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On the stable approximate solution of the ill-posed boundary value problem for the Laplace equation with homogeneous conditions of the second kind on the edges at inaccurate data on the approximated boundary

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Abstract. In this paper, we consider the ill-posed continuation problem for harmonic functions from an ill-defined boundary in a cylindrical domain with homogeneous boundary conditions of the second type on the side faces. The value of the function and its normal derivative (Cauchy conditions) is known approximately on an approximated surface of arbitrary shape bounding the cylinder. In this case, the Cauchy problem for the Laplace equation has the property of instability with respect to the error in the Cauchy data, that is, it is ill-posed. On the basis of an idea about the source function of the original problem, the exact solution is represented as a sum of two functions, one of which depends explicitly on the Cauchy conditions, and the second one can be obtained as a solution of the Fredholm integral equation of the first kind in the form of Fourier series on the eigenfunctions of the second boundary value problem for the Laplace equation. To obtain an approximate stable solution of the integral equation, the Tikhonov regularization method is applied when the solution is obtained as an extremal of the Tikhonov functional. For an approximated surface, we consider the calculation of the normal to this surface and its convergence to the exact value depending on the error with which the original surface is given. The convergence of the obtained approximate solution to the exact solution is proved when the regularization parameter is compared with the errors in the data both on the inexactly specified boundary and on the value of the original function on this boundary. A numerical experiment is carried out to demonstrate the effectiveness of the proposed approach for a special case, for a flat boundary and a specific initial heat source (a set of sharpened sources).

Key words and phrases: ill-posed problem, Tikhonov regularization method, Cauchy problem for the Laplace equation, integral equation of the first kind

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Introduction

Among non-invasive diagnostic methods [1], thermal imaging stands out for its efficiency and accuracy with proper data processing. When carried out with the help of a thermal imager, a thermogram of the surface of the object can be obtained, showing the heat distribution on the surface of the object under investigation in the infrared range.

Corrected for various interferences in heat exchange processes and surface inhomogeneities of the observed object, the thermogram image conveys the structure of the heat-generating object, which makes it possible to assess various abnormalities in the state of the patient's internal organs during medical diagnosis.

The paper proposes a method for correcting the image on the thermogram based on a mathematical model that considers a homogeneous heat-conducting body in the form of a rectangular cylinder containing heat sources with a distribution density function independent of time, bounded by an arbitrary surface S on which a boundary condition of the third kind corresponding to convective heat exchange with the medium is set. The proposed model considers the case when there is no heat exchange on the lateral faces of the body — homogeneous boundary conditions of the second kind take place.

As a result of processing, the temperature distribution function on the plane (corrected thermogram) near the heat sources is constructed as a result of continuation of the temperature distribution from an arbitrary surface from which the original thermogram is taken. The corrected thermogram more accurately conveys the structure of the heat sources than the image on the original thermogram.

When obtaining the corrected thermogram as a result of processing, approaches similar to the continuation of gravitational fields in geophysics problems were used [2–4].

To obtain the result, the inverse problem to the mixed boundary value problem for the Poisson equation is solved, since the goal is to obtain information about inaccessible heat sources from data on a given surface. The inverse problem is ill-posed, because small errors in the initial data (in the initial thermogram, in the data on the surface) can lead to significant distortions of the result. To construct its stable approximate solution, the Tikhonov regularisation method is used [5–7].

1. Problem statement

Physical model: we consider a homogeneous heat-conducting body in the form of a rectangular cylinder bounded by the surface S and containing heat sources with a distribution density function independent of time.

These heat sources create a stationary temperature distribution in the body.

The object of study is the density function of heat source distribution.

On the surface S there is convective heat exchange with the medium described by the given function.

On the side faces of the cylinder we assume that there is no heat exchange.

Mathematical model: in a rectangular cylinder

$$D^{\infty} = \{(x, y, z) : 0 < x < l_x, 0 < y < l_y, -\infty < z < +\infty\} \tag{1}$$

we consider a cylindrical domain

$$D(F, +\infty) = \{(x, y, z) : 0 < x < l_x, 0 < y < l_y, F(x, y) < z < +\infty\}$$

bounded by a surface

$$S = \{(x, y, z) : 0 < x < l_x, 0 < y < l_y, z = F(x, y) < H\},\ F \in C^2. \tag{2}$$

In the domain $D(F, +\infty)$ we consider the following mixed boundary value problem for the Poisson equation

$$\begin{aligned} &\Delta u(M) = \rho(M),\ M \in D(F, +\infty) \\ &\frac{\partial u}{\partial n}\Big|_S = h(U_0 - u)\Big|_S = g, \\ &\frac{\partial u}{\partial n}\Big|_{x=0,\,l_x} = 0, \quad \frac{\partial u}{\partial n}\Big|_{y=0,\,l_y} = 0, \\ &u \text{ bounded when } z \to +\infty. \end{aligned} \tag{3}$$

Problem (3) corresponds to a stable temperature distribution created by heat sources with the distribution density function ρ.

On the surface S a third boundary condition is set and corresponds to a convective heat exchange with a medium of temperature U_0 with a constant coeffcient h. In this case we will consider the temperature of the medium as $U_0 = 0$. On the side faces $D(F, +\infty)$ there is no heat transfer — boundary conditions of the second kind take place.

We assume that the functions ρ, g are such that the solution of problem (3) exists in $C^2(D(F, \infty)) \cap C^1\overline{(D(F, \infty))}$. In particular, solving problem (3) allows us to find $u|_S$.

In addition, we assume that the support density ρ is in the $z > H$ domain.

Statement of the inverse problem:

Let ρ not be known.

But $u|_S = f$ is the initial thermogram.

We consider the surface S to be arbitrary

$$S = \{(x, y, z) : 0 < x < l_x, 0 < y < l_y, z = F(x, y) < H\},\ F \in C^2.$$

We need to find a continuous function ρ.

To solve the inverse problem, we apply the approach [2] used in geophysics problems.

The source of information about the density ρ will be the function $u|_{z=H}$ on the plane $z = H$, which is closer to the support of the density ρ than the surface S.

Since the support of the function ρ is outside the domain

$$D(F, H) = \{(x, y, z) : 0 < x < l_x, 0 < y < l_y, F(x, y) < z < H\}$$

then the solution of problem (3) satisfies the Laplace equation in this domain.

Assume that the functions f and g are taken from the set of solutions to the forward problem (3), so the solution to the inverse problem exists in $C^2(D(F, H)) \cap C^1\overline{(D(F, H))}$.

Then we obtain the continuation problem

$$\begin{aligned} &\Delta u(M) = 0,\ M \in D(F, H), \\ &u|_S = f, \quad \frac{\partial u}{\partial n}\Big|_S = -hu\Big|_S = g, \\ &\frac{\partial u}{\partial n}\Big|_{x=0,\,l_x} = 0, \quad \frac{\partial u}{\partial n}\Big|_{y=0,\,l_y} = 0. \end{aligned} \tag{4}$$

from the boundary S with homogeneous boundary conditions of the second kind on the side faces of D.

