, as shown in the figure 4. The plane Oxy parallel to $\mathbb{R}\mathrm{P}^2$ corresponds to a perfect straight line on which all ideal points lie. A model of an ideal straight line can be a circle lying infinitely far away in the plane $\mathbb{R}\mathrm{P}^2$.

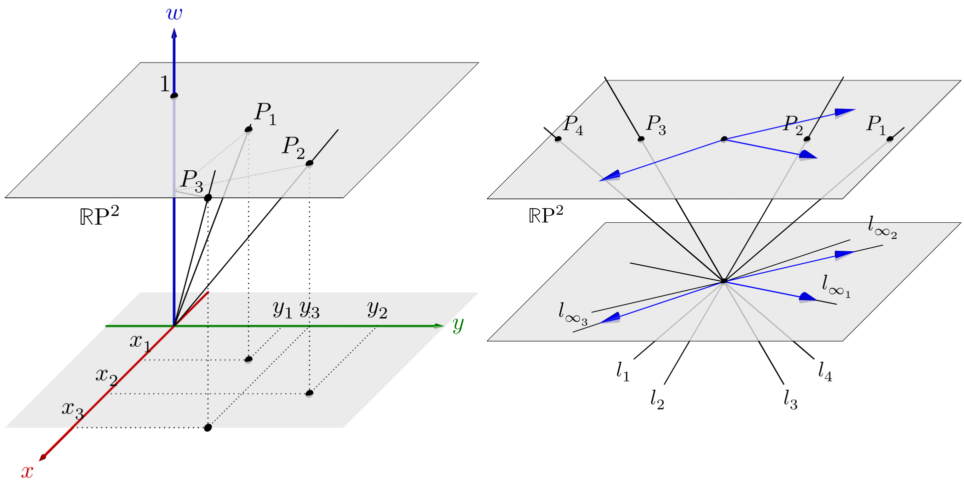

Figure 3. The model of the projective plane $\mathbb{R}P^2$. Each point in the plane corresponds to a line of $\mathbb{R}^3$ space passing through the origin. The z coordinate is traditionally denoted by the symbol w

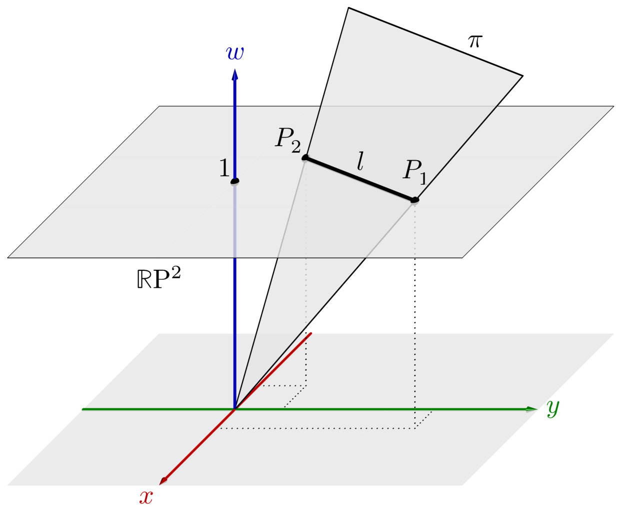

Figure 4. The model of the projective plane $\mathbb{R}P^2$. Each straight plane corresponds to a plane of space $\mathbb{R}^3$ passing through the origin. In this case, the plane is shown as a triangle

2.3. A model of three-dimensional projective space $\mathbb{R}\mathrm{P}^3$

Finally, let's consider the model of the projective space $\mathbb{R}\mathrm{P}^3$. Unfortunately, this model cannot be fully visualized, since it uses the four-dimensional space $\mathbb{R}^4$, where a Cartesian coordinate system with the axes Ox, Oy, Oz and Ow is introduced. Some intuitive understanding can be gained if we draw analogies with the projective plane, instead of which a hyperplane of dimension 3 is drawn through the point $w = 1$, which we will identify with $\mathbb{R}\mathrm{P}^3$.

- Endpoints in $\mathbb{R}\mathrm{P}^3$ correspond to lines intersecting $\mathbb{R}\mathrm{P}^3$, and ideal points correspond to lines parallel to $\mathbb{R}$.

- Finite lines correspond to planes intersecting $\mathbb{R}P^3$, and ideal lines correspond to planes parallel to $\mathbb{R}P^3$.
- The finite planes correspond to hyperplanes of dimension 3 intersecting $\mathbb{R}P^3$ along planes (of dimension 2). The ideal plane corresponds to the hyperplane $Oxyz$ parallel to $\mathbb{R}P^3$.

Some idea of a hyperplane of dimension 3 can be obtained if we imagine that an ordinary plane (of dimension 2) is intersected by some other plane that cuts off a straight line on the plane, that is, a geometric object one dimension smaller than itself. Similarly, a hyperplane of dimension 3 intersects another hyperplane of dimension 3 and cuts off the usual planes of dimension 2 on it, which in some way are located in "volume" $\mathbb{R}P^3$.

Although a full-fledged visual representation is impossible in this case, it is still possible to work with $\mathbb{R}P^3$ using algebra, in particular using homogeneous coordinates. Each point in this space can be matched in the same way as for $\mathbb{R}P^1$ and $\mathbb{R}P^2$ with homogeneous coordinates of the form $(x : y : z : 1)$ in the case of endpoints and the form $(x : y : z : 0)$ for ideal points. Unlike the projective plane, $\mathbb{R}P^3$ contains planes and infinitely many ideal straight lines, all of which lie on an infinitely distant ideal plane, which can be conventionally modeled as a sphere with infinite radius.

To distinguish the vectors of the projective space $\mathbb{R}P^3$ from the vectors of the Euclidean affine space $\mathbb{R}^3$, we will use bold font with the addition of an arrow icon at the top, for example:

$$\vec{\boldsymbol{p}} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = (\boldsymbol{p} \mid 1) = (x : y : z : 1), \quad \vec{\boldsymbol{q}} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = (\boldsymbol{q} \mid w) = (x : y : z : w).$$

A point in $\mathbb{R}P^3$ is considered normalized if $w = 1$. Geometrically, this means that it lies in a hyperplane drawn through the fourth axis Ow.

The introduction of homogeneous coordinates into the projective space model makes it possible to represent not only points, but also straight lines and planes in linear algebraic form. In other words — construct *projective analytic geometry*. The linear-algebraic representation of lines and planes with the introduced homogeneous coordinates will be called *homogeneous representation* of lines and planes. For the case of a straight line, the terms *Plucker representation* of a straight line or *Plucker coordinates* of a straight line are also used. Both of these terms in this article will be synonymous with the homogeneous representation of a straight line. For the case of a plane, the term Plucker coordinates is not usually used.

3. A line on plane

Before proceeding to the representation of a line in a projective form, let us list the main ways of algebraic representation of a line on an ordinary Euclidean plane. These methods are studied in standard analytical geometry courses, so we will focus only on the main points that are important for the topic of the article. For additional information, we refer the reader, for example, to [36, Chapter 5, §1] and [37].

Consider a line l on a plane. Let's use the letter P to denote an arbitrary point on a line with the radius vector $\boldsymbol{p} = (x, y)^T$. Let us know some fixed point P_0 with a radius vector $\boldsymbol{p}_0 = (x_0, y_0)^T$, and the guiding vector $\boldsymbol{v}$ is also specified. You can write *parametric equation* in a line l:

$$\boldsymbol{r}(t) = \begin{bmatrix} x \\ y \end{bmatrix} = \boldsymbol{p}_0 + \boldsymbol{v}t = \begin{bmatrix} x_0 + v_x t \\ y_0 + v_y t \end{bmatrix}.$$

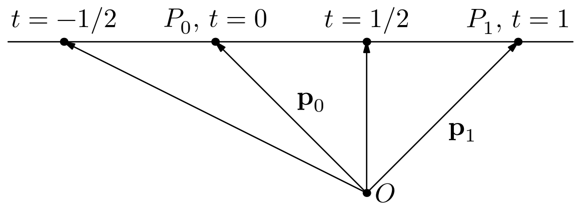

Figure 5. Illustration of the linear interpolation process. By setting the segment P_1P_2, you can move between points P_1 and P_2 by changing the parameter from 0 to 1. With parameter values outside the segment $[0, 1]$, the point will slide along a line (P_1P_2) going beyond the segment

Where $\boldsymbol{r}(t)$ is the radius vector of a line point, t is a parameter running through all real numbers. The guiding vector $\boldsymbol{v}$ can be geometrically interpreted as a tangent vector to a line that lies on l, since the tangent to the l line coincides with the line itself.

The guide vector can be calculated if any two points of a line are known, for example P_0 and P_1, then $\boldsymbol{v} = \boldsymbol{p}_1 - \boldsymbol{p}_0$. The order of the radius vectors in the difference affects the direction of movement of the point. Usually, a guiding vector is chosen so that as t increases, the point moves from left to right, and as it decreases from right to left.

Substitute $\boldsymbol{v} = \boldsymbol{p}_1 - \boldsymbol{p}_0$ into the parametric equation and rearrange the terms:

$$\boldsymbol{r}(t) = \boldsymbol{p}_0 + (\boldsymbol{p}_1 - \boldsymbol{p}_0)t = \boldsymbol{p}_0(1 - t) + \boldsymbol{p}_1 t.$$

In this form, the equation of a line defines the segment between the points P_0 and P_1, if $0 \leqslant t \leqslant 1$, as shown in the figure 5. The term *lerp* is well-established in computer geometry from the phrase linear interpolation (**l**inear int**erp**olation). This is the simplest version of interpolation, but it is very widely used in animation, curve drawing, etc.

The guide vector $\boldsymbol{v}$ can have an arbitrary length, since it is enough to specify any two points of a line P_1 and P_2 to calculate it, however, it is convenient to use a single guide vector $\|\boldsymbol{v}\| = 1$ for calculations, which can always be obtained by reparametrization:

$$\boldsymbol{r}(s) = \boldsymbol{p}_0 + \boldsymbol{v}\frac{t}{\|\boldsymbol{v}\|} = \boldsymbol{p}_0 + \boldsymbol{v}s, \quad s = \frac{t}{\sqrt{v_x^2 + v_y^2}}.$$

The parameter s is called *natural parameter* and is interpreted as the length of a line measured from some fixed point, in this case from P_0, since for $s = 0$ we get $r(0) = \boldsymbol{p}_0$. Sometimes the letter l is used to denote a natural parameter.

Directly from the parametric equation, one can obtain the *canonical equation* of a line:

$$\left.\begin{aligned} t &= \frac{x - x_0}{v_x} \\ t &= \frac{y - y_0}{v_y} \end{aligned}\right\} \Rightarrow \frac{x - x_0}{v_x} = \frac{y - y_0}{v_y}.$$

If we put $\boldsymbol{v} = (x_1 - x_0, y_1 - y_0)T$, then we can write the equation of a line passing through two points P_0 and P_1:

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0}.$$

Now consider a normal drawn to a line at some point and having a guiding vector $\boldsymbol{N} = (A, B)$. By definition, a normal is perpendicular to a line, hence $\boldsymbol{N} \perp \boldsymbol{v} = \boldsymbol{p} - \boldsymbol{p}_0$:

$$(\boldsymbol{N}, \boldsymbol{p} - \boldsymbol{p}_0) = 0 \Leftrightarrow (\boldsymbol{N}, \boldsymbol{p}) - (\boldsymbol{N}, \boldsymbol{p}_0) = 0 \Leftrightarrow Ax + By - Ax_0 - By_0 = 0.$$

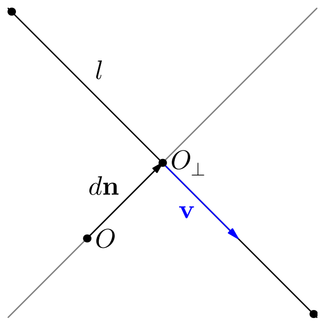

Figure 6. The geometric meaning of the normal equation of a line on a plane. The figure shows a line l, its guiding vector $\boldsymbol{v}$, the normal vector $d\boldsymbol{n}$, where d is the directional distance from the origin to the line or, in other words, the length of the perpendicular $\boldsymbol{OO}_{\perp}$

Denoting $C = -Ax_0 - By_0$, we write *general equation* of a line on a plane:

$$Ax + By + C = 0.$$

It can also be rewritten in a different form by dividing all terms by $-C$ and writing:

$$-\frac{A}{C}x - \frac{B}{C}y - 1 = 0 \Leftrightarrow \frac{x}{-C/A} - \frac{y}{-C/B} = 1 \Rightarrow \frac{x}{a} - \frac{y}{b} = 1, \quad a = -C/A, \quad b = -C/B.$$

This form of the equation is called the *equation in segments*.