Note that in problem (4) the Cauchy conditions on the surface S of the form (2) are given, i.e., the boundary values f of the desired function u and the values of its normal derivative are given, so problem (4) has a single solution.

The boundary $z = H$ of the domain $D(F,H)$ is free, and, as a Cauchy problem for the Laplace equation, is unstable with respect to errors in the data, i.e., ill-posed.

In the inverse problem, the function f corresponds to the original thermogram obtained with a thermal imager.

The function $u|_{z=H}$ will be considered as a corrected thermogram, i.e., a source of more accurate information about the density ρ.

2. Exact solution of the inverse problem

Based on the approach of [8], an explicit representation of the exact solution of problem (4) is constructed similarly to [9].

Let us consider the source function $\varphi(M,P)$ of the Neumann problem for the Laplace equation in an infinite cylinder D^∞ of the form (1), i.e. the solution of the problem

$$\begin{gathered}\Delta w(P) = -\delta_{MP},\ P \in D^\infty,\\ \frac{\partial w}{\partial n}\Big|_{x=0,\,l_x} = 0, \quad \frac{\partial w}{\partial n}\Big|_{y=0,\,l_y} = 0,\\ \frac{\partial w}{\partial z} \to \frac{1}{l_x l_y} \text{ at } z \to +\infty, \quad \frac{\partial w}{\partial z} \to 0 \text{ at } z \to -\infty.\end{gathered} \tag{5}$$

for which the necessary solvability condition is fulfilled

$$\int\limits_S \frac{\partial w}{\partial n} dS - \int\limits_V \Delta w\, dV = 0.$$

The source function $\varphi(M,P)$ of problem (5) can be represented as

$$\varphi(M,P) = \frac{1}{4\pi r_{MP}} + W(M,P)$$

where r_{MP} is the distance between points M and P, $W(M,P)$ is a harmonic function on P. The source function can also be obtained [9] as a Fourier series under the condition $z_M < \min\limits_{(x,y)} F(x,y) < z_P$

$$\begin{gathered}\tilde{\varphi}(M,P) = \frac{1}{2l_x l_y} C + \frac{2}{l_x l_y} \sum_{n,m=0,n^2+m^2\neq 0}^{\infty} \varepsilon_n \varepsilon_m \frac{e^{-k_{nm}|z_M - z_P|}}{k_{nm}} \times \\ \times \cos\frac{\pi n x_M}{l_x} \cos\frac{\pi m y_M}{l_y} \cos\frac{\pi n x_P}{l_x} \cos\frac{\pi m y_P}{l_y}\end{gathered} \tag{6}$$

where

$$k_{nm} = \pi\sqrt{\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2}}, \qquad \varepsilon_n = \begin{cases} 1 & \text{n} \neq 0, \\ 0,5 & \text{n} = 0. \end{cases} \tag{7}$$

Taking into account homogeneous boundary conditions for $\tilde{\varphi}$ and u on the side faces of the cylindrical domain $D(F,H)$, we obtain

$$\begin{aligned} u(M) = &\int\limits_S \left[g(P)\tilde{\varphi}(M,P) - f(P)\frac{\partial\tilde{\varphi}}{\partial n_P}(M,P)\right] d\sigma_P + \\ &+ \int\limits_{\Pi(H)} \left[\frac{\partial u}{\partial n}(P)\tilde{\varphi}(M,P) - u(P)\frac{\partial\tilde{\varphi}}{\partial n_P}(M,P)\right] d\sigma_P,\end{aligned}$$

where

$$\Pi(H) = \{(x, y, z) : 0 < x < l_x, 0 < y < l_y, z = H\} \tag{8}$$

By introducing the notations

$$\Phi(M) = -\int\limits_S \left[g(P)\tilde{\varphi}(M,P) - f(P)\frac{\partial\tilde{\varphi}}{\partial n_P}(M,P)\right] d\sigma_P,\ M \in D(-\infty, H), \tag{9}$$

$$v(M) = \int\limits_{\Pi(H)} \left[\frac{\partial u}{\partial n}(P)\tilde{\varphi}(M,P) - u(P)\frac{\partial\tilde{\varphi}}{\partial n_P}(M,P)\right] d\sigma_P,\ M \in D(-\infty, H), \tag{10}$$

we obtain the solution of problem (4) in the form

$$u(M) = v(M) - \Phi(M), \quad M \in D(F, H) \tag{11}$$

where $\Phi(M)$ is computed from known functions f and g and can be considered as a known function.

The function $v(M)$ can be viewed as a solution to the problem

$$\begin{aligned}
&\Delta v(M) = 0,\ M \in D(-\infty, H),\\
&v|_{z=H} = v_H, \quad \frac{\partial v}{\partial n}\Big|_{x=0,l_x} = 0, \quad \frac{\partial v}{\partial n}\Big|_{y=0,\,l_y} = 0,\\
&v \text{ is bounded at } z \to -\infty,
\end{aligned}$$

which can be obtained by the Fourier method, and the function v can be expressed through v_H

$$v(M) = \sum_{n,m=0,\ n^2+m^2\neq 0}^{\infty} (\tilde{v}_H)_{nm} e^{k_{nm}(z_M - H)} \cos\frac{\pi n x_M}{l_x}\cos\frac{\pi m y_M}{l_y}, \tag{12}$$

$$(\tilde{v}_H)_{nm} = \frac{4\varepsilon_n\varepsilon_m}{l_x l_y}\int\limits_0^{l_x}\int\limits_0^{l_y} v_H(x,y)\cos\frac{\pi n x}{l_x}\cos\frac{\pi m y}{l_y} dx dy. \tag{13}$$

The function v_H satisfies the Fredholm integral equation of the first kind, taking into account homogeneous boundary conditions for $\tilde{\varphi}$ and u, and notations (9) and (10), we obtain

$$v(M) = \Phi(M), \quad M \in D(-\infty, F) \tag{14}$$

Let $a < \min\limits_{(x,y)} F(x, y)$ and $M \in \Pi(a)$, where $\Pi(a)$ is a domain of the form (8) at $z = a$, then from (14) and (12) we obtain a system of equations with respect to the Fourier coefficients of the function v_H

$$\sum_{n,m=0,\ n^2+m^2\neq 0}^{\infty} (\tilde{v}_H)_{nm} e^{k_{nm}(a-H)} \cos\frac{\pi n x_M}{l_x}\cos\frac{\pi m y_M}{l_y} = \Phi(M). \tag{15}$$