In the equation $Ax + By + C = 0$, we divide all terms by the norm of the guide vector of the normal $\|\boldsymbol{N}\| = \sqrt{A^2 + B^2}$ and write:

$$\frac{A}{\sqrt{A^2+B^2}}x + \frac{B}{\sqrt{A^2+B^2}}y + \frac{C}{\sqrt{A^2+B^2}} = 0, \quad \boldsymbol{n} = \left(\frac{A}{\sqrt{A^2+B^2}}, \frac{B}{\sqrt{A^2+B^2}}\right) = (n_x, n_y).$$

We have introduced the notation $\boldsymbol{n}$ for the unit direction vector of the normal (ort of the normal) to the line. In addition, since

$$\frac{C}{\sqrt{A^2+B^2}} = \frac{-Ax_0 - By_0}{\sqrt{A^2+B^2}} = -\frac{A}{\sqrt{A^2+B^2}}x_0 - \frac{B}{\sqrt{A^2+B^2}}y_0 = -(\boldsymbol{n}, \boldsymbol{p}_0) = d.$$

as a result, the general equation of the line is written as

$$n_x x + n_y y + d = 0.$$

In this form, the equation of a line is called *normalized equation*. The coefficient d has the geometric meaning of the distance from the line to the origin. In general, the distance from a point to a line is defined as the length of the perpendicular lowered from the point to the line. The direction of the perpendicular coincides with the direction of the normal vector, since the normal is also perpendicular to the line.

The components of the unit normal vector can be written using trigonometric functions. Since $\|\boldsymbol{n}\|^2 = n_x^2 + n_y^2 = 1$, there will always be a θ such that $n_x = \cos\theta$, $n_y = \sin\theta$, where $\theta = \measuredangle(\boldsymbol{e}_x, \boldsymbol{n})$ — the value of the angle between the ort $\boldsymbol{e}_x$ of the Ox axis and the vector $\boldsymbol{n}$. For an unnormalized vector $\boldsymbol{N}$, you can also write $\boldsymbol{N} = \|\boldsymbol{N}\|(\cos\theta, \sin\theta)$ or else $A = \|\boldsymbol{N}\|\cos\theta$, $B = \|\boldsymbol{N}\|\sin\theta$.

It should be emphasized here that the vector $\boldsymbol{n}$ is a free vector that defines the direction of the normal. When visualizing $\boldsymbol{n}$, it can be moved both along the normal and the normal itself can be moved along a line. The specific image method depends on what exactly we want to illustrate. For example, in the figure 6, the vector $d\boldsymbol{n}$ is shifted away from the origin, since it indicates the point of the plane closest to the origin.

It is interesting to note that the equation $Ax + By + D = 0$ is already a projective equation of a line on the plane $\mathbb{R}\mathrm{P}^2$, since within the framework of this model, lines on the projective plane are three-dimensional planes passing through the origin, which are given by the equation $Ax + By + Dw = 0$. If we put $w = 1$, we get just the general equation of the line. For a line on the projective plane, the term Plucker coordinates is not used, but their analog is the numbers A, B, C or in normalized form n_x, n_y, d.

4. A line in space

4.1. The parametric equation

Here we briefly recall how the *parametric equation* of a line is written in Euclidean space, and immediately proceed to write the Plucker coordinates of a line, that is, the linear algebraic representation of a line in a projective space. For information about the canonical equation of a line, we refer the reader to [36, Ch. 5, §4].

Let there also be some point P_0 lying on the line l and the guiding vector of the line $\boldsymbol{v}$, then the parametric equation will be written as

$$\boldsymbol{l}(t) = \boldsymbol{p}_0 + \boldsymbol{v}t = \begin{cases} x_0 + v_x t, \\ y_0 + v_y t, \\ z_0 + v_z t. \end{cases}$$

Where $\boldsymbol{l}(t)$ is the radius vector of an arbitrary point P belonging to a line. By writing the guiding vector through the radius of the vectors of the two points of the line, we obtain *canonical equation*:

$$\boldsymbol{p} - \boldsymbol{p}_0 = (\boldsymbol{p}_1 - \boldsymbol{p}_0)t \Rightarrow \frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} = \frac{z - z_0}{z_1 - z_0}.$$

Just as for the plane case, a formula can be derived from the parametric equation for linear interpolation of a segment lying between points with radius vectors $\boldsymbol{p}_0$ and $\boldsymbol{p}_1$:

$$\boldsymbol{l}(t) = \boldsymbol{p}_0(1 - t) + \boldsymbol{p}_1 t.$$

The question arises, is it possible to write down the equivalent of the normal equation of a line on a plane for the three-dimensional case? Even from visual geometric considerations, it becomes clear that it is not possible to do this, since the distance from the origin and the guiding vector of the normal do not uniquely define a line in space, unlike in the flat case.

4.2. The moment and coordinates of the Plucker line in space

Consider a line l, the points of which are set parametrically using the radius vector $\boldsymbol{l}(t) = \boldsymbol{p} + \boldsymbol{v}t$, where $\boldsymbol{p}$ is the radius vector of some point P belonging to the line l, and $\boldsymbol{v}$ is the guiding vector of the line. Let's introduce the vector $\boldsymbol{m}$, which we define as

$$\boldsymbol{m} = \boldsymbol{p} \times \boldsymbol{v}$$

and let's call *the moment of the line l*.

It can be shown that the moment $\boldsymbol{m}$ does not depend on the choice of a point on a line. To do this, find the vector product $\boldsymbol{l}(t) \times \boldsymbol{v}$:

$$\boldsymbol{l}(t) \times \boldsymbol{v} = (\boldsymbol{p} + \boldsymbol{v}t) \times \boldsymbol{v} = \boldsymbol{p} \times \boldsymbol{v} + \boldsymbol{v} \times \boldsymbol{v}t = \boldsymbol{p} \times \boldsymbol{v}.$$

Therefore, the moment $\boldsymbol{m}$ characterizes a line and does not depend on the choice of a point on it, but depends on the guiding vector $\boldsymbol{v}$. In this case, the vector $\boldsymbol{v}$ is *sliding*, that is, its origin is not tied to a single point and can move along the line l in any direction, including in the opposite direction. To emphasize the importance of the vector $\boldsymbol{v}$, the following terminology is used: the point P is called the *application point* of the vector $\boldsymbol{v}$, and the line itself l is called the *line vector* $\boldsymbol{v}$. We also note that the moment $\boldsymbol{m}$ is the main object in the theory of screws [26, 28].

The line is completely characterized by its guiding vector $\boldsymbol{v}$ and the moment $\boldsymbol{m}$. These two vectors allow you to define a line without reference to a specific point in space, as is done in the two-dimensional case using the normal equation of a line. The six parameters $\{v_x, v_y, v_z \mid m_x, m_y, m_z\}$ are called *Plucker coordinates* of a line in honor of Julius Plucker (1801-1868) [38, Ch. 3, §3] [29, 30]. The Plucker coordinates of a line are a linear algebraic representation of a line within a system of homogeneous coordinates, and just like homogeneous coordinates, they are defined up to a common factor. As a result, we get the opportunity to define any line in a homogeneous form.

$$\{\boldsymbol{v} \mid \boldsymbol{m}\} = \{v_x, v_y, v_z \mid m_x, m_y, m_z\} = \{\boldsymbol{v} \mid \boldsymbol{p} \times \boldsymbol{v}\}. \tag{1}$$

The notation for writing Plucker coordinates in the form of curly brackets is used in the book [30], and we will follow them in this tutorial as well.

The moment of a line can also be calculated using two points $P_1, P_2 \in l$ with radius vectors $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$. To do this, note that $\boldsymbol{p}_2 - \boldsymbol{p}_1$ is the guiding vector of a line, hence

$$\boldsymbol{m} = \boldsymbol{p}_1 \times (\boldsymbol{p}_2 - \boldsymbol{p}_1) = \boldsymbol{p}_1 \times \boldsymbol{p}_2 - \boldsymbol{p}_1 \times \boldsymbol{p}_1 = \boldsymbol{p}_1 \times \boldsymbol{p}_2 \Longrightarrow$$

$$\boldsymbol{m} = \boldsymbol{p}_1 \times \boldsymbol{p}_2 \tag{2}$$

This entry corresponds to the formula (B) of the table 1.

Emphasis should be placed on the fact that the moment $\boldsymbol{m}$ and the guiding vector $\boldsymbol{v}$ are mutually orthogonal, which follows directly from our definition of the moment as the vector product of $\boldsymbol{p} \times \boldsymbol{v}$. Ratio

$$(\boldsymbol{v}, \boldsymbol{m}) = v_x m_x + v_y m_y + v_z m_z = 0,$$

It is called *Plucker equations* or *Plucker condition*. It imposes a restriction on the parameters $v_x, v_y, v_z, m_x, m_y, m_z$ so that not every six numbers can set a line, but only those for which the specified equality holds. The condition itself is a second-order algebraic expression, which means that four parameters are sufficient to define a line.

If the coordinates of the two points through which the line l passes are given in a homogeneous form, that is, P_1 as $(\boldsymbol{p}_1 \mid w_1)$ and P_2 as $(\boldsymbol{p}_2 \mid w_2)$, then normalize the coordinates of the point and calculate the guide vector, which is written as $\boldsymbol{v} = \boldsymbol{p}_2/w_2 - \boldsymbol{p}_1/w_1$, $w_1 w_2 \boldsymbol{v} = w_1 \boldsymbol{p}_2 - w_2 \boldsymbol{p}_1$. The moment of the line is calculated as follows:

$$\frac{\boldsymbol{p}_1}{w_1} \times w_1 w_2 \boldsymbol{v} = \frac{\boldsymbol{p}_1}{w_1} \times (w_1 \boldsymbol{p}_2 - w_2 \boldsymbol{p}_1) = \boldsymbol{p}_1 \times \boldsymbol{p}_2 - \frac{w_2}{w_1} \boldsymbol{p}_1 \times \boldsymbol{p}_1 = \boldsymbol{p}_1 \times \boldsymbol{p}_2 = \boldsymbol{m}.$$

Therefore, in a homogeneous form, the line can be written as

$$\{w_1 \boldsymbol{p}_2 - w_2 \boldsymbol{p}_1 \mid \boldsymbol{p}_1 \times \boldsymbol{p}_2\}, \tag{3}$$

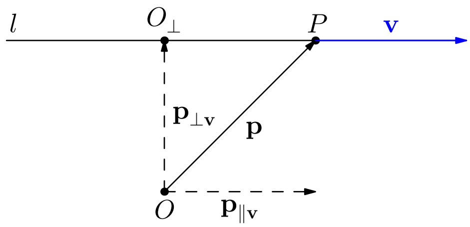

Figure 7. The distance from the origin O to the line l

which corresponds to formula (D) from table 1.

This formula allows you to calculate the Plucker coordinates of a line passing through non-matching points $(\boldsymbol{u}_1 \mid 0)$ and $(\boldsymbol{u}_2 \mid 0)$:

$$\{\mathbf{0} \mid \boldsymbol{u}_1 \times \boldsymbol{u}_2\}.$$

Such a line is called *improper line* or *ideal line*. In the visual arts, the horizon line corresponds to such line. The guiding vector of an improper line, therefore, is equal to the zero vector $\mathbf{0}$, and the moment can be nonzero.

4.3. Distance from a point to a line

Consider the problem of finding the distance from the origin point O to the line l, for which we must find the radius vector $\boldsymbol{OO}_\perp$, where the point $O_\perp$ is the base of the perpendicular omitted from O onto the line l as shown in the figure 7. The vector $\boldsymbol{OO}_\perp$ is found as the perpendicular part of the radius of the vector $\boldsymbol{p}$ relative to the guiding vector $\boldsymbol{v}$ of the line. Note also that the choice of the point of the line P is arbitrary. Using the formula $\boldsymbol{a} \times \boldsymbol{b} \times \boldsymbol{c} = \boldsymbol{b}(\boldsymbol{a}, \boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a}, \boldsymbol{b})$ for a triple vector product, write an expression for $\boldsymbol{OO}_\perp$

$$\boldsymbol{OO}_\perp = \boldsymbol{p}_{\perp \boldsymbol{v}} = \boldsymbol{p} - \boldsymbol{p}_{\parallel \boldsymbol{v}} = \boldsymbol{p} - \frac{(\boldsymbol{p}, \boldsymbol{v})}{\|\boldsymbol{v}\|^2}\boldsymbol{v} = \frac{\boldsymbol{p}\|\boldsymbol{v}\|^2 - (\boldsymbol{p}, \boldsymbol{v})\boldsymbol{v}}{\|\boldsymbol{v}\|^2} = \frac{\boldsymbol{v} \times \boldsymbol{p} \times \boldsymbol{v}}{\|\boldsymbol{v}\|^2}.$$

Substitute $\boldsymbol{p} \times \boldsymbol{v} = \boldsymbol{m}$ and write:

$$\boldsymbol{OO}_\perp = \frac{\boldsymbol{v} \times \boldsymbol{p} \times \boldsymbol{v}}{\|\boldsymbol{v}\|^2} = \frac{\boldsymbol{v} \times \boldsymbol{m}}{\|\boldsymbol{v}\|^2}.$$

You can write the coordinates of the point $O_\perp$ in a homogeneous form. Since the radius vector is known, the homogeneous coordinates will look like:

$$\left(\frac{\boldsymbol{v} \times \boldsymbol{m}}{\|\boldsymbol{v}\|^2} \;\Big|\; 1\right) = (\boldsymbol{v} \times \boldsymbol{m} \mid (\boldsymbol{v}, \boldsymbol{v})), \tag{4}$$

which proves the formula (G) from the table 1.