Using (13), equation (15) can also be written as an integral equation of the first kind

$$\int\limits_{\Pi(H)} G(M,P) v_H(P) dx_P dy_P = \Phi(M),\ M \in \Pi(a), \tag{16}$$

where the kernel of the integral operator has the form

$$\begin{aligned}
G(M,P) = \frac{4}{l_x l_y}&\sum_{n,m=0,\ n^2+m^2\neq 0}^{\infty} \varepsilon_n\varepsilon_m e^{-k_{nm}(H-a)}\times\\
&\times\cos\frac{\pi n x_M}{l_x}\cos\frac{\pi m y_M}{l_y}\cos\frac{\pi n x_P}{l_x}\cos\frac{\pi m y_P}{l_y}.
\end{aligned} \tag{17}$$

Equation (16) will also be written in the form

$$Gv_H = \Phi(a).$$

From equation (16), taking into account the expansion (17) at $z_M = a$, we obtain the relation between the Fourier coefficients of the single solution v_H and the Fourier coefficients of the right-hand side

$$(\tilde{v}_H)_{nm} e^{-k_{nm}(H-a)} = \tilde{\Phi}_{nm}(a), \tag{18}$$

where $\tilde{\Phi}_{nm}(a)$ are the Fourier coefficients of the function $\Phi(M)|_{M\in\Pi(a)}$:

$$\tilde{\Phi}_{nm}(a) = \frac{4\varepsilon_n\varepsilon_m}{l_x l_y} \int\limits_{\Pi(a)} \Phi(x,y,a) \cos\frac{\pi n x}{l_x} \cos\frac{\pi m y}{l_y} dxdy.$$

Substituting the Fourier coefficients $(\tilde{v}_H)_{nm}$ from (18) into the series (12), we obtain the function v in the domain $D(-\infty, H)$

$$v(M) = \sum_{n,m=0,\ n^2+m^2\neq 0}^{\infty} \tilde{\Phi}_{nm}(a) e^{k_{nm}(z_M-a)} \cos\frac{\pi n x_M}{l_x} \cos\frac{\pi m y_M}{l_y}. \tag{19}$$

The series (19), like series (12), converges uniformly in $D(-\infty, H-\varepsilon)$ for any $\varepsilon > 0$, if the solution of problem (4) exists given f and g.

Formula (11), where the functions v and Φ are of the form (19) and (9), respectively, gives an explicit expression for the solution of problem (4).

$$\begin{gathered} u(M) = \sum_{n,m=0,\ n^2+m^2\neq 0}^{\infty} \tilde{\Phi}_{nm}(a) e^{k_{nm}(z_M-a)} \cos\frac{\pi n x_M}{l_x} \cos\frac{\pi m y_M}{l_y} + \\ + \int\limits_S \left[g(P)\tilde{\varphi}(M,P) - f(P)\frac{\partial\tilde{\varphi}}{\partial n_P}(M,P) \right] d\sigma_P. \end{gathered} \tag{20}$$

3. Approximately given surface S. Calculation of the normal to the surface

Formula (20) gives an explicit expression for the solution of problem (4).

Since the surface S of the form (2)

$$S = \{(x,y,z) : 0 < x < l_x, 0 < y < l_y, z = F(x,y) < H\}$$

is given by the equation $z = F(x,y)$, the function f, defined on S, can be viewed as a function of the variables x and y on the rectangle Π:

$$\Pi = \{(x,y) : 0 < x < l_x, 0 < y < l_y\}. \tag{21}$$

In applied problems, the surface S, which is a part of the boundary of the domain, can be defined on the basis of measurements, i.e. approximated. If the surface S is defined with an error, the calculation of the integral (9) is complicated by the necessity to calculate the normal to such a surface. The problem of calculating the normal to the surface, in other words, the gradient of a function given approximatively, is ill-posed as a problem of numerical differentiation.

To compute the function Φ (9), it is necessary to compute [10] the vector function $\mathbf{n}_1$ of the normal to the surface S, which is the gradient of the function $F(x;y)-z$ of the following form

$$\mathbf{n}_1=(F'_x,F'_y,-1)=iF'_x+jF'_y-k=grad(F(x,y)-z)=\nabla_{xy}F-k$$

$$\frac{\partial\tilde{\varphi}}{\partial n}=(\mathbf{n},\nabla\tilde{\varphi}),\quad \mathbf{n}=\frac{\mathbf{n}_1}{n_1}$$

$$d\sigma_P=n_1(x_P,y_P)dx_Pdy_P$$

The integral (9) can be represented as

$$\Phi(M)=-\int\limits_{\Pi}[g(P)\tilde{\varphi}(M,P)-f(P)\left(\nabla_P\tilde{\varphi}(M,P),\mathbf{n}(P)\right)]\,n_1(P)dx_Pdy_P,$$

$$\Phi(M)=-\int\limits_{\Pi}[g(P)\tilde{\varphi}(M,P)n_1(P)-f(P)\left(\nabla_P\tilde{\varphi}(M,P),n_1(P)\right)]\,dx_Pdy_P. \tag{22}$$

Let us assume that the surface S is defined with some error, namely: instead of the exact function F, there is a function F^μ such that

$$\|F^\mu-F\|_{L_2(\Pi)}\le\mu. \tag{23}$$

As an approximation to the function $\nabla_{xy}F$, calculated from the known function F^μ, related to F by the condition (23), consider the gradient from the extremum of the functional

$$N^\beta[W]=\|W-F^\mu\|^2_{L_2(\Pi)}+\beta\|\nabla W\|^2_{L_2(\Pi)} \tag{24}$$

where Π is a plane of the form (21).

We will consider such surfaces S for which

$$F|_{x=0,\,l_x}=F|_{y=0,\,l_y}=0.$$

This condition, in particular, takes place in the case when S can be considered as a perturbation of the main plane $z=0$. Then the extremal of the functional (24) satisfies the Euler equation

$$\begin{gathered}-\beta\Delta W+W=F^\mu,\\ W|_{x=0,\,l_x}=W|_{y=0,\,l_y}=0.\end{gathered} \tag{25}$$

Solving the problem (25) by Fourier method, we obtain:

$$W^\mu_\beta(x,y)=\sum_{n,m=1}^{\infty}\frac{\tilde{F}^\mu_{nm}}{1+\beta k^2_{nm}}\sin\frac{\pi nx}{l_x}\sin\frac{\pi my}{l_y}. \tag{26}$$

It is easy to see that the series (26) converges uniformly to Π (21).

As an approximation of the gradient of the function F^μ we will consider the vector function

$$\begin{gathered}\nabla_{xy}W^\mu_\beta(x,y)=\sum_{n,m=1}^{\infty}\frac{\tilde{F}^\mu_{nm}}{1+\beta k^2_{nm}}\times\\ \times\left(\mathbf{i}\frac{\pi n}{l_x}\cos\frac{\pi nx}{l_x}\sin\frac{\pi my}{l_y}+\mathbf{j}\frac{\pi m}{l_y}\cos\frac{\pi my}{l_y}\sin\frac{\pi nx}{l_x}\right).\end{gathered} \tag{27}$$

The series (27) is also uniformly convergent on Π (21).