Considering that $\boldsymbol{v} \perp \boldsymbol{m}$ by virtue of the definition of the vector product, we calculate the length of the vector $\boldsymbol{OO}_\perp$

$$d_{OO_\perp} = \|\boldsymbol{OO}_\perp\| = \frac{\|\boldsymbol{v} \times \boldsymbol{m}\|}{\|\boldsymbol{v}\|} = \frac{\|\boldsymbol{v}\|\|\boldsymbol{m}\| \sin\frac{\pi}{2}}{\|\boldsymbol{v}\|^2} = \frac{\|\boldsymbol{m}\|}{\|\boldsymbol{v}\|}. \tag{5}$$

Note that if a straight line passes through the origin, then the distance $d_{OO_\perp} = 0$ and $\|\boldsymbol{m}\|/\|\boldsymbol{v}\| = 0$, therefore $\boldsymbol{m} = 0$. In Plucker coordinates, such a straight line is written as

$$\{\boldsymbol{v} \mid 0\}. \tag{6}$$

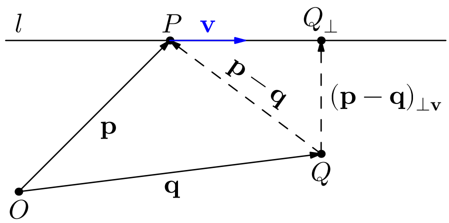

Figure 8. The distance from an arbitrary point Q to a straight line l. Vector $\boldsymbol{u} = \boldsymbol{p} - \boldsymbol{q}$

Indeed, a straight line passing through O is completely determined by its guiding vector, since in parametric form the radius vector of its points is written as $\boldsymbol{l}(t) = \boldsymbol{0} + \boldsymbol{v}t = \boldsymbol{v}t$.

Now let's complicate the problem and calculate the distance from an arbitrary point $Q \notin l$ to the straight line l. To do this, we find the length of the segment $QQ_\perp$, where $Q_\perp$ is the foot of the perpendicular dropped from the point Q onto the straight line l as this is shown in the figure 8.

We also use the radius vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ of the points P and Q, respectively, as well as the vector $\boldsymbol{u} = \boldsymbol{p} - \boldsymbol{q}$. The perpendicular vector $\boldsymbol{QQ}_\perp$ is calculated as the perpendicular part of the vector $\boldsymbol{u}$ relative to the guiding vector of the straight line $\boldsymbol{v}$. You can enter the moment of a straight line relative to the point Q as follows:

$$\boldsymbol{m}_Q = (\boldsymbol{p} - \boldsymbol{q}) \times \boldsymbol{v} = \boldsymbol{p} \times \boldsymbol{v} - \boldsymbol{q} \times \boldsymbol{v} = \boldsymbol{m}_O - \boldsymbol{q} \times \boldsymbol{v},$$

where $\boldsymbol{m}_O$ is the moment of a straight line relative to the origin. Note that

$$(\boldsymbol{m}_Q, \boldsymbol{v}) = (\boldsymbol{m}_O, \boldsymbol{v}) - (\boldsymbol{q} \times \boldsymbol{v}, \boldsymbol{v}) = 0, \text{ так как } \boldsymbol{m}_O \perp \boldsymbol{v} \text{ и } \boldsymbol{v} \perp \boldsymbol{q} \times \boldsymbol{v},$$

therefore, the moment $\boldsymbol{m}_Q$ is orthogonal to the vector $\boldsymbol{v}$.

Using the moment $\boldsymbol{m}_Q$, we can calculate the vector $\boldsymbol{QQ}_\perp$

$$\boldsymbol{QQ}_\perp = (\boldsymbol{p} - \boldsymbol{q})_{\perp\boldsymbol{v}} = (\boldsymbol{p} - \boldsymbol{q}) - \frac{(\boldsymbol{p} - \boldsymbol{q}, \boldsymbol{v})}{\|\boldsymbol{v}\|^2}\boldsymbol{v} = \frac{\boldsymbol{v} \times (\boldsymbol{p} - \boldsymbol{q}) \times \boldsymbol{v}}{\|\boldsymbol{v}\|^2} = \frac{\boldsymbol{v} \times \boldsymbol{m}_Q}{\|\boldsymbol{v}\|^2}.$$

To find the length of the vector $\boldsymbol{QQ}_\perp$, we write it as follows

$$\boldsymbol{QQ}_\perp = (\boldsymbol{p} - \boldsymbol{q}) - (\boldsymbol{p} - \boldsymbol{q})_{\|\boldsymbol{v}} = \boldsymbol{u} - \boldsymbol{u}_{\|\boldsymbol{v}}$$

and find the square of the norm $\|\boldsymbol{p} - \boldsymbol{q}\|$

$$\|\boldsymbol{p} - \boldsymbol{q}\|^2 = (\boldsymbol{u} - \boldsymbol{u}_{\|\boldsymbol{v}}, \boldsymbol{u} - \boldsymbol{u}_{\|\boldsymbol{v}}) = (\boldsymbol{u}\boldsymbol{u}) - (\boldsymbol{u}, \boldsymbol{u}_{\|\boldsymbol{v}}) - (\boldsymbol{u}_{\|\boldsymbol{v}}, \boldsymbol{u}) + (\boldsymbol{u}_{\|\boldsymbol{v}}, \boldsymbol{u}_{\|\boldsymbol{v}}) = \|\boldsymbol{u}\|^2 + \left\|\boldsymbol{u}_{\|\boldsymbol{v}}\right\|^2 - 2(\boldsymbol{u}, \boldsymbol{u}_{\|\boldsymbol{v}})$$

Since $\boldsymbol{u}_{\|\boldsymbol{v}} = \dfrac{(\boldsymbol{u}, \boldsymbol{v})\boldsymbol{v}}{\|\boldsymbol{v}\|^2}$, then

$$(\boldsymbol{u}, \boldsymbol{u}_{\|\boldsymbol{v}}) = \frac{(\boldsymbol{u}, \boldsymbol{v})}{\|\boldsymbol{v}\|^2}(\boldsymbol{u}, \boldsymbol{v}) = \frac{(\boldsymbol{u}, \boldsymbol{v})^2}{\|\boldsymbol{v}\|^2},$$

$$\left\|\boldsymbol{u}_{\|\boldsymbol{v}}\right\|^2 = \left(\frac{(\boldsymbol{u}, \boldsymbol{v})}{\|\boldsymbol{v}\|^2}\boldsymbol{v}, \frac{(\boldsymbol{u}, \boldsymbol{v})}{\|\boldsymbol{v}\|^2}\boldsymbol{v}\right) = \frac{(\boldsymbol{u}, \boldsymbol{v})^2}{\|\boldsymbol{v}\|^4}(\boldsymbol{v}, \boldsymbol{v}) = \frac{(\boldsymbol{u}, \boldsymbol{v})^2}{\|\boldsymbol{v}\|^4}\|\boldsymbol{v}\|^2 = \frac{(\boldsymbol{u}, \boldsymbol{v})}{\|\boldsymbol{v}\|^2}.$$

Let now substitute this formula into the expression for $\|\boldsymbol{u}\|^2$ to get:

$$\|\boldsymbol{u}\|^2 = \|\boldsymbol{u}\|^2 + \frac{(\boldsymbol{u}, \boldsymbol{v})}{\|\boldsymbol{v}\|^2} - 2\frac{(\boldsymbol{u}, \boldsymbol{v})^2}{\|\boldsymbol{v}\|^2} = \|\boldsymbol{u}\|^2 - \frac{(\boldsymbol{u}, \boldsymbol{v})^2}{\|\boldsymbol{v}\|^2} = \frac{\|\boldsymbol{u}\|^2\|\boldsymbol{v}\|^2 - (\boldsymbol{u}, \boldsymbol{v})^2}{\|\boldsymbol{v}\|^2} = \frac{\|\boldsymbol{u} \times \boldsymbol{v}\|^2}{\|\boldsymbol{v}\|^2}.$$

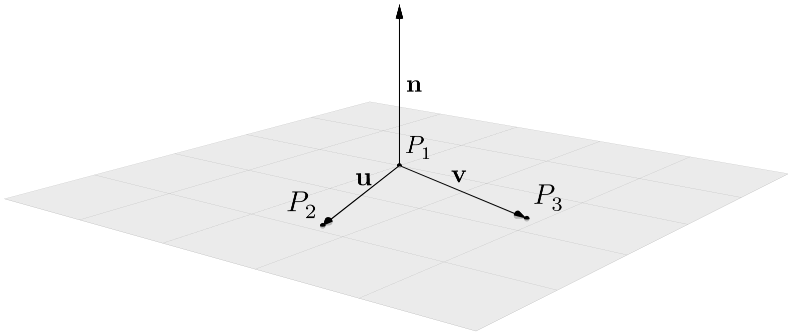

Figure 9. The plane π and its guide vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. The normal vector $\boldsymbol{n}$ is perpendicular to the guide vectors

In the last step, we again used the Lagrange identity of the vector product. Since $\boldsymbol{u}\times\boldsymbol{v} = (\boldsymbol{p}-\boldsymbol{q})\times\boldsymbol{v} = \boldsymbol{p}\times\boldsymbol{v} - \boldsymbol{q}\times\boldsymbol{v} = \boldsymbol{m} - \boldsymbol{q}\times\boldsymbol{v} = \boldsymbol{m} + \boldsymbol{v}\times\boldsymbol{q}$, then we got the formula for calculating the distance from an arbitrary point Q to a straight line $\{\boldsymbol{v} \mid \boldsymbol{m}\}$ (formula (P) in the table 1):

$$d_{QQ_\perp} = \frac{\|\boldsymbol{m} + \boldsymbol{v}\times\boldsymbol{q}\|}{\|\boldsymbol{v}\|} = \frac{\|(\boldsymbol{p}-\boldsymbol{q})\times\boldsymbol{v}\|}{\|\boldsymbol{v}\|}. \tag{7}$$

5. Plane equation

5.1. Parametric and general equations of the plane

Consider here *the parametric equation of the plane*. For information about other classical forms of the plane equation, we refer the reader to [36, Ch. 5, §3]. The parametric equation is given as follows:

$$\boldsymbol{r}(s,t) = \boldsymbol{p}_1 + (\boldsymbol{p}_2 - \boldsymbol{p}_1)s + (\boldsymbol{p}_3 - \boldsymbol{p}_1)t,$$

where $\boldsymbol{p}_1$, $\boldsymbol{p}_2$ and $\boldsymbol{p}_3$ are the radius vectors of three points, s and t are some real numbers that are parameters. Unlike a line, a plane requires two parameters, since it is a two-dimensional object.

The differences $\boldsymbol{u} = \boldsymbol{p}_2 - \boldsymbol{p}_1$ and $\boldsymbol{v} = \boldsymbol{p}_3 - \boldsymbol{p}_1$ can be interpreted as tangent vectors to the plane, since

$$\boldsymbol{r}_s = \frac{\partial \boldsymbol{r}}{\partial s} = \boldsymbol{p}_2 - \boldsymbol{p}_1, \quad \boldsymbol{r}_t = \frac{\partial \boldsymbol{r}}{\partial t} = \boldsymbol{p}_3 - \boldsymbol{p}_1.$$

However, the tangent plane coincides with the π plane itself, which is why these tangent vectors lie completely in π and are the *guide vectors* of the plane.

Normal to a plane is a line perpendicular to any line lying completely on the plane and, therefore, perpendicular to the plane itself. The guiding vector of the normal is denoted as $\boldsymbol{N}$ and is called *the normal vector of the plane*. Its normalized version is more often used $\boldsymbol{n} = \boldsymbol{N}/\|\boldsymbol{N}\|$— the unit normal vector of the plane.