Let F^- be an odd-periodic continuation of a function F with period $2l_x$, on the variable x and with period $2l_y$ on the variable y, i.e.

$$F^-(-x,y) = -F^-(x,y), \quad (x,y) \in \Pi,$$
$$F^-(x,-y) = -F^-(x,y), \quad F^-(-x,y) = F^-(x,-y),$$
$$F^-(x+2l_x n, y+2l_y m) = F^-(x,y) = F(x,y).$$

Similarly to [11–13] it can be proved

Theorem 1. *Let* $F^- \in C^2(R^2)$, $\beta = \beta(\mu) > 0$, $\beta(\mu) \to 0$ *and* $\mu/\sqrt{\beta(\mu)} \to 0$ *if* $\mu \to 0$. *Then*

$$\|\nabla_{xy} W^{\mu}_{\beta(\mu)} - \nabla_{xy} F\|_{L_2(\Pi)} \le \frac{\mu}{2\sqrt{\beta}} + \frac{\sqrt{\beta}}{2}\|\Delta F\|_{L_2(\Pi)} \to 0 \text{ at } \mu \to 0.$$

Based on the theorem, we can use formula (27) for approximate calculation of the normal to the surface:

$$\mathbf{n}^{\mu}_{1,\beta} = \nabla_{xy} W^{\mu}_{\beta} - k. \tag{28}$$

It follows from the proof of the theorem that under the conditions formulated in the theorem

$$\|\mathbf{n}^{\mu}_{1,\beta} - \mathbf{n}_1\|_{L_2(\Pi)} = \|\nabla_{xy} W^{\mu}_{\beta} - \nabla_{xy} F\|_{L_2(\Pi)} \le \frac{\mu}{2\sqrt{\beta}} + \frac{\sqrt{\beta}}{2}\|\Delta F\|_{L_2(\Pi)}$$

the maximum in β of the expression on the right-hand side is reached when $\beta(\mu) = \frac{\mu}{\|\Delta F\|}$, and thus, denoting by (28)

$$\mathbf{n}^{\mu}_1 = \mathbf{n}^{\mu}_{1,\beta} = \nabla_{xy} W^{\mu}_{\beta(\mu)} - k, \tag{29}$$

we obtain:

$$\|\mathbf{n}^{\mu}_1 - \mathbf{n}_1\|_{L_2(\Pi)} \le \sqrt{\|\Delta F\|\mu} \xrightarrow[\mu\to 0]{} 0. \tag{30}$$

It is also not difficult to obtain an approximate estimate

$$\|W_{\beta(\mu)} - F\|_{L_2(\Pi)} \le 2\mu.$$

The surface defined by the equation $z = W^{\mu}_{\beta(\mu)}(x,y)$ we denote as

$$S^{\mu} = \{(x,y,z) : 0 < x < l_x, 0 < y < l_y, z = W^{\mu}_{\beta(\mu)}(x,y)\}.$$

4. Stable approximate solution in case of inaccurate data on the approximated boundary

Now let the functions f and g in problem (4) be approximated, namely: let f^{δ} and g^{δ} be given such that

$$\|f^{\delta} - f\|_{L_2(\Pi)} \le \delta, \quad \|g^{\delta} - g\|_{L_2(\Pi)} \le \delta. \tag{31}$$

Then the function Φ of the form (22) at exactly given surface S can be calculated with some error:

$$\begin{aligned} \Phi^{\delta}(M) = -\int_{\Pi} \big[& g^{\delta}(P)\tilde{\varphi}(M,P)n_1(P) - \\ & -f^{\delta}(P)\left(\nabla_P \tilde{\varphi}(M,P), \mathbf{n}^{\mu}_1(P)\right)\big]_{P\in S} dx_P dy_P. \end{aligned} \tag{32}$$

Obtaining an approximate solution of problem (4) under conditions (31) and its convergence to the exact solution (20) for a precisely defined surface S is considered in [9].

When approximating the surface S (23)

$$\|F^{\mu} - F\|_{L_2(\Pi)} \le \mu$$

the right-hand side of (16) can be calculated approximating by formula (32) taking into account (29):

$$\begin{aligned} \Phi^{\delta,\mu}(M) = -\int\limits_{\Pi} \big[g^{\delta}(P)\tilde{\varphi}(M,P)n_1^{\mu}(P) - \\ -f^{\delta}(P)\left(\nabla_P\tilde{\varphi}(M,P), \mathbf{n}_1^{\mu}(P)\right)\big]_{P=P(x,y,W_{\beta}^{\mu})\in S^{\mu}}\, dx_P dy_P. \end{aligned} \tag{33}$$

Let us evaluate the difference

$$\begin{aligned} |\Phi^{\delta,\mu}(M) - \Phi(M)| \le |\Phi^{\delta,\mu}(M) - \Phi^{\delta}(M)| + \\ + |\Phi^{\delta}(M) - \Phi(M)|, \quad M \in \Pi(a), \end{aligned} \tag{34}$$

where $\Phi^{\delta,\mu}$, Φ^{δ}, Φ are functions of the form (33), (32), (22), respectively. We evaluate the first difference in (34) by subtracting and adding the function:

$$\begin{aligned} \Phi_1^{\delta,\mu}(M) = \int\limits_0^{l_x} dx \int\limits_0^{l_y} dy \left[g^{\delta}(x,y)\tilde{\varphi}(M,P^{\mu})n_1(x,y) - \right. \\ \left. -f^{\delta}(x,y)\left(\nabla_P\tilde{\varphi}(M,P^{\mu}), \mathbf{n}_1\right)\right], \; P^{\mu} = (x,y,W_{\beta}^{\mu}(x,y)), \; M = (x_M, y_M, a) \end{aligned}$$

differing from the function $\Phi^{\delta,\mu}(M)$ by the exact normal:

$$\begin{aligned} |\Phi^{\delta,\mu}(M) - \Phi^{\delta}(M)| = |\Phi^{\delta,\mu}(M) - \Phi_1^{\delta,\mu}(M) + \Phi_1^{\delta,\mu}(M) - \Phi^{\delta}(M)| = \\ = \left| \int\limits_0^{l_x} dx \int\limits_0^{l_y} dy \left[f^{\delta}(x,y)\left(\mathbf{n}_1^{\mu}(x,y) - \mathbf{n}_1(x,y), \nabla_P\tilde{\varphi}(M,P^{\mu})\right) - \right.\right. \\ -g^{\delta}(x,y)\tilde{\varphi}(M,P^{\mu})\left(n_1^{\mu}(x,y) - n_1(x,y)\right) + \\ +f^{\delta}(x,y)\left(\mathbf{n}_1(x,y), \nabla_P\left(\tilde{\varphi}(M,P^{\mu}) - \tilde{\varphi}(M,P)\right)\right) - \\ \left.\left. - g^{\delta}(x,y)\left(\tilde{\varphi}(M,P^{\mu}) - \tilde{\varphi}(M,P)\right)n_1(x,y)\right] \right|. \end{aligned}$$