The plane π can be defined using the vector $\boldsymbol{N}$ and one fixed point belonging to the plane P_0 with the radius vector $\boldsymbol{p}_0 = (x_0, y_0, z_0)^T$. Let $\boldsymbol{p} = (x, y, z)^T$ be the radius vector of an arbitrary point in the plane, then $\boldsymbol{p} - \boldsymbol{p}_0$ also belongs to the plane, since the beginning and end of the vector lie in π. It follows from the perpendicularity of the normal vector to the plane that

$$(\boldsymbol{N}, \boldsymbol{p} - \boldsymbol{p}_0) = 0 \Rightarrow (\boldsymbol{N}, \boldsymbol{p}) - (\boldsymbol{N}, \boldsymbol{p}_0) = 0.$$

Table 1

Formulas in terms of homogeneous coordinates [30, Table 3.1]. The duality relation holds between formulas D and E, F and L, G and M, H and N. To simplify the search for the formula output, the first column contains the number under which it appears in the main text

		Formula	**Description**
A	(1)	$\{\boldsymbol{v} \mid \boldsymbol{p} \times \boldsymbol{v}\}$	A line passing through the point P in the direction of $\boldsymbol{v}$.
B	(2)	$\{\boldsymbol{p}_2 - \boldsymbol{p}_1 \mid \boldsymbol{p}_1 \times \boldsymbol{p}_2\}$	A line passing through two points P_1 and P_2.
C	(6)	$\{\boldsymbol{p} \mid \boldsymbol{0}\}$	A line passing through the origin and the point P.
D	(3)	$\{w_1\boldsymbol{p}_2 - w_2\boldsymbol{p}_1 \mid \boldsymbol{p}_1 \times \boldsymbol{p}_2\}$	A line passing through two points $\vec{\boldsymbol{p}}_1 = (\boldsymbol{p}_1 \mid w_1)$ and $\vec{\boldsymbol{p}}_2 = (\boldsymbol{p}_2 \mid w_2)$.
E	(12)	$\{\boldsymbol{n}_1 \times \boldsymbol{n}_2 \mid d_1\boldsymbol{n}_2 - d_2\boldsymbol{n}_1\}$	A line of intersection of two layers $[\boldsymbol{n}_1 \mid_1]$ and $[\boldsymbol{n}_2 \mid_2]$.
F	(13)	$(\boldsymbol{m} \times \boldsymbol{n} + d\boldsymbol{v} \mid -(\boldsymbol{n}, \boldsymbol{v}))$	The point of intersection of the plane $[\boldsymbol{n} \mid]$ and the line $\{\boldsymbol{v} \mid \boldsymbol{m}\}$.
F.a	(21)	$[\boldsymbol{m}_1 \times \boldsymbol{m}_2 \mid (\boldsymbol{v}_2, \boldsymbol{m}_1)]$	The point of intersection of two lines $\{\boldsymbol{v}_1 \mid \boldsymbol{m}_1\}$ и $\{\boldsymbol{v}_2 \mid \boldsymbol{m}_2\}$
F.b	(11)	$\big(d_1\boldsymbol{n}_3 \times \boldsymbol{n}_2 + d_2\boldsymbol{n}_1 \times \boldsymbol{n}_3 + d_3\boldsymbol{n}_2 \times \boldsymbol{n}_1 \mid (\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3)\big)$	The point of intersection of three planes $[\boldsymbol{n}_1 \mid d_1]$, $[\boldsymbol{n}_2 \mid d_2]$ и $[\boldsymbol{n}_3 \mid d_3]$
G	(4)	$(\boldsymbol{v} \times \boldsymbol{m} \mid (\boldsymbol{v}, \boldsymbol{v}))$	The point closest to the origin on the line $\{\boldsymbol{v} \mid \boldsymbol{m}\}$.
H	(14)	$(-d\boldsymbol{n} \mid (\boldsymbol{n}, \boldsymbol{n}))$	The point closest to the origin on the plane $[\boldsymbol{n} \mid d]$.
I	(16)	$[\boldsymbol{v} \times \boldsymbol{u} \mid -(\boldsymbol{u}, \boldsymbol{m})]$	A plane containing a line $\{\boldsymbol{v} \mid \boldsymbol{m}\}$ and a direction $\boldsymbol{u}$.
J	(17)	$[\boldsymbol{v} \times \boldsymbol{p} + \boldsymbol{m} \mid -(\boldsymbol{p}, \boldsymbol{m})]$	A plane containing a line $\{\boldsymbol{v} \mid \boldsymbol{m}\}$ and a point $(\boldsymbol{p} \mid 1)$.
K	(18)	$[\boldsymbol{m} \mid 0]$	A plane containing a line $\{\boldsymbol{v} \mid \boldsymbol{m}\}$ and the origin.
L	(15)	$[\boldsymbol{v} \times \boldsymbol{p} + w\boldsymbol{m} \mid -(\boldsymbol{p}, \boldsymbol{m})]$	A plane containing a line $\{\boldsymbol{v} \mid \boldsymbol{m}\}$ and a point $(\boldsymbol{p} \mid w)$.
L.a	(19)	$[\boldsymbol{v} \times \boldsymbol{u} \mid (\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{p})]$	The plane containing the point $(\boldsymbol{p} \mid 1)$ and the directions $\boldsymbol{v}$ and $\boldsymbol{u}$.
M		$[\boldsymbol{m} \times \boldsymbol{v} \mid (\boldsymbol{m}, \boldsymbol{m})]$	The plane with a line $\{\boldsymbol{v} \mid \boldsymbol{m}\}$, the furthest from O.
N		$[-w\boldsymbol{p} \mid (\boldsymbol{p}, \boldsymbol{p})]$	The plane with the point $(\boldsymbol{p} \mid w)$ furthest from O.
O	(20)	$\dfrac{\lvert(\boldsymbol{v}_1, \boldsymbol{m}_2) + (\boldsymbol{v}_2, \boldsymbol{m}_1)\rvert}{\lVert\boldsymbol{v}_1 \times \boldsymbol{v}_2\rVert}$	The distance between lines $\{\boldsymbol{v}_1 \mid \boldsymbol{m}_1\}$ and $\{\boldsymbol{v}_2 \mid \boldsymbol{m}_2\}$.
P	(7)	$\dfrac{\lvert\boldsymbol{v} \times \boldsymbol{p} + \boldsymbol{m}\rvert}{\lVert\boldsymbol{v}\rVert}$	The distance between the line $\{\boldsymbol{v} \mid \boldsymbol{m}\}$ and point $(\boldsymbol{p} \mid 1)$.
Q	(5)	$\lVert\boldsymbol{m}\rVert / \lVert\boldsymbol{v}\rVert$	The distance from the line $\{\boldsymbol{v} \mid \boldsymbol{m}\}$ to the origin.
R	(9)	$\dfrac{\lvert(\boldsymbol{n}, \boldsymbol{p}) + d\rvert}{\lVert\boldsymbol{n}\rVert}$	Distance from the plane $[\boldsymbol{n} \mid d]$ to the point $(\boldsymbol{p} \mid 1)$.
S	(8)	$\lvert d\rvert / \lVert\boldsymbol{n}\rVert$	The distance from the plane $[\boldsymbol{n} \vert d]$ to the origin.

Table 2

Normalization of a point, line, and plane

	General form	**Normalized form**
Point	$(\boldsymbol{p} \mid w)$	$(\boldsymbol{p}/w \mid 1)$
Line	$(\boldsymbol{v} \mid \boldsymbol{m})$	$[\boldsymbol{v}/\lVert\boldsymbol{v}\rVert \mid \boldsymbol{m}/\lVert\boldsymbol{v}\rVert]$
Plane	$[\boldsymbol{n} \mid d]$	$[\boldsymbol{n}/\lVert\boldsymbol{n}\rVert \mid d/\lVert\boldsymbol{n}\rVert]$

Table 3

Duality of a point, a line, and a plane

Original	**Dual**
Point $(\boldsymbol{p} \mid w)$	Plane $[\boldsymbol{p} \mid w]$
Line $\{\boldsymbol{v} \mid \boldsymbol{m}\}$	Line $\{\boldsymbol{m} \mid \boldsymbol{v}\}$
Plane $[\boldsymbol{n} \mid d]$	Point $(\boldsymbol{n} \mid d)$

Since the point P_0 is fixed, then $(\boldsymbol{N}, \boldsymbol{p}_0) = \text{const}$.

Let's introduce the components of the normal vector $\boldsymbol{N} = (A, B, C)$ and denote

$$D = -(\boldsymbol{N}, \boldsymbol{p}_0),$$

then, in the Cartesian coordinate system, the plane will be uniquely defined by a linear equation of the following form:

$$Ax + By + Cz + D = 0.$$

The resulting equation is called *general equation of the plane*. It is assumed that A, B, C do not simultaneously vanish, that is, $A^2 + B^2 + C^2 \neq 0$.

5.2. The projective representation of the plane

In the $\mathbb{RP}^3$ model of a three-dimensional projective space, planes are modeled using hyperplanes of the $\mathbb{R}^4$ space. When using homogeneous coordinates in the $\mathbb{R}^4$ space, a Cartesian coordinate system with $Oxyzw$ axes is introduced, and a three-dimensional projective space is modeled as a hyperplane passing through a point $(0, 0, 0, 1)$ that is, through a point on the Oz axis, parallel to the hyperplane $Oxyz$. The hyperplanes $Ax + By + Cz + Dw = 0$ passing through the coordinate center and intersecting the plane $w = 1$ cut off the three-dimensional planes defined by the equation $Ax + By + Cz + D = 0$. In fact, this is the general equation of the plane from classical analytical geometry, which we discussed above. Let's now consider its normalized version.

Instead of the vector $\boldsymbol{N}$, you can use the unit normal vector $\boldsymbol{n}$, the components of which are calculated as follows:

$$\boldsymbol{n} = \frac{\boldsymbol{N}}{\lVert\boldsymbol{N}\rVert} = \left(\frac{A}{\sqrt{A^2 + B^2 + C^2}}, \frac{B}{\sqrt{A^2 + B^2 + C^2}}, \frac{B}{\sqrt{A^2 + B^2 + C^2}}\right) = (n_x, n_y, n_z), \quad n_x^2 + n_y^2 + n_z^2 = 1.$$

The values n_x, n_y, n_z are uniquely determined by the guiding cosines $n_x = \cos\alpha$, $n_y = \cos\beta$, $n_z = \cos\gamma$, where α, β, γ are the angles between the vector $\boldsymbol{n}$ and the axes Ox, Oy and Oz. You can also normalize D by dividing it by the norm of the normal vector $\boldsymbol{N}$

$$d = \frac{D}{\sqrt{A^2 + B^2 + C^2}}.$$

As a result, the general equation of the plane will be written as *normal equation of the plane* in the following form:

$$n_x x + n_y y + n_z z + d = \cos\alpha x + \cos\beta y + \cos\gamma z + d = (\boldsymbol{n}, \boldsymbol{p}) + d = 0.$$

If instead of the radius vector $\boldsymbol{p} = (x, y, z)^T$ of a three-dimensional Euclidean space defining the coordinates of a point $P \in \pi$, enter the radius vector $\vec{\boldsymbol{p}} = (\boldsymbol{p} \mid 1) = (x : y : z : 1)$ of a projective space defining the projective coordinates of a point P, then the normal equation of the plane can be rewritten as a scalar product of the vector $\vec{\boldsymbol{p}}$ and the vector $\vec{\boldsymbol{\pi}} = [\boldsymbol{n} \mid d] = (n_x : n_y : n_z : d)$:

$$(\vec{\boldsymbol{p}}, \vec{\boldsymbol{\pi}}) = \begin{bmatrix} n_x & n_y & n_z & d \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = n_x x + n_y y + n_z z + d = 0.$$

Writing down the coefficients of the normal equation of the plane in the form of $[\boldsymbol{n} \mid d]$, we follow the notation adopted in [30].

The vector $\vec{\boldsymbol{\pi}} = [\boldsymbol{n} \mid d]$ for a plane is an analog of homogeneous coordinates for a point and allows you to write formulas related to the plane in a homogeneous form. Indeed, multiplying the vector $\vec{\boldsymbol{n}}$ by a scalar does not change the equation of the plane and you can always return to the normalized form by dividing all the components of the vector by $\|\boldsymbol{n}\|$:

$$\left[\frac{\boldsymbol{n}}{\|\boldsymbol{n}\|} \;\middle|\; \frac{d}{\|\boldsymbol{n}\|} \right]. \tag{8}$$

The value of $|d|/\|\boldsymbol{n}\|$ is equal to the distance from the origin to the plane. We have introduced the vector $\boldsymbol{n}$ as a unit, however, an error may accumulate in the process of computer calculations and $\boldsymbol{n}$ will cease to be a unit. Therefore, the formula (8) is divided by $\|\boldsymbol{n}\|$, which provides renormalization, eliminating the accumulated error.

5.3. The distance from the point to the plane

Let's return to the general equation of the π plane and write it in a normalized form:

$$(\boldsymbol{n}, \boldsymbol{p}) + d = 0, \quad n_x x + n_y y + n_z z + d = 0, \quad \|\boldsymbol{n}\| = 1.$$

The geometric meaning of the value d is the projection of the radius vector of an arbitrary point P_1 lying on a plane onto the unit vector of the normal $\boldsymbol{n}$, as shown in the figure 10. The distance from an arbitrary point in space to the plane is defined as the length of the perpendicular lowered from this point onto the plane. The direction of the perpendicular to the plane coincides with the direction of the normal vector $\boldsymbol{n}$, so d can also be interpreted as the distance from the origin to the plane.

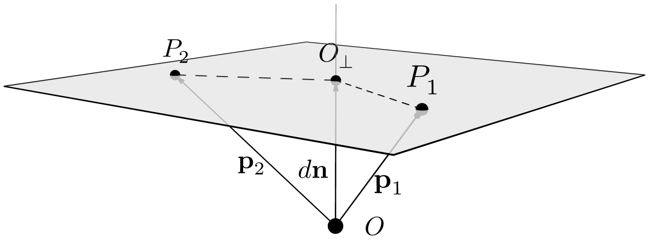

Figure 10. The geometric meaning of d is the directional distance from the origin to the plane. Radius vector $\boldsymbol{OO}_\perp = -d\boldsymbol{n}$, where $\boldsymbol{n}$ is the unit normal vector

We show that the value of d does not depend on the choice of a point on the plane. Any point in the plane is defined using the parametric equation radius vector

$$\boldsymbol{p}(t,s) = \boldsymbol{p}_1 + \boldsymbol{u}s + \boldsymbol{v}t,$$

where $\boldsymbol{u} = \boldsymbol{p}_2 - \boldsymbol{p}_1$ and $\boldsymbol{v} = \boldsymbol{p}_3 - \boldsymbol{p}_1$ are the guiding vectors of the plane perpendicular to the normal vector $\boldsymbol{n}$. Calculate the length of the projection of an arbitrary point $(\boldsymbol{p},\boldsymbol{n})$:

$$(\boldsymbol{p},\boldsymbol{n}) = (\boldsymbol{p}_1 + \boldsymbol{u}s + \boldsymbol{v}t, \boldsymbol{n}) = (\boldsymbol{p}_1,\boldsymbol{n}) + \cancelto{0}{(\boldsymbol{u},\boldsymbol{n})s} + \cancelto{0}{(\boldsymbol{v},\boldsymbol{n})t} = (\boldsymbol{p}_1,\boldsymbol{n}).$$

Since the choice of the point P_1 is also arbitrary, the parameter d is uniquely determined for this particular plane. So, the radius vector of the point $O_\perp$ — projections of the origin on the plane π can be calculated as $\boldsymbol{OO}_\perp = -d\boldsymbol{n}$ (see also (14)).

Let Q be an arbitrary point in space, and $\boldsymbol{q} = (x,y,z)^T$ be its radius vector. The value $\delta(\boldsymbol{q}) = n_x x + n_y y + n_z z + d$ is called *deviation of a point from the plane* and has the geometric meaning of the oriented distance from the point Q to the plane. The orientation of the distance makes it possible to determine by the sign on which side of the plane the point is located.

$$\delta(\boldsymbol{q}) = (\boldsymbol{n},\boldsymbol{q}) + d = \begin{cases} > 0, & \text{is the point is in front of the plane,} \\ = 0, & \text{is the point belongs to the plane,} \\ < 0, & \text{is the point is behind the plane.} \end{cases}$$

To clarify, the phrase “the point is in front of the plane” means that an observer at the point can see the front of the plane, and the phrase “the point is behind the plane” means that the same observer sees the inside of the plane. If the plane forms the face of some complex three-dimensional object, then in this case the observer is inside this object.

If we consider the homogeneous coordinates of the point Q given by the vector $\vec{\boldsymbol{q}} = (\boldsymbol{q} \mid 1) = (x : y : z : 1)$, then the deviation of the point Q from the plane reduces to finding the dot product of the vectors $\boldsymbol{q}$ and $\vec{\boldsymbol{\pi}}$:

$$\delta(\vec{\boldsymbol{q}}) = n_x x + n_y y + n_z z + d = \begin{bmatrix} n_x & n_y & n_z & d \end{bmatrix} \begin{bmatrix} q_x \\ q_y \\ q_z \\ 1 \end{bmatrix} = (\vec{\boldsymbol{\pi}}, \vec{\boldsymbol{q}}). \tag{9}$$

Note that in the table 1, this formula is given in a normalized form on the assumption that the normal vector $\boldsymbol{n}$ will not necessarily be singular.

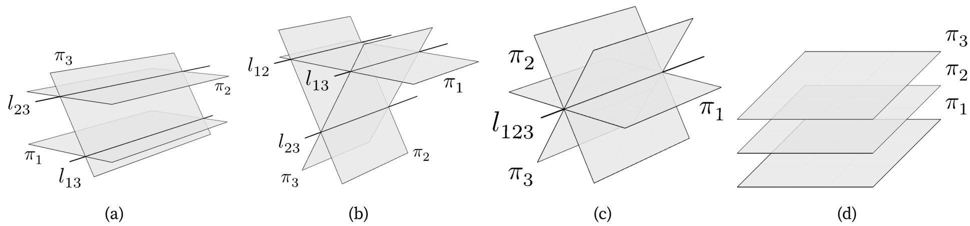

Figure 11. Options for intersecting planes at an incorrect point. In variant (a), the lines l_{13} and l_{23} are parallel and intersect at an irregular point, hence the planes intersect at the same point. Similarly, in case (b), three parallel lines intersect at an irregular point. In Figure (c), the planes intersect along their own straight line l_{123}, which contains, among other things, an irregular point at which the planes intersect. Finally, in (d), parallel planes intersect in an irregular straight line

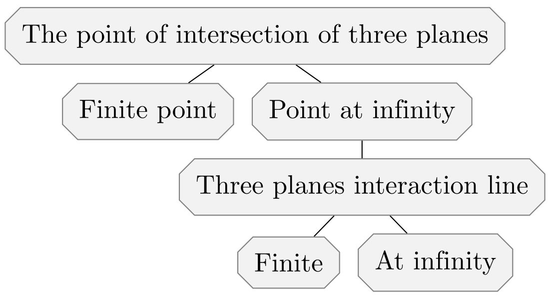

Figure 12. Variants of the intersection of three planes in a projective space

The direction of the normal vector to the plane is extremely important in computer graphics, since modeling the reflection of light from three-dimensional objects depends on it. The normal vector allows you to specify the *front side of the plane*, that is, the side from which the normal vector is directed (the vector is directed into the eye of the observer looking at the plane) and the *back side of the plane*, opposite to the front. The front side can also be called face side if the plane forms the face of an object.

5.4. Intersection of two and three planes

In ordinary three-dimensional Euclidean space, three planes can be in six different positions.

1. Have a single common point, figure 13.
2. Have a single common straight line, Figure 11 (c).
3. Intersect in pairs, as shown in the figure, figure 11 (b).
4. Two planes can be parallel and intersect with the third, as shown in the figure, figure 11 (a).
5. All three planes can be mutually parallel, figure 11 (d).
6. All planes can match.

If we consider the location of three planes in a projective space, then the number of options is sharply reduced, since in a projective space any planes intersect, however, the intersection point can be either its own or improper. The general scheme is shown in the diagram 12.

Let's define three planes in a homogeneous form: $\vec{\pi}_1 = [\boldsymbol{n}_1 \mid d_1]$, $\vec{\pi}_2 = [\boldsymbol{n}_2 \mid d_2]$ and $\boldsymbol{\pi}_3 = [\boldsymbol{n}_3 \mid d_3]$. It is necessary to find a point $P = \pi_1 \cap \pi_2 \cap \pi_3$ whose homogeneous coordinates $(\boldsymbol{p} \mid 1) = (x : y : z : 1)$ satisfy each of the three equations of the planes π_1, π_2 and π_3:

$$(\vec{\pi}_i, \vec{\boldsymbol{p}}) = [\boldsymbol{n}_i \mid 1] \begin{bmatrix} \boldsymbol{p} \\ 1 \end{bmatrix} = 0 \;\Leftrightarrow\; (\boldsymbol{n}_i, \boldsymbol{p}) + d_i = 0, \;\; i = 1, 2, 3.$$

Let's consider the matrix N, the rows of which make up the components of the normal vectors $\boldsymbol{n}_1$, $\boldsymbol{n}_2$ and $\boldsymbol{n}_3$ and use it to write a system of linear equations for the components of the radius vector $\boldsymbol{p}$:

$$N = \begin{bmatrix} \boldsymbol{n}_1 \\ \boldsymbol{n}_2 \\ \boldsymbol{n}_3 \end{bmatrix} = \begin{bmatrix} n_{1x} & n_{1y} & n_{1z} \\ n_{2x} & n_{2y} & n_{2z} \\ n_{3x} & n_{3y} & n_{3z} \end{bmatrix} \Longrightarrow \begin{bmatrix} n_{1x} & n_{1y} & n_{1z} \\ n_{2x} & n_{2y} & n_{2z} \\ n_{3x} & n_{3y} & n_{3z} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = - \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \quad \boldsymbol{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

To solve this system, we will find the inverse matrix N^{-1}, for which we will use the formula to find the attached matrix $N^{\vee}$ using the cross product and the mixed product to find the determinant $\det N$, after which we write

$$N^{-1} = \frac{N^{\vee}}{\det N} = \frac{[\boldsymbol{n}_2 \times \boldsymbol{n}_3 \;\; \boldsymbol{n}_3 \times \boldsymbol{n}_1 \;\; \boldsymbol{n}_1 \times \boldsymbol{n}_2]}{(\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3)}, \quad N^{\vee} = \begin{bmatrix} | & | & | \\ \boldsymbol{n}_2 \times \boldsymbol{n}_3 & \boldsymbol{n}_3 \times \boldsymbol{n}_1 & \boldsymbol{n}_1 \times \boldsymbol{n}_2 \\ | & | & | \end{bmatrix},$$

$$(\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3) = \det N = \begin{vmatrix} n_{1x} & n_{1y} & n_{1z} \\ n_{2x} & n_{2y} & n_{2z} \\ n_{3x} & n_{3y} & n_{3z} \end{vmatrix},$$

where the vectors $\boldsymbol{n}_2 \times \boldsymbol{n}_3$, $\boldsymbol{n}_3 \times \boldsymbol{n}_1$ and $\boldsymbol{n}_1 \times \boldsymbol{n}_2$ form the columns of the matrix $N^{\vee}$. Next, you can write the solution of a system of linear equations in the following form:

$$\boldsymbol{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{(\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_2)} \begin{bmatrix} | & | & | \\ \boldsymbol{n}_2 \times \boldsymbol{n}_3 & \boldsymbol{n}_3 \times \boldsymbol{n}_1 & \boldsymbol{n}_1 \times \boldsymbol{n}_2 \\ | & | & | \end{bmatrix} \begin{bmatrix} -d_1 \\ -d_2 \\ -d_3 \end{bmatrix} = \frac{-d_1 \boldsymbol{n}_2 \times \boldsymbol{n}_3 - d_2 \boldsymbol{n}_3 \times \boldsymbol{n}_1 - d_3 \boldsymbol{n}_1 \times \boldsymbol{n}_2}{(\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3)}.$$

Finally, we write down the formula for the radius vector of the point P of the intersection of three planes $\pi_1 : [\boldsymbol{n}_1 \mid d_1]$, $\pi_2 : [\boldsymbol{n}_2 \mid d_2]$ and $\pi_3 : [\boldsymbol{n}_3 \mid d_3]$:

$$\boldsymbol{p} = \frac{d_1 \boldsymbol{n}_3 \times \boldsymbol{n}_2 + d_2 \boldsymbol{n}_1 \times \boldsymbol{n}_3 + d_3 \boldsymbol{n}_2 \times \boldsymbol{n}_1}{(\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3)}. \tag{10}$$

In this form, the formula can be found, for example, in [22, p. 56] or in [30, p. 129]. In homogeneous coordinates, the same formula can be written without the division operation by moving the denominator to the place of the w coordinate:

$$\vec{\boldsymbol{p}} = \big(d_1 \boldsymbol{n}_3 \times \boldsymbol{n}_2 + d_2 \boldsymbol{n}_1 \times \boldsymbol{n}_3 + d_3 \boldsymbol{n}_2 \times \boldsymbol{n}_1 \mid (\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3)\big).$$

To solve the system of equations $(n, * p) = 0$, you can use the more familiar Kramer method.