Replacing the modulus by the sum of moduli and evaluating the difference of functions using the Lagrange formula, we obtain:

$$\begin{aligned} |\Phi^{\delta,\mu}(M) - \Phi^{\delta}(M)| \le \int\limits_0^{l_x} dx \int\limits_0^{l_y} dy \left[|f^{\delta}(x,y)| \cdot |\mathbf{n}_1^{\mu}(x,y) - \mathbf{n}_1(x,y)| \cdot |\nabla_P\tilde{\varphi}(M,P^{\mu})| + \right. \\ + |g^{\delta}(x,y)| \cdot |\tilde{\varphi}(M,P^{\mu})| \cdot |n_1^{\mu}(x,y) - n_1(x,y)| + \\ + |f^{\delta}(x,y)| \cdot |\mathbf{n}_1(x,y)| \cdot \left|\frac{\partial}{\partial z_P}\nabla_P\tilde{\varphi}(M,P^*)\right| \cdot |W_{\beta}^{\mu} - F| + \\ \left. + |g^{\delta}(x,y)| \cdot \left|\frac{\partial\tilde{\varphi}(M,P^*)}{\partial z_P}\right| |W_{\beta}^{\mu} - F| \cdot |n_1(x,y)| \right] \le \end{aligned}$$

taking out the maxima, we use the Cauchy–Bunyakovsky inequality:

$$\le \left(\max_{M\in\Pi(a)} |\nabla_P \tilde{\varphi}(M,P^\mu)| \cdot \|f^\delta\| + \max_{M\in\Pi(a)} |\tilde{\varphi}(M,P^\mu)| \cdot \|g^\delta\| \right) \|\mathbf{n}_1^\mu - \mathbf{n}_1\| +$$
$$+ \left(\max_{M\in\Pi(a)} \left|\frac{\partial}{\partial z_P}\nabla_P \tilde{\varphi}(M,P^*)\right| \cdot |n_1| \|f^\delta\| + \max_{M\in\Pi(a)} \left|\frac{\partial}{\partial z_P}\tilde{\varphi}(M,P^*)\right| \cdot |n_1| \|g^\delta\| \right) \|W_\beta^\mu - F\| \le$$

By virtue of the inequalities $\|f^\delta\| - \|f\| \le |\|f^\delta\| - \|f\|| \le \|f^\delta - f\| \le \delta$ we obtain $\|f^\delta\| \le \|f\| + \delta$ and thus,

$$\le \left(\max_{M\in\Pi(a)} |\nabla_P \tilde{\varphi}(M,P^*)| \cdot (\|f\| + \delta) + \max_{M\in\Pi(a)} |\tilde{\varphi}(M,P^\mu)| \cdot (\|g\| + \delta) \right) \|\mathbf{n}_1^\mu - \mathbf{n}_1\| +$$
$$+ \left(\max_{M\in\Pi(a)} \left|\frac{\partial}{\partial z_P}\nabla_P \tilde{\varphi}(M,P^*)\right| \cdot |n_1| (\|f\| + \delta) + \max_{M\in\Pi(a)} \left|\frac{\partial}{\partial z_P}\tilde{\varphi}(M,P^*)\right| \cdot |n_1| (\|g\| + \delta) \right) \|W_\beta^\mu - F\|.$$

The maximums are evaluated by constants. Since we are interested in the behaviour of the regularized solution of problem (4) when $\delta \to 0$, we can assume that $\delta \le \delta_0$, and thus, taking into account (30)

$$|\Phi^{\delta,\mu}(M) - \Phi^\delta(M)|_{M\in\Pi(a)} \le C_1 \|\mathbf{n}_1^\mu - \mathbf{n}_1\| + C_2 \|W_\beta^\mu - F\|.$$

Consider the difference

$$\|W_\beta^\mu - F\| \le \|W_\beta^\mu - W_\beta\| + \|W_\beta - F\|, \tag{35}$$

where W_β is calculated by formula (26) at $\mu = 0$

$$W_\beta = \sum_{n,m=1}^{\infty} \frac{\tilde{F}_{nm}}{1 + \beta k_{nm}^2} \sin\frac{\pi n x}{l_x} \sin\frac{\pi m y}{l_y}.$$

The evaluation of the first difference in (35) gives:

$$\|W_\beta^\mu - W_\beta\|^2 = \int_0^{l_x} dx \int_0^{l_y} dy \left| \sum_{n,m=1}^{\infty} \frac{\tilde{F}_{nm}^\mu - \tilde{F}_{nm}}{1 + \beta k_{nm}^2} \sin\frac{\pi n x}{l_x} \sin\frac{\pi m y}{l_y} \right|^2.$$

Using the orthogonality of the trigonometric system, we obtain:

$$\|W_\beta^\mu - W_\beta\|^2 = \frac{l_x l_y}{4} \sum_{n,m=1}^{\infty} \frac{(\tilde{F}_{nm}^\mu - \tilde{F}_{nm})^2}{(1 + \beta k_{nm}^2)^2} \le$$
$$\le \frac{l_x l_y}{4} \sum_{n,m=1}^{\infty} (\tilde{F}_{nm}^\mu - \tilde{F}_{nm})^2 = \|\tilde{F}_{nm}^\mu - \tilde{F}_{nm}\|^2 \le \mu^2.$$

Similarly, to evaluate the second difference in (35) when $\beta = \frac{\mu}{\|\Delta F\|}$ we obtain:

$$\|W_\beta^\mu - F\|^2 \le \mu^2.$$

Combining the estimates, we obtain

$$\|W_\beta - F\| \le 2\mu. \tag{36}$$

From (30), (35) and (36) we obtain

$$\begin{aligned} |\Phi^{\delta,\mu}(M) - \Phi^\delta(M)| &\le C\|\mathbf{n}_1^\mu - \mathbf{n}_1\| + C\|W_\beta^\mu - F\| \le \\ &\le C\mu + C\sqrt{\mu} \le C\sqrt{\mu}(1 + C\mu) = C_1\sqrt{\mu}, \quad M \in \Pi(a). \end{aligned} \tag{37}$$

To evaluate the second difference, we obtain the same way as in [14]:

$$\begin{gathered}|\Phi^{\delta}(M)-\Phi(M)| = \left| \int\limits_{\Pi} \left[(f^{\delta}(P) - f(P))(\mathbf{n}_1(P), \nabla_P \tilde{\varphi}(M,P)) - \right.\right. \\ \left.\left. - (g^{\delta}(P) - g(P))\tilde{\varphi}(M,P) n_1(P) \right]_{P\in S} dx_P dy_P \right| \leq \\ \leq Const_{n_1} \max_{M\in\Pi(a)} \|\nabla\tilde{\varphi}(M)\|_{L_2(S)} \cdot \|f^{\delta} - f\|_{L_2(\Pi)} + \\ + Const^1_{n_1} \max_{M\in\Pi(a)} \|\tilde{\varphi}(M)\|_{L_2(S)} \cdot \|g^{\delta} - g\|_{L_2(\Pi)} \leq C_2\delta, \quad M \in \Pi(a).\end{gathered} \tag{38}$$