$$\begin{cases} n_{1x}x + n_{1y}y + n_{1z}z + d_1 = 0, \\ n_{2x}x + n_{2y}y + n_{2z}z + d_2 = 0, \\ n_{3x}x + n_{3y}y + n_{3z}z + d_3 = 0. \end{cases} \Leftrightarrow \begin{cases} n_{1x}x + n_{1y}y + n_{1z}z = -d_1, \\ n_{2x}x + n_{2y}y + n_{2z}z = -d_2, \\ n_{3x}x + n_{3y}y + n_{3z}z = -d_3. \end{cases}$$

$$\Delta = \begin{vmatrix} n_{1x} & n_{1y} & n_{1z} \\ n_{2x} & n_{2y} & n_{2z} \\ n_{3x} & n_{3y} & n_{3z} \end{vmatrix} \quad \Delta_x = \begin{vmatrix} -d_1 & n_{1y} & n_{1z} \\ -d_2 & n_{2y} & n_{2z} \\ -d_3 & n_{3y} & n_{3z} \end{vmatrix} \quad \Delta_y = \begin{vmatrix} n_{1x} & -d_1 & n_{1z} \\ n_{2x} & -d_2 & n_{2z} \\ n_{3x} & -d_3 & n_{3z} \end{vmatrix} \quad \Delta_z = \begin{vmatrix} n_{1x} & n_{1y} & -d_1 \\ n_{2x} & n_{2y} & -d_2 \\ n_{3x} & n_{3y} & -d_3 \end{vmatrix}.$$

As a result of solving this system, we obtain a point in homogeneous coordinates in the form:

$$\vec{\boldsymbol{p}} = \left(\frac{\Delta_x}{\Delta}, \frac{\Delta_y}{\Delta}, \frac{\Delta_z}{\Delta} \;\middle|\; 1\right) = (\Delta_x, \Delta_y, \Delta_z \mid \Delta).$$

We get that:

$$d_1 \boldsymbol{n}_3 \times \boldsymbol{n}_2 + d_2 \boldsymbol{n}_1 \times \boldsymbol{n}_3 + d_3 \boldsymbol{n}_2 \times \boldsymbol{n}_1 = \Delta_x \boldsymbol{e}_x + \Delta_y \boldsymbol{e}_y + \Delta_z \boldsymbol{e}_z.$$

Let's write down once again the formula for calculating the homogeneous coordinates of the intersection point of three planes $\pi_1 : [\boldsymbol{n}_1 \mid d_1]$, $\pi_2 : [\boldsymbol{n}_2 \mid d_2]$ and $\pi_3 : [\boldsymbol{n}_3 \mid d_3]$ in a homogeneous form in two versions:

$$\vec{\boldsymbol{p}} = \big(d_1 \boldsymbol{n}_3 \times \boldsymbol{n}_2 + d_2 \boldsymbol{n}_1 \times \boldsymbol{n}_3 + d_3 \boldsymbol{n}_2 \times \boldsymbol{n}_1 \mid (\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3)\big) = (\Delta_x, \Delta_y, \Delta_z \mid \Delta). \tag{11}$$

The first option has an advantage, since it is written in a non-component form, and in the second option, the components participate in calculating the determinants Δ_x, Δ_y and Δ_z of the system.

For a full-fledged analytical study of the relative position of the three planes, we should first consider the problem of the location of the two planes, which we will do next. In the meantime, it follows directly from (11) that if $\Delta = (\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3) \neq 0$, then the intersection point is its own (terminal), and if $(\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3) = 0$, then the point is incorrect and this case covers all the remaining 5 plane locations.

Two planes in a projective space relative to each other can be in the following positions.

- Have their own common straight line (intersect along the end line).
- Should have a common irregular straight line (be parallel and intersect in a perfect straight line).
- Match.

When considering planes in a projective space, it is not necessary to consider the first two cases separately.

Find the straight line $l = \pi_1 \cap \pi_2$, which is obtained when two planes intersect $\pi_1 : [\boldsymbol{n}_1 \mid d_1]$, and $\pi_2 : [\boldsymbol{n}_2 \mid d_2]$. The guiding vector of such a straight line must be perpendicular to both the normal vector $\boldsymbol{n}_1$ and the normal vector $\boldsymbol{n}_2$, therefore it can be calculated as $\boldsymbol{v} = \boldsymbol{n}_1 \times \boldsymbol{n}_2$, which means shown on the left side of the figure 13a.

To find the point of this straight line, take the third plane π, with a normal vector equal to the vector $\boldsymbol{n}_3 = \boldsymbol{v}$ and passing through the origin, that is, $d_3 = 0$. Then, using the formula (10), we immediately get:

$$\boldsymbol{p} = \frac{d_1 \boldsymbol{v} \times \boldsymbol{n}_2 + d_2 \boldsymbol{n}_1 \times \boldsymbol{v}}{(\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{v})},$$

where the triple product is simplified to the square of the norm of the cross product $\|\boldsymbol{n}_1 \times \boldsymbol{n}_2\|^2$

$$(\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{v}) = (\boldsymbol{n}_1, \boldsymbol{n}_2 \times \boldsymbol{v}) = (\boldsymbol{n}_1 \times \boldsymbol{n}_2, \boldsymbol{v}) = (\boldsymbol{v}, \boldsymbol{v}) = \|\boldsymbol{v}\|^2 = \|\boldsymbol{n}_1 \times \boldsymbol{n}_2\|^2.$$

Let's finally write it down:

$$\boldsymbol{p} = \frac{d_1 \boldsymbol{v} \times \boldsymbol{n}_2 + d_2 \boldsymbol{n}_1 \times \boldsymbol{v}}{\|\boldsymbol{v}\|^2}, \text{ where } \boldsymbol{v} = \boldsymbol{n}_1 \times \boldsymbol{n}_2.$$

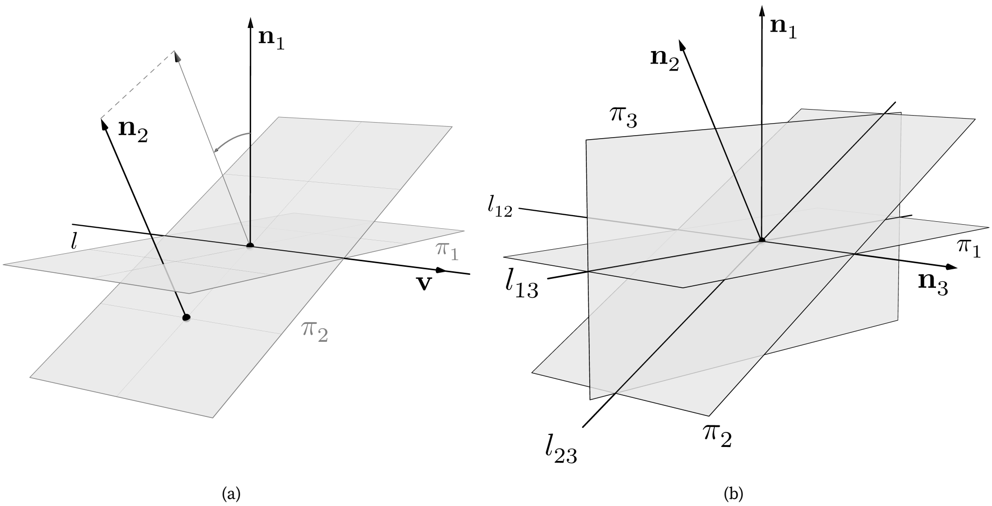

Figure 13. In Figure (a), the line of intersection of two planes with the guide vector $\boldsymbol{v} = \boldsymbol{n}_1 \times \boldsymbol{n}_2$. The triple of vectors $\langle \boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{v} \rangle$ is right, since the rotation from $\boldsymbol{n}_1$ to $\boldsymbol{n}_2$ from the end of $\boldsymbol{v}$ occurs counterclockwise. In Figure (b), three planes intersect along straight lines $l_{12} = \pi_1 \cos \pi_2$, $l_{13} = \pi_1 \cos \pi_3$, $l_{23} = \pi_2 \cos \pi_3$. The third plane passes through the origin and its normal vector is perpendicular to $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ and coincides with the guiding vector $\boldsymbol{v}$ of the line l_{12} of the intersection of π_1 and π_2.

To write the line l in Plucker coordinates, calculate the moment $\boldsymbol{m}$

$$\boldsymbol{m} = \boldsymbol{p} \times \boldsymbol{v} = \frac{d_1 \boldsymbol{v} \times \boldsymbol{n}_2 \times \boldsymbol{v} + d_2 \boldsymbol{n}_1 \times \boldsymbol{v} \times \boldsymbol{v}}{\|\boldsymbol{v}\|^2},$$

Using the “bac minus cab” property of the cross product, we transform the numerator to a simpler form:

$$\begin{aligned} &\boldsymbol{v} \times \boldsymbol{n}_2 \times \boldsymbol{v} = \boldsymbol{n}_2 \|\boldsymbol{v}\|^2 - \boldsymbol{v}(\boldsymbol{v}, \boldsymbol{n}) = \boldsymbol{n}_2 \|\boldsymbol{v}\|^2, \\ &\boldsymbol{n}_1 \times \boldsymbol{v} \times \boldsymbol{v} = \boldsymbol{v}(\boldsymbol{v}, \boldsymbol{n}) - \boldsymbol{n}_1(\boldsymbol{v}, \boldsymbol{v}) = -\boldsymbol{n}_1 \|\boldsymbol{v}\|^2, \end{aligned} \Rightarrow$$

$$\Rightarrow d_1 \boldsymbol{v} \times \boldsymbol{n}_2 \times \boldsymbol{v} + d_2 \boldsymbol{n}_1 \times \boldsymbol{v} \times \boldsymbol{v} = d_1 \boldsymbol{n}_2 \|\boldsymbol{v}\|^2 - d_2 \boldsymbol{n}_1 \|\boldsymbol{v}\|^2 = (d_1 \boldsymbol{n}_2 - d_2 \boldsymbol{n}_1)\|\boldsymbol{v}\|^2.$$

We used the fact that $\boldsymbol{v} \perp \boldsymbol{n}_1$ and $\boldsymbol{v} \perp \boldsymbol{n}_2$ since $\boldsymbol{v} = \boldsymbol{n}_1 \times \boldsymbol{n}_2$ which means $(\boldsymbol{v}, \boldsymbol{n}_1) = (\boldsymbol{v}, \boldsymbol{n}_2)$. The formula for the moment is simplified.

$$\boldsymbol{m} = \frac{(d_1 \boldsymbol{n}_2 - d_2 \boldsymbol{n}_1)\|\boldsymbol{v}\|^2}{\|\boldsymbol{v}\|^2} = d_1 \boldsymbol{n}_2 - d_2 \boldsymbol{n}_1.$$

As a result, the line of intersection of the planes $\pi_1 : \; [\boldsymbol{n}_1 \;|_1]$ and $\pi_2 : \; [\boldsymbol{n}_2 \;|_2]$ in Plucker coordinates is written as

$$\{\boldsymbol{v} \mid \boldsymbol{m}\} = \{\boldsymbol{n}_1 \times \boldsymbol{n}_2 \mid d_1 \boldsymbol{n}_2 - d_2 \boldsymbol{n}_1\} \tag{12}$$

5.5. Intersection of a straight line and a plane

Consider the problem of the intersection of the plane $\pi : \; [\boldsymbol{n} \mid d]$ and the straight line l, given in parametric form by the radius vector $\boldsymbol{l}(t) = \boldsymbol{p} + \boldsymbol{v}t$. Let’s immediately exclude the case when the straight line is parallel to the plane, that is, $(\boldsymbol{n}, \boldsymbol{v}) = 0$, $\boldsymbol{n} \perp \boldsymbol{v}$ and there is no intersection point.