From (37) and (38), we obtain for the estimate (34):

$$\max_{M\in\Pi(a)} |\Phi^{\delta,\mu}(M) - \Phi(M)| \leq C_1\sqrt{\mu} + C_2\delta = \Delta(\mu,\delta) \xrightarrow[\substack{\mu\to 0\\ \delta\to 0}]{} 0 \tag{39}$$

Thus, the right part of the integral equation (16) is known with some error Δ, having the structure (39). The stable approximate solution of the problem (4) is constructed on the basis of the search for extrema of the Tikhonov functional [5], and can be obtained in the form of

$$u_{\alpha}^{\delta,\mu}(M) = v_{\alpha}^{\delta,\mu}(M) - \Phi^{\delta,\mu}(M),\ M \in D(H,F) \tag{40}$$

where $\Phi^{\delta,\mu}$ is a function of the form (33), and $v_{\alpha}^{\delta,\mu}$ (19) has the form:

$$v_{\alpha}^{\delta,\mu}(M) = \sum_{n,m=0,\ n^2+m^2\neq 0}^{\infty} \frac{\tilde{\Phi}_{nm}^{\mu,\delta}(a) e^{k_{nm}(z_M - a)}}{1 + \alpha e^{2k_{nm}(H-a)}} \cos\frac{\pi n x_M}{l_x} \cos\frac{\pi m y_M}{l_y}. \tag{41}$$

Here $\tilde{\Phi}_{nm}^{\mu,\delta}(a)$ is the Fourier coefficients of the function $\Phi_{nm}^{\mu,\delta}(a)\big|_{M\in\Pi(a)}$:

$$\tilde{\Phi}_{nm}^{\mu,\delta}(a) = \frac{4\varepsilon_n\varepsilon_m}{l_x l_y} \int\limits_0^{l_x} dx \int\limits_0^{l_y} dy\, \Phi^{\delta,\mu}(x,y,a) \cos\frac{\pi n x}{l_x} \cos\frac{\pi n y}{l_y}$$

and α is the regularisation parameter. According to the notations introduced above, the value of a is chosen such that

$$a < \min_{(x,y)\in\Pi} F(x,y).$$

For the considered boundary conditions of the second kind on the side faces of the cylinder, taking into account [8], there is a theorem of convergence of the approximate solution to the exact one when the regularisation parameter, consistent with the accuracy of the initial data, tends to zero.

Theorem 2. *Let the solution of problem* (4) *exist in the domain* $D(H,F)$, $\alpha = \alpha(\Delta)$, $\alpha(\Delta) \to 0$, $\Delta/\sqrt{\alpha(\Delta)} \to 0$ *at* $\Delta \to 0$. *Then the function* $u_{\alpha(\Delta)}$ *of the form* (40), *where according to* (39) $\Delta = \Delta(\mu,\delta) = C_1\sqrt{\mu} + C_2\delta$, *converges uniformly to the exact solution of problem* (4) *at* $\delta \to 0$, $\mu \to 0$ *in the domain* $D(F+\varepsilon, H-\varepsilon)$, *where* $\varepsilon > 0$ *is some fixed arbitrarily small number.*

Proof. Let's evaluate the difference

$$|u_{\alpha(\delta)}^{\delta,\mu} - u| \leq |v_{\alpha}^{\delta,\mu} - v| + |\Phi^{\delta,\mu} - \Phi|$$

in the domain $G(H+\varepsilon, F-\varepsilon)$.

The second difference is evaluated similarly to (38) when replacing $\Pi(a)$ by $D(H+\varepsilon, F-\varepsilon)$.

For the difference $v_\alpha^\delta - v$ we obtain

$$|v_\alpha^\delta - v| \le |v_\alpha^\delta - v_\alpha| + |v_\alpha - v|,$$

where v_α is a function of the form (41) at $\delta = 0$.

We estimate $v_\alpha^\delta - v_\alpha$ in $D(H+\varepsilon, F-\varepsilon)$.

$$|v_{\alpha(M)}^\delta - v_{\alpha(M)}| \le \left| \sum_{n,m=0}^{\infty} \frac{e^{k_{nm}(z_M - a)}}{1+\alpha e^{2k_{nm}(H-a)}} \right| \cdot 4 \max_{P\in\Pi(a)} |\Phi^\delta(P) - \Phi(P)| \le$$
$$\le C_1 \cdot \delta \sum_{n,m=0}^{\infty} \frac{e^{k_{nm}(H-\varepsilon-a)}}{1+\alpha e^{2k_{nm}(H-a)}} \le C_1 \cdot \delta \cdot \max_{x'} \left[\frac{e^{x'}}{1+\alpha e^{2x'}} \right] \sum_{n,m=0}^{\infty} e^{-k_{nm}\varepsilon} \le C_2 \cdot \frac{\Delta(\mu,\delta)}{\sqrt{\alpha}}.$$

For the difference $v_\alpha - v$ we obtain

$$|v_\alpha - v| = \left| \sum_{n,m=0}^{\infty} \frac{\alpha e^{2k_{nm}(H-a)} e^{\pi k_{nm}(z_M - H)}}{1+\alpha e^{2k_{nm}(H-a)}} (v_H)_{nm} \cos\frac{\pi n x_M}{l_x} \frac{\pi m y_M}{l_y} \right| \le$$
$$\le \left[\sum_{n,m=0}^{\infty} \frac{\alpha e^{2k_{nm}(H-a)}}{1+\alpha e^{2k_{nm}(H-a)}} e^{-\varepsilon k_{nm}} \right]^{\frac{1}{2}} \cdot \|v_H\|_{L_2}.$$

Since the parameter-dependent series is majorised by a convergent numerical series

$$\sum_{n,m=0}^{\infty} e^{-\varepsilon k_{nm}}$$

then a limit transition on α is possible, and thus,

$$|v_\alpha^{\delta,\mu} - v| \to 0 \text{ при } \alpha(\delta) \to 0.$$

Using 2, the convergence of the approximate solution (40) to the exact solution (20) of the problem (4) of continuation from the boundary S (2) is proved.

5. Numerical solution of the inverse problem for the case of flat boundary

Let us demonstrate the effectiveness of the proposed approach of solving the problem (4) of continuation from the boundary S, on which the third boundary condition corresponding to convective heat exchange with the medium of temperature U_0 with a constant coefficient h is defined

$$\left.\frac{\partial u}{\partial n}\right|_S = g = h(U_0 - f)|_S$$

and the surface S itself is the plane $\Pi(0)$ $z = 0$ for the following conditions: $U_0 = 0$, $h = 0.4$, $l_x = 60$, $l_y = 60$, $H = 1.5$

Let the function $\rho(M)$ in the direct problem (3) correspond to three point sources in the plane $\Pi(H)$: $(x_1, y_1) = (30, 32)$, $(x_2, y_2) = (30, 30)$, $(x_3, y_3) = (32, 30)$.

For this case, using the results of [15], the function specifying the boundary condition for problem (3) with accuracy up to a constant can be obtained in the form of

$$f(x,y)=\sum_{n,m=0}^{\infty}\sum_{i=0}^{3}q_i\varepsilon_n\varepsilon_m\frac{e^{-k_{nm}H}}{k_{nm}+h}\cos\frac{\pi m x_i}{l_x}\cos\frac{\pi m y_i}{l_y}\cos\frac{\pi m x}{l_x}\cos\frac{\pi m y}{l_y}, \tag{42}$$

where k_{nm}, ε_n and ε_m are calculated by formula (7) and $q_i = 100$, $i = 1,2,3$.

We will solve the inverse continuation problem (4), assuming that the value of the function on the boundary f^δ is given approximated, its values will be determined on the basis of the function $f(x,y)$ (42) and a randomly given relative error within 3%.

In the applied approach, the solution can be obtained by applying (40) and (41).

According to (32)

$$\Phi^\delta(M)=-\int\limits_{\Pi(0)}\big[g^\delta(P)\tilde\varphi(M,P)n_1(P)-$$
$$-f^\delta(P)\left(\nabla_P\tilde\varphi(M,P),\mathbf{n}_1^\mu(P)\right)\big]_{P\in S}dx_Pdy_P$$

where $\mathbf{n}_1=(F'_x,F'_y,-1)$ and $n_1=|\mathbf{n}_1|=\sqrt{1+(F'_x)^2+(F'_y)^2}$.

For the assumption that S is a plane $z=F(x,y)=0$, $\mathbf{n}_1=(0,0,-1)$ and $n_1=1$.

According to (6) the source function with accuracy up to constant

$$\tilde\varphi(M,P)=\frac{2}{l_xl_y}\sum_{n,m=0,n^2+m^2\neq0}^{\infty}\varepsilon_n\varepsilon_m\frac{e^{-k_{nm}|z_M-z_P|}}{k_{nm}}\cos\frac{\pi n x_M}{l_x}\cos\frac{\pi m y_M}{l_y}\cos\frac{\pi n x_P}{l_x}\cos\frac{\pi m y_P}{l_y}$$

and

$$grad_P\varphi(M,P)|_{P\in S}=\frac{-2}{l_xl_y}\sum_{n,m=0}^{\infty}\varepsilon_n\varepsilon_m e^{-k_{nm}|z_M-z_P|}\cos\frac{\pi n x_M}{l_x}\cos\frac{\pi m y_M}{l_y}\cos\frac{\pi n x_P}{l_x}\cos\frac{\pi m y_P}{l_y}.$$

To obtain numerical results, problems (3), (4) are discretised.

We will assume that the rectangles $\Pi(0)$, $\Pi(H)$ and $\Pi(a)$, $a=-0.6$, are covered by a uniform grid $(N_x+1)\times(N_y+1)$ of points such that

$$x_i=i\frac{l_x}{N_x},\ i=0,\dots,N_x,$$
$$y_j=j\frac{l_y}{N_y},\ j=0,\dots,N_y$$

We will consider $N_x=N_y=60$.

As a result of descretisation, using the approach [16], we obtain

$$\tilde\Phi^\delta_{nm}(a)=\left[1+\frac{h}{k_{nm}}\right]\frac{2}{N_xN_y}\varepsilon_n\varepsilon_m e^{k_{nm}a}\sum_{i=0}^{N_x-1}\sum_{j=0}^{N_y-1}f^\delta(x_i,y_i)\cos\frac{\pi n i}{N_x}\cos\frac{\pi m j}{N_y} \tag{43}$$

$$v^N_\delta(x_i,y_j,H)=-\sum_{m=0}^{N_x-1}\sum_{n=0}^{N_y-1}\frac{\tilde\Phi^\delta_{nm}(a)e^{k_{nm}(H-a)}}{1+\alpha e^{2k_{nm}(H-a)}}\cos\frac{\pi n i}{N_x}\cos\frac{\pi m j}{N_y},\quad i=0,\dots,N_x,\quad j=0,\dots,N_y. \tag{44}$$

And, thus, according to (40), as a result of function recovery at $z=H$ we obtain

$$u^N_\delta(H)=v^N_\delta(H)-\Phi^N(H).$$

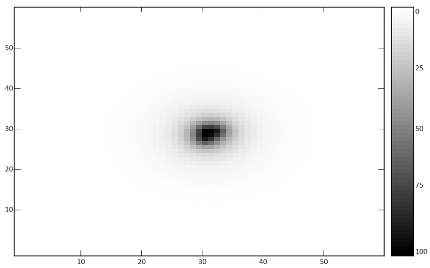

Figure 1. Initial thermogram on the surface S

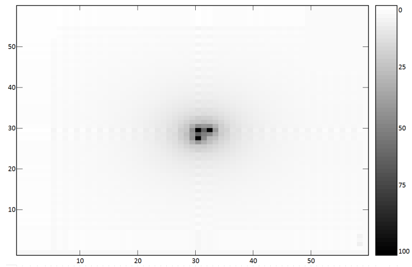

Figure 2. Adjusted thermogram obtained as an approximate solution of the inverse problem $u|_{z=H}$

The results of calculations are shown in Fig. 1 and Fig. 2.

Fig. 1 shows the initial data of the inverse problem — the function f^δ calculated by the discrete analogue of the formula (42) with the addition of a randomly specified error within 3%. The three sources are perceived as a single unit.

Fig. 2 shows the result of function recovery using (43), (44). Three sources are clearly visible.

When computing (44) discrete Fourier series, the algorithms described in [17–20] can be used.

The value of the obtained solution is calculated for the boundary conditions of the problem (4) with accuracy to a constant. Accordingly, Fig. 1 and Fig. 2 show the values normalised from 0 to 100.

Conclusion

When solving the inverse problem (4) of continuation from the boundary S, the function f can be interpreted as the original image obtained with the thermal imager or as the original thermogram.

The thermogram obtained with the help of a thermal imager reproduces with a certain degree of reliability the image of the structure of heat sources located inside the body. Then the solution of the inverse problem obtained as a result of the proposed approach can be considered as a mathematical processing of the thermogram, the obtained function $u|_{z=H}$ represents the temperature distribution on the plane located closer to the investigated heat sources than the initial surface S, we can expect a more accurate reproduction of the image of the sources on the calculated thermogram $u|_{z=H}$.