In the case of non-parallel lines and planes, they will necessarily intersect at some point Q with homogeneous coordinates $(\boldsymbol{q} \mid 1)$, where $\boldsymbol{q}$ is the radius vector of a point in Euclidean space. Since the point belongs to a straight line, then $\boldsymbol{q} = \boldsymbol{p} + \boldsymbol{v}t$ for a certain value of the parameter t. The task is to find the value of this parameter. To do this, we substitute the radius vector of the point Q into the normal equation of the plane, or, in other words, we find the dot product of homogeneous vectors:

$$[\boldsymbol{n} \mid d]\begin{bmatrix}\boldsymbol{q}\\ 1\end{bmatrix} = (\boldsymbol{n}, \boldsymbol{q}) + d = 0 \Rightarrow (\boldsymbol{n}, \boldsymbol{p}) + t(\boldsymbol{n}, \boldsymbol{v}) + d = 0 \Rightarrow t = -\frac{(\boldsymbol{n}, \boldsymbol{p}) + d}{(\boldsymbol{n}, \boldsymbol{v})}.$$

Substituting this value of t into the parametric equation of the straight line, we write:

$$\boldsymbol{q} = \boldsymbol{p} - \frac{(\boldsymbol{n}, \boldsymbol{p}) + d}{(\boldsymbol{n}, \boldsymbol{v})}\boldsymbol{v}.$$

Let's transform this formula so as to express the moment $\boldsymbol{m}$ of a straight line:

$$(\boldsymbol{n}, \boldsymbol{v})\boldsymbol{q} = \boldsymbol{p}(\boldsymbol{n}, \boldsymbol{v}) - (\boldsymbol{n}, \boldsymbol{p})\boldsymbol{v} - d\boldsymbol{v} = -(\boldsymbol{v}(\boldsymbol{n}, \boldsymbol{p}) - \boldsymbol{p}(\boldsymbol{n}, \boldsymbol{v}) + d\boldsymbol{v}) = -\boldsymbol{n} \times \boldsymbol{v} \times \boldsymbol{p} - d\boldsymbol{v}.$$

Here we used the "bac minus cab" property of the cross product again. Now consider that $\boldsymbol{m} = \boldsymbol{p} \times \boldsymbol{v}$ and write:

$$-\boldsymbol{n} \times \boldsymbol{v} \times \boldsymbol{p} - d\boldsymbol{v} = \boldsymbol{n} \times \boldsymbol{m} - d\boldsymbol{v} \Rightarrow (\boldsymbol{n}, \boldsymbol{v})\boldsymbol{q} = \boldsymbol{n} \times \boldsymbol{m} - d\boldsymbol{v} = -(\boldsymbol{m} \times \boldsymbol{n} + d\boldsymbol{v}) \Rightarrow \boldsymbol{q} = -\frac{(\boldsymbol{m} \times \boldsymbol{n} + d\boldsymbol{v})}{(\boldsymbol{n}, \boldsymbol{v})}.$$

The homogeneous coordinates of the point Q can be written as

$$\left(-\frac{(\boldsymbol{m} \times \boldsymbol{n} + d\boldsymbol{v})}{(\boldsymbol{n}, \boldsymbol{v})} \;\middle|\; 1\right) = (-(\boldsymbol{m} \times \boldsymbol{n} + d\boldsymbol{v}) \mid (\boldsymbol{n}, \boldsymbol{v})) = ((\boldsymbol{m} \times \boldsymbol{n} + d\boldsymbol{v}) \mid -(\boldsymbol{n}, \boldsymbol{v})) \tag{13}$$

If the lines are parallel, then $(\boldsymbol{n}, \boldsymbol{v}) = 0$ and the intersection point becomes ideal $(\boldsymbol{m} \times \boldsymbol{n} + d\boldsymbol{v} \mid 0)$ i.e. the lines intersect in an infinitely distant point and the resulting coordinates indicate the direction where this point is located.

5.6. The point of the plane closest to the origin

If the plane is specified as $[\boldsymbol{n} \mid]$. The point of the plane closest to the origin lies on a straight line passing through the origin perpendicular to the plane. Such a straight line is defined by the guiding vector $\boldsymbol{v} = \boldsymbol{n}$ and a certain moment $\boldsymbol{m} = \boldsymbol{p} \times \boldsymbol{v}$, where $\boldsymbol{p}$ is the radius vector of an arbitrary point on a straight line. Since the straight line passes through the origin, you can choose $\boldsymbol{p} = \boldsymbol{0}$ and calculate the moment $\boldsymbol{m} = \boldsymbol{0} \times \boldsymbol{v} = \boldsymbol{0}$. Therefore, the line can be written as $\{\boldsymbol{v} \mid \boldsymbol{0}\}$. Using the formula (13) we write

$$(\boldsymbol{m} \times \boldsymbol{n} + d\boldsymbol{v} \mid -(\boldsymbol{n}, \boldsymbol{v})) = (d\boldsymbol{n} \mid -(\boldsymbol{n}, \boldsymbol{n})) = (d\boldsymbol{n} \mid -\|\boldsymbol{n}\|^2) = (-d\boldsymbol{n} \mid \|\boldsymbol{n}\|^2).$$

As a result, we obtain a formula for calculating the homogeneous coordinates of the point in the plane closest to the origin.

$$(-d\boldsymbol{n} \mid \|\boldsymbol{n}\|^2). \tag{14}$$

5.7. A plane passing through a straight line and a point

Let the plane π contain a straight line l defined by the Plucker coordinates in the form $\{\boldsymbol{v} \mid \boldsymbol{m}\}$, as well as a point P with homogeneous coordinates $(\boldsymbol{p} \mid w)$. The radius vector of this point in $\mathbb{R}^3$ will have the form $\boldsymbol{p}/w$. As the second point of the plane, we can take any point lying on a straight line, the equation of which we know. This point is the point of the straight line closest to the origin, which is calculated using the formula (4) $(\boldsymbol{v} \times \boldsymbol{m} \mid (\boldsymbol{v}, \boldsymbol{v}))$. Let's denote this point as Q and calculate the vector $\boldsymbol{PQ} = \boldsymbol{u}$:

$$\boldsymbol{u} = \frac{\boldsymbol{p}}{w} - \boldsymbol{q} = \frac{\boldsymbol{p}}{w} - \frac{\boldsymbol{v} \times \boldsymbol{m}}{\|\boldsymbol{v}\|^2}.$$

Now we can find the normal vector of the plane π as $\boldsymbol{n} = \boldsymbol{v} \times \boldsymbol{u}$, since $\boldsymbol{v} \perp \pi$ and $\boldsymbol{u} \perp \pi$.

$$\boldsymbol{n} = \boldsymbol{v} \times \boldsymbol{u} = \frac{\boldsymbol{v} \times \boldsymbol{p}}{w} - \frac{\boldsymbol{v} \times \boldsymbol{v} \times \boldsymbol{m}}{\|\boldsymbol{v}\|^2} = \frac{\boldsymbol{v} \times \boldsymbol{p}}{w} + \frac{\boldsymbol{m}\|\boldsymbol{v}\|^2}{\|\boldsymbol{v}\|^2} = \frac{\boldsymbol{v} \times \boldsymbol{p}}{w} + \boldsymbol{m},$$

where we used the "bac minus cab" property of the cross product. As a result, we come to the expression for $\boldsymbol{n}$:

$$w\boldsymbol{n} = \boldsymbol{v} \times \boldsymbol{p} + w\boldsymbol{m}.$$

The distance from the origin to the plane is found as the length of the projection of the point $\boldsymbol{p}/w$ onto the vector $\boldsymbol{n}$:

$$\left\|\frac{1}{w}\boldsymbol{p}_{\|\boldsymbol{n}}\right\| = \frac{1}{w}(\boldsymbol{p}, \boldsymbol{n}) = \frac{1}{w}\left(\boldsymbol{p}, \boldsymbol{m} + \frac{\boldsymbol{v} \times \boldsymbol{p}}{w}\right) = \frac{1}{w}(\boldsymbol{p}, \boldsymbol{m}) + \frac{1}{w}\cancelto{0}{\left(\boldsymbol{p}, \frac{\boldsymbol{v} \times \boldsymbol{p}}{w}\right)} = \frac{1}{w}(\boldsymbol{p}, \boldsymbol{m})$$

and the directional distance from the plane to the straight line can be calculated as $d = -\dfrac{(\boldsymbol{p}, \boldsymbol{m})}{w}$.

Now we can write the plane in homogeneous coordinates as

$$[\boldsymbol{n} \mid d] = \left[\frac{\boldsymbol{v} \times \boldsymbol{p}}{w} + \boldsymbol{m} \,\middle|\, -\frac{(\boldsymbol{p}, \boldsymbol{m})}{w}\right],$$

multiplying by w, we finally write down the formula L from the table 1

$$[\boldsymbol{v} \times \boldsymbol{p} + w\boldsymbol{m} \mid -(\boldsymbol{p}, \boldsymbol{m})]. \tag{15}$$

If instead of a point on the plane, the direction is known, that is, a point in homogeneous coordinates $(\boldsymbol{u} \mid 0)$, then replacing the point $(\boldsymbol{p} \mid w)$ with $(\boldsymbol{u} \mid 0)$, we'll write it down immediately

$$[\boldsymbol{v} \times \boldsymbol{u} \mid -(\boldsymbol{u}, \boldsymbol{m})]. \tag{16}$$

If the coordinates of a point are given as $(\boldsymbol{p} \mid 1)$, that is, $w = 1$, then we also directly obtain a homogeneous representation of the plane from the formula (15)

$$[\boldsymbol{v} \times \boldsymbol{p} + \boldsymbol{m} \mid -(\boldsymbol{p}, \boldsymbol{m})]. \tag{17}$$

If the plane passes through the origin, then also substituting $(\boldsymbol{0} \mid 1)$ using the formula (15), we get

$$[\boldsymbol{m} \mid 0]. \tag{18}$$

Let's prove another formula that is not in the original table, but which can be useful when defining the plane π through vector guides. Let us know one point P of the plane $\vec{\boldsymbol{p}} = (\boldsymbol{p} \mid 1)$ and two guide vectors $\vec{\boldsymbol{v}} = (\boldsymbol{v} \mid 0)$ and $\vec{\boldsymbol{u}} = (\boldsymbol{u} \mid 0)$. Then, using the formula (1), we obtain the Plucker coordinates of a straight line lying in the plane $\{\boldsymbol{v} \mid \boldsymbol{p} \times \boldsymbol{v}\} \subset \pi$, and from (16) the final expression for the plane π in homogeneous form:

$$[\boldsymbol{v} \times \boldsymbol{u} \mid -(\boldsymbol{u}, \boldsymbol{p} \times \boldsymbol{v})] = [\boldsymbol{v} \times \boldsymbol{u} \mid (\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{p})], \tag{19}$$

where $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{p})$ is the triple product of three vectors.

5.8. The position of straight lines in space

In a projective space, two straight lines l_1 and l_2 can be in three positions relative to each other.

- Lines can *intersect*, that is, they do not lie in the same plane.
- Lines can lie in the same plane (*coplanar* lines). In this case:
  - lines intersect at their own point;
  - lines intersect at an irregular point (parallel).

The distance between the lines is calculated as the length of the mutual perpendicular. If the lines intersect, the length of the perpendicular will be zero.

When using Plucker coordinates, the distance between two intersecting lines can be obtained as the distance between two parallel planes. Consider the lines $\{\boldsymbol{v}_1 \mid \boldsymbol{m}_1\}$ and $\{\boldsymbol{v}_2 \mid \boldsymbol{m}_2\}$ and construct two planes π_1 and π_2, where the plane π_1 contains the straight line $\{\boldsymbol{v}_1 \mid \boldsymbol{m}_1\}$ and the direction given by the vector $\boldsymbol{v}_2$. In turn, the plane π_2 contains the line $\{\boldsymbol{v}_2 \mid \boldsymbol{m}_2\}$ and the vector $\boldsymbol{v}_1$. The normal vectors of the planes match and are calculated as $\boldsymbol{v}_1 \times \boldsymbol{v}_2 = \boldsymbol{n}_1 = \boldsymbol{n}_2$.

We use the formula (16), where for π_1 we put $\boldsymbol{v} = \boldsymbol{v}_1$, $\boldsymbol{u} = \boldsymbol{v}_2$, and for π_2, on the contrary: $\boldsymbol{v} = \boldsymbol{v}_2$, $\boldsymbol{u} = \boldsymbol{v}_1$, as a result:

$$\pi_1 : [\boldsymbol{v}_1 \times \boldsymbol{v}_2 \mid -(\boldsymbol{v}_2, \boldsymbol{m}_1)], \quad \pi_2 : [\boldsymbol{v}_2 \times \boldsymbol{v}_1 \mid -(\boldsymbol{v}_1, \boldsymbol{m}_2)].$$

To find the distance from the origin to the planes, let's go to the normalized view by dividing by the norm of the normal vector $\|\boldsymbol{v}_1 \times \boldsymbol{v}_2\|$:

$$\pi_1 : \left[\frac{\boldsymbol{v}_1 \times \boldsymbol{v}_2}{\|\boldsymbol{v}_1 \times \boldsymbol{v}_2\|} \,\middle|\, -\frac{(\boldsymbol{v}_2, \boldsymbol{m}_1)}{\|\boldsymbol{v}_1 \times \boldsymbol{v}_2\|}\right], \quad \pi_2 : \left[-\frac{\boldsymbol{v}_1 \times \boldsymbol{v}_2}{\|\boldsymbol{v}_1 \times \boldsymbol{v}_2\|} \,\middle|\, -\frac{(\boldsymbol{v}_1, \boldsymbol{m}_2)}{\|\boldsymbol{v}_1 \times \boldsymbol{v}_2\|}\right] = \left[\frac{\boldsymbol{v}_1 \times \boldsymbol{v}_2}{\|\boldsymbol{v}_1 \times \boldsymbol{v}_2\|} \,\middle|\, \frac{(\boldsymbol{v}_1, \boldsymbol{m}_2)}{\|\boldsymbol{v}_1 \times \boldsymbol{v}_2\|}\right].$$

The distances from the planes π_1 and π_2 to the origin will be respectively:

$$d_1 = -\frac{(\boldsymbol{v}_2, \boldsymbol{m}_1)}{\|\boldsymbol{v}_1 \times \boldsymbol{v}_2\|}, \quad d_2 = -\frac{(\boldsymbol{v}_1, \boldsymbol{m}_2)}{\|\boldsymbol{v}_1 \times \boldsymbol{v}_2\|}.$$

The distance between the planes is calculated as the difference between d_2 and d_1:

$$d = |d_2 - d_1| = \left|\frac{(\boldsymbol{v}_1, \boldsymbol{m}_2) + (\boldsymbol{v}_2, \boldsymbol{m}_1)}{\|\boldsymbol{v}_1 \times \boldsymbol{v}_2\|}\right|. \tag{20}$$

By the condition of constructing planes, this is the distance between the lines l_1 and l_2. Value $M = (\boldsymbol{v}_1, \boldsymbol{m}_2) + (\boldsymbol{v}_2, \boldsymbol{m}_1)$ is called *mutual moment* of two straight lines l_1 and l_2.

- If $M > 0$, then turn from l_1 to l_2 — right.
- If $M < 0$, then turn from l_1 to l_2 — left.
- If $M = 0$, then the lines lie in the same plane or, in other words, *are coplanar*.

The condition of line coplanarity $M = 0$ can be given a simple geometric interpretation. We have found the formula for the distance between straight lines as the distance between two parallel planes. If it turns to zero, then the planes coincide, and the mutual moment is zero.:

$$M = (\boldsymbol{v}_1, \boldsymbol{m}_2) + (\boldsymbol{v}_2, \boldsymbol{m}_1) = 0.$$

Meeting this condition allows us to assert that the lines lie in the same plane, but does not allow us to determine the point of their intersection (proper or improper).

To find the intersection point of the lines, we use the projective formula (11) and consider three planes: the first plane π_1 passes through l_1 and the origin, the second plane π_2— through l_2 and the origin, and finally the third plane π_3 contains the straight line l_1 and the direction $\boldsymbol{v}_2$. From the formula (16) we get the Plucker coordinates for π_3, and from (18) for π_1 and π_2:

$$\vec{\pi}_1 = [\boldsymbol{m}_1 \mid 0], \quad \vec{\pi}_2 = [\boldsymbol{m}_2 \mid 0], \quad \vec{\pi}_3 = [\boldsymbol{v}_1 \times \boldsymbol{v}_2 \mid -(\boldsymbol{v}_2, \boldsymbol{m}_1)].$$

Now we substitute the components of the planes in (11), which will give the following expression:

$$[-(\boldsymbol{v}_2, \boldsymbol{m}_1)\boldsymbol{m}_2 \times \boldsymbol{m}_1 \mid (\boldsymbol{v}_1 \times \boldsymbol{v}_2, \boldsymbol{m}_1 \times \boldsymbol{m}_2)].$$

Simplify $(\boldsymbol{v}_1 \times \boldsymbol{v}_2, \boldsymbol{m}_1 \times \boldsymbol{m}_2)$ using the Lagrange property of the vector product:

$$(\boldsymbol{v}_1 \times \boldsymbol{v}_2, \boldsymbol{m}_1 \times \boldsymbol{m}_2) = (\boldsymbol{v}_1, \boldsymbol{m}_1)(\boldsymbol{v}_2, \boldsymbol{m}_2) - (\boldsymbol{v}_1, \boldsymbol{m}_2)(\boldsymbol{v}_2, \boldsymbol{m}_1) = -(\boldsymbol{v}_1, \boldsymbol{m}_2)(\boldsymbol{v}_2, \boldsymbol{m}_1),$$

since the moment and the guiding vector of the straight line are orthogonal, then $(\boldsymbol{v}_1, \boldsymbol{m}_1) = (\boldsymbol{v}_2, \boldsymbol{m}_2) = 0$. Next, we use the equality of the mutual moment of the curves to zero: $(\boldsymbol{v}_1, \boldsymbol{m}_2) + (\boldsymbol{v}_2, \boldsymbol{m}_1) = 0$, whence $(\boldsymbol{v}_1, \boldsymbol{m}_2) = -(\boldsymbol{v}_2, \boldsymbol{m}_1)$. Finally we get:

$$(\boldsymbol{v}_1 \times \boldsymbol{v}_2, \boldsymbol{m}_1 \times \boldsymbol{m}_2) = (\boldsymbol{v}_2, \boldsymbol{m}_1)^2 = (\boldsymbol{v}_1, \boldsymbol{m}_2)^2.$$

Using the uniformity of coordinates, we divide all coordinates by $(\boldsymbol{v}_2, \boldsymbol{m}_1)$:

$$[-(\boldsymbol{v}_2, \boldsymbol{m}_1)\boldsymbol{m}_2 \times \boldsymbol{m}_1 \mid (\boldsymbol{v}_2, \boldsymbol{m}_1)^2] = [-\boldsymbol{m}_2 \times \boldsymbol{m}_1 \mid (\boldsymbol{v}_2, \boldsymbol{m}_1)] = [\boldsymbol{m}_1 \times \boldsymbol{m}_2 \mid (\boldsymbol{v}_2, \boldsymbol{m}_1)].$$

We have obtained that the intersection point of two straight lines $\{\boldsymbol{v}_1 \mid \boldsymbol{m}_1\}$ and $\{\boldsymbol{v}_2 \mid \boldsymbol{m}_2\}$ in homogeneous coordinates is calculated as

$$[\boldsymbol{m}_1 \times \boldsymbol{m}_2 \mid (\boldsymbol{v}_2, \boldsymbol{m}_1)] = [\boldsymbol{m}_2 \times \boldsymbol{m}_1 \mid (\boldsymbol{v}_1, \boldsymbol{m}_2)] \tag{21}$$

6. Conclusion

We have outlined the basics of the $\mathbb{R}\mathrm{P}^3$ model of projective space in an analytical form. As noted above, there are two aspects of this work that may be of value.

- Filling the gap in educational literature and specialized monographs. Analytical projective geometry is actively used for applied purposes in robotics and computer graphics, and a detailed description of its basic aspects is extremely useful in methodological and pedagogical plans.
- Summary of the basic formulas in the form of a table, which is an extended table from [30]. Based on this table, you can implement everything you need to work with projective points, straight lines, and planes in the form of software structures.

The latter aspect is covered in detail in the article [39].

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References

1. Sulanke, R. & Onishchik, A. L. *Projective and Cayley-Klein Geometries* 450 pp. (Springer, 2006).
2. Rosov, N. K. Felix Klein and his Erlangen program. Russian. *Mat. Pros.* **3,** 49–55 (3 2019).

3. Volberg, O. A. *Basic ideas of projective geometry* 3rd ed. Russian. 188 pp. (State Educational and Pedagogical Publishing House of thy Ministry of Education of the RSFSR, Moscow, 1949).
4. Coxeter, H. S. M. *The Real Projective Plane* (1955).
5. Ponarin, Y. P. *Affine and projective geometry* Russian. 288 pp. (MCCME, Moscow, 2019).
6. Prasolov, V. V. & Tikhomirov, V. M. *Geometry* Russian. 336 pp. (MCCME, Moscow, 2019).
7. Young, J. W. *Projective Geometry* 184 pp. (1938).
8. Shcherbakov, R. N. & Pichurin, L. F. *From projective geometry to non-Euclidean geometry (around the absolute)* Russian. 158 pp. (Prosveshchenie, Moscow, 1979).
9. Baer, R. *Linear Algebra and Projective Geometry* 399 pp. (New York, 1952).
10. Artin, E. *Geometric Algebra* 214 pp. (Interscience Publishers, New York, 1957).
11. Busemann, H. & Kelly, P. J. *Projective geometry and projective metrics* 352 pp. (Courier Corporation, New York, 2012).
12. Roger, D. F. & Adams, J. A. *Mathematical elements for computer graphics* 611 pp. (1990).
13. Ammeraal, L. *Programming Principles in Computer Graphics* 233 pp. (John Wiley and Sons, 1992).
14. Shikin, E. V. & Boreskov, A. V. *Computer graphics* Polygonal models. 464 pp. (Dialog-MIFI, Moscow, 2001).
15. Marschner, S. & Shirley, P. *Fundamentals of Computer Graphics* 5th ed. 717 pp. (CRC Press, 2022).
16. Vince, J. *Rotation transforms for computer graphics* 1st ed. 232 pp. (Springer-Verlag London, 2011).
17. Golovanov, N. N. *Geometric modeling* Russian. 472 pp. (Fizmatlit, Moscow, 2002).
18. Dunn, F. & Parberry, I. *3D Math Primer for Graphics and Game Development* 2nd ed. 846 pp. (CRC Press, 2011).
19. Nikulin, E. A. *Computer geometry and machine graphics algorithms* Russian. 560 pp. (BHV-Petersburg, Saint-Petersburg, 2003).
20. Cisarzh, V. V. & Marusik, R. I. *Mathematical methods of computer graphics* Russian. 464 pp. (Fakt, Kiev, 2004).
21. Siciliano, B., Sciavicco, L., Villani, L. & Oriolo, G. *Robotics. Modelling, Planning and Control* 632 pp. (Springer-Verlag, London, 2010).
22. Faux, I. D. & Pratt, M. J. *Computational Geometry for Design and Manufacture* 329 pp. (1980).
23. De Berg, M., Cheong, O., van Kreveld, M. & Overmars, M. *Computational Geometry* Algorithms and Applications (Springer Berlin Heidelberg, 2008).
24. Boreskov, A. V. *Computer graphics programming. Modern OpenGL* Russian. 372 pp. (DMK Press, Moscow, 2019).
25. Manevich, V. A., Kotov, I. I. & Zengin, A. *Analytical geometry with image theory* Russian. 304 pp. (Vysshaya shkola, Moscow, 1969).
26. Dimentberg, F. M. *The Screw Calculus and Its Applications in Mechanics* 162 pp. (Foreign Technology Division translation, 1965).
27. Kotelnikov, A. P. *The helical number and some of its applications to geometry and mechanics* Russian. 222 pp. (ComKniga, Moscow, 2019).
28. Chelnokov, Y. Biquaternion solution of the kinematic control problem for the motion of a rigid body and its application to the solution of inverse problems of robot-manipulator kinematics. *Mechanics of Solids* **48,** 512. doi:10.3103/S0025654413010044 (Apr. 2013).
29. Hartley, R. & Zisserman, A. *Multiple View Geometry in Computer Vision* 2nd ed. 655 pp. (Cambridge University Press, Cambridge, 2004).
30. Lengyel, E. *Foundations of Game Engine Development.* 1: *Mathematics* 4 vols. 195 pp. (Terathon Software LLC, Lincoln, California, 2016).
31. Hammerlindl, A., Bowman, J. & Prince, T. *Asymptote. The Vector Graphics Language* https://asymptote.sourceforge.io/ (2023).

32. Kryachkov, Y. G. *Asymptote for beginners. Creating drawings in the language of vector graphics Asymptote* Russian. http://mif.vspu.ru/books/ASYfb.pdf (2020).
33. Volchenko, Y. M. *Scientific graphics in the Asymptote language* Russian. http://www.math.volchenko.com/AsyMan.pdf (2020).
34. Tedeev, V. A. *From painting to projective geometry* Russian. 231 pp. (Visha Shkola, Moscow, 1988).
35. Hartshorne, R. *Foundations of projective geometry* 161 pp. (W. A. Benjamin inc., New York, 1967).
36. Ilyin, V. A. & Poznyak, E. G. *Analytical geometry* Russian. 232 pp. (Nauka, Moscow, 1981).
37. Fedorchuk, V. V. *Analytical Geometry and Linear Algebra course* Russian. 328 pp. (MSU, Moscow, 1990).
38. Savelov, A. A. *Flat curves. Taxonomy, properties, and applications* Russian. 293 pp. (State Publishing House of Physico-mathematical Literature, Moscow, 1960).
39. Gevorkyan, M. N., Korolkova, A. V., Kulyabov, D. S. & Sevastyanov, L. A. Implementation of Analytic Projective Geometry for Computer Graphics. *Programming and Computer Software* **50,** 153–165. doi:10.1134/S0361768824020075 (Apr. 2024).

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Аналитическая проективная геометрия для компьютерной графики

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Аннотация. Мотивом к написанию данной работы послужила разработка авторами курса по компьютерной геометрии для студентов физико-математических специальностей. Под термином «компьютерная геометрия» здесь и далее понимаются математические основы машинной графики. Важно отдельно подчеркнуть, что разрабатываемый курс должен быть рассчитан на студентов второго года обучения и, следовательно, от них можно требовать лишь предварительное знание стандартного курса алгебры и математического анализа. Это накладывает определённые ограничения на излагаемый материал. При изучении тематической литературы было выяснено, что стандартом де факто в современной компьютерной графике стало использование проективного пространства и однородных координат. Однако авторы столкнулись с проблемой методологического характера — практически полным отсутствием подходящей учебной литератогы как на русском, так и на английском языках. Для представления собранной авторами информации по данному вопросу и была написана данная работа.

Ключевые слова: проективная геометрия, система Asymptote, координаты Плюккера, собственные и несобственные точки, прямые и плоскости