The above calculations show the effectiveness of the proposed method based on the stable solution of the inverse continuation problem (40) and (41) and its applicability for processing thermographic images, in particular, in medicine [1].

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References

1. Ivanitskii, G. R. Thermovision in medicine. Russian. *Vestnik RAN* **76,** 44–53 (2006).
2. Tikhonov, A. N. & Glasko, V. B. Use of the regularization method in non-linear problems. *U.S.S.R. Comput. Math. Math. Phys.* **5,** 93–107. doi:10.1016/0041-5553(65)90150-3 (1965).
3. Ivanov, V. K., Vasin, V. V. & Tanana, V. P. *The theory of linear ill-posed problems and its applications* Russian (Nauka, Moscow, 1978).
4. Vasin, V. V. The stable evaluation of a derivative in space $C(-\infty, +\infty)$. *U.S.S.R. Comput. Math. Math. Phys.* **13,** 16–24. doi:10.1016/0041-5553(73)90002-5 (1973).
5. Tihonov, A. N. & Arsenin, V. Y. *Methods for solving ill-posed problems* Russian (Nauka, Moscow, 1979).
6. Tihonov, A. N., Glasko, V. B., Litvinenko, O. K. & Melihov, V. R. On the continuation of the potential towards disturbing masses based on the regularization method. Russian. *Vestnik RAN* **1,** 30–48 (1968).
7. Laneev, E. B., Chernikova, N. Y. & Baaj, O. Application of the minimum principle of a Tikhonov smoothing functional in the problem of processing thermographic data. *Advances in Systems Science and Applications* **1,** 139–149. doi:10.25728/assa.2021.21.1.1055 (2021).
8. Baaj, O., Chernikova, N. Y. & Laneev, E. B. Correction of Thermographic Images Based on the MIinimization Method of Tikhonov Functional. *Yugoslav Journal of Operations Research* **32,** 407–424. doi:10.2298/YJOR211015026B (2022).
9. Laneev, E. B. & Klimishin, A. V. On an approximate solution to an ill-posed mixed boundary value problem for the Laplace equation in a cylindrical domain with homogeneous conditions of the second kind on the lateral surface of the cylinder. Russian. *Russian Universities Reports. Mathematics* **29,** 164–175. doi:10.20310/2686-9667-2024-29-146-164-175 (2024).
10. Laneev, E. B. & Baaj, O. On a stable calculation of the normal to a surface given approximately. *Discrete and Continuous Models and Applied Computational Science* **3,** 228–241. doi:10.22363/2658-4670-2023-31-3-228-241 (2023).
11. Laneev, E. B. & Muratov, M. N. On the stable solution of a mixed boundary value problem for the Laplace equation with an approximately given boundary. Russian. *Vestnik RUDN. Seriya Matematika* **1,** 102–111 (2002).
12. Laneev, E. B. & Muratov, M. N. Ob odnoy obratnoy zadache k kraevoy zadache dlya uravneniya Laplasa s usloviem tret'ego roda na netochno zadannoy granitse. Russian. *Vestnik RUDN. Seriya Matematika* **1,** 100–110 (2003).
13. Chernikova, N. Y., Laneev, E. B., Muratov, M. N. & Ponomarenko, E. Y. On an Inverse Problem to a Mixed Problem for the Poisson Equation. *Mathematical Analysis With Applications. CONCORD-90 2018* **318,** 141–146 (2020).
14. Laneev, E. B. & Bhuvana, V. Ob ustojchivom reshenii odnoj smeshannoj zadachi dlya uravneniya Laplasa. Russian. *Vestnik RUDN. Seriya Prikladnaya matematika i informatika* **1,** 128–133 (1999).
15. Laneev, E. B., Lesik, P. A., Klimishin, A. V., Kotyukov, A. M. & Romanov A. A. and, K. A. G. On a stable approximate solution of an ill-posed boundary value problem for the metaharmonic equation. Russian. *Russian Universities Reports. Mathematics* **25,** 156–164. doi:10.20310/2686-9667-2020-25-130-156-164 (2020).
16. Laneev, E. B., Mouratov, M. N. & Zhidkov, E. P. Discretization and its proof for numerical solution of a Cauchy problem for Laplace equation with inaccurately given Cauchy conditions on an inaccurately defined arbitrary surface. *Phys. Part. Nuclei Lett* **5,** 164–167. doi:10.1134/S1547477108030059 (2008).

17. Baaj, O. On the application of the Fourier method to solve the problem of correction of thermographic images. *Discrete and Continuous Models and Applied Computational Science* **30,** 205–216. doi:10.22363/2658-4670-2022-30-3-205-216 (2022).
18. Hamming, R. W. *Numerical methods for scientists and engineers* (McGraw-Hill Book Company, New York, 1962).
19. Laneev, E. B. & Baaj, O. On a modification of the Hamming method for summing discrete Fourier series and its application to solve the problem of correction of thermographic images. *Discrete and Continuous Models and Applied Computational Science* **30,** 342–356. doi:10.22363/2658-4670-2022-30-4-342-356 (2022).
20. Morozov, V. A. On a stable method for computing the values of unbounded operators. Russian. *Dokladi AN SSSR* **185,** 267–270 (1969).

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Об устойчивом приближённом решении некорректно поставленной краевой задачи для уравнения Лапласа с однородными условиями второго рода на краях при неточных данных на приближённо заданной границе

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Аннотация. В работе рассматривается некорректно поставленная задача продолжения гармонических функций с неточно заданной границы в цилиндрической области с однородными краевыми условиями второго рода на боковых гранях. Значение функции и её нормальной производной (условия Коши) — известны приближённо на приближённо заданной поверхности произвольного вида, ограничивающей цилиндр. В данном случае задача Коши для уравнения Лапласа обладает свойством неустойчивости по отношению к погрешности в данных Коши, т. е. является некорректно поставленной. На основе представлений о функции источника исходной задачи, точное решение представляется в виде суммы двух функций, одна из которых явно зависит от условий Коши, вторая может быть получена как решение интегрального уравнения Фредгольма первого рода в виде ряда Фурье по собственным функциям второй краевой задачи для уравнения Лапласа. Для получения приближённого устойчивого решения интегрального уравнения применён метод регуляризации Тихонова, когда решение получается как экстремаль функционала Тихонова. Для приближённо заданной поверхности рассматривается вычисление нормали к этой поверхности и её сходимость к точному значению в зависимости от погрешности, с которой задана исходная поверхность. Доказывается сходимость полученного приближённого решения к точному решению при сопоставлении параметра регуляризации с ошибками в данных как по неточно заданной границе, так и по значению исходной функции на этой границе. Проводится численный эксперимент, который демонстрирует эффективность предложенного подхода для частного случая — для плоской границы и конкретного исходного источника тепла (набора точеных источников).

Ключевые слова: некорректно поставленная задача, метод регуляризации Тихонова, задача Коши для уравнения Лапласа, интегральное уравнение первого рода