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Convergence of Fourier Method connected with Orthogonal Splines

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Abstract. Fourier method and Fourier series have wide fields of application. The use of the theory of orthogonal splines, created by the author of this article and developed in the last thirty years, has led to significant progress in a number of numerical and analytical methods of deformable solid mechanics and mathematical physics. In particular, the generalized Fourier method associated with the use of finite Fourier series and orthogonal splines was successfully applied earlier by the author in solving parabolic initial boundary value problems for regions with curved boundaries. Recent article proposes further development and novel full research of the algorithm of this method, designed to solve parabolic initial boundary value problems in non-canonical domains. The method gives approximate analytical solutions in form of finite Fourier series whose structure is similar to that of partial sums of an infinite Fourier series for an exact solution. Full investigation of the method's convergence presented in this article is based on the theory of finite difference methods. As a number of grid nodes in a region increases, such finite Fourier series approach an exact solution of a parabolic initial boundary value problem. Investigation of convergence shows efficiency of the novel algorithm of the generalized Fourier method in solving parabolic initial boundary value problems for non-canonical regions.

Keywords: parabolic initial boundary value problems, curvilinear boundary, non-canonical regions, method of variable separation, finite Fourier series; orthogonal splines

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Оригинальная статья

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Сходимость метода Фурье, связанного с ортогональными сплайнами

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Аннотация. Применение теории ортогональных сплайнов, созданной автором данной статьи и получившей развитие в последние тридцать лет, привело к существенному прогрессу в алгоритмах ряда численных и аналитических методов механики деформируемого твердого тела и математической физики. В частности, обобщенный метод Фурье, связанный с использованием конечных рядов Фурье и ортогональных сплайнов, был успешно применен ранее автором данной статьи при решении параболических начально-краевых задач для областей с криволинейными границами. В данной статье предлагается дальнейшее развитие и новое всестороннее исследование алгоритма этого метода Фурье, предназначенного для решения параболических начально-краевых задач в неканонических областях. Этот метод дает приближенные аналитические решения в виде конечного ряда Фурье, структура которого аналогична структуре частных сумм бесконечного ряда Фурье точного решения. Полное исследование сходимости этого метода, представленное в данной статье, основано на теории конечно-разностных методов. По мере увеличения числа узлов сетки в области такие конечные ряды Фурье приближаются к точному решению параболической начально-краевой задачи. Исследование сходимости показывает эффективность нового алгоритма обобщенного метода Фурье при решении параболических начально-краевых задач для неканонических областей.

Ключевые слова: параболические начально-краевые задачи, криволинейная граница, неканонические области, метод разделения переменных, конечные ряды Фурье, ортогональные сплайны

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1. Introduction

The modified Fourier method [1] was proposed and investigated earlier [1] in parabolic initial boundary value problems for regions with a noncanonical curvilinear boundaries. The method [1] connected with explicit difference scheme is similar to another variant of Fourier method connected with implicit difference scheme and proposed here. Convergence of approximate analytic solutions was obtained in [1] only with respect to eigenvalues and functions in the framework of the Sturm-Liouville problem. Here is proposed full investigation of convergence of approximate solutions obtained in form of finite Fourier series for novel variant of Fourier method connected with orthogonal splines in parabolic initial boundary value problems. An estimate is obtained, which shows a high rate of convergence of such finite Fourier series to exact solutions of problems for regions with a noncanonical curvilinear boundaries.

The method of separation of variables (Fourier method) allows finding solutions in analytical form of many initial boundary value problems. The method is connected with the Sturm-Liouville problem and in many cases of initial boundary value problems with using of special functions. Implementation of classical Fourier method for many types of initial boundary value problems, including problems to which all parts of the boundary of a canonical region are coordinate lines or surfaces, meets with significant difficulties. One way to expand the scope of the classical Fourier method is to solve mathematical questions related to structure of boundary conditions [2]. Special functions appear in the algorithm of the Fourier method when a Sturm-Liouville problem is solved in curvilinear coordinate systems in cases of canonical regions whose boundaries are coordinate lines or surfaces. In the general case of initial boundary value problems for noncanonical regions with curvilinear boundaries, the use of special functions is inefficient. The classical Fourier method is applicable only in initial boundary value problems for canonical regions of classical shape, in particular, in solving contact problems [3] for elastic bodies. The applications of the classical Fourier method are given, for example, in the articles [4–5].

Other directions of development of different methods for solving initial boundary value problems for not canonical regions with curvilinear boundaries are associated, first, with the application of other methods, for example [6–10], and, secondly, with a modification of the Fourier method itself.

Finite difference methods [6–9] and finite element methods [10] have wide scopes. But numerical methods [6–10] not give solutions in form of Fourier series.

Fourier series are used in many applications, in particular, [11–12]. Scope of spline approximations, for example [13], also is enough wide. The generalized Fourier method associated with the use of orthogonal splines was proposed for parabolic initial boundary value problems in the article [1]. It gives solutions in form of finite Fourier series. This method, thanks to orthogonal splines, has expanded scope which contains initial boundary value problems for noncanonical regions with any curvilinear boundaries. Used in [1] and here finite Fourier series, based on orthogonal splines, shows high efficiency [14] also in problems of approximation of functions in regions with curvilinear boundaries and generates fast algorithm of approximations.

The numerical solutions of some initial boundary value problems for noncanonical regions with curvilinear boundaries shown high computational possibilities of the method [1]. Investigations of the method [1] are continued here, efficiency of the similar generalized Fourier method is demonstrated in parabolic initial boundary value problems for noncanonical regions. The investigation of convergence of such method is proposed here. This study is based on the theory of finite difference methods.

2. Problem Statement

The parabolic initial boundary value problem

$$
\begin{aligned}
& L[u] = \frac{\partial u}{\partial t} \quad \forall\ (x,y) \in S,\ \forall\ t \geq 0; \\
& L[u] = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right); \\
& u|_{t=0} = \varphi(x,y) \quad \forall\ (x,y) \in S; \\
& u|_{\partial S} = 0 \quad \forall\ t \geq 0;
\end{aligned} \tag{2.1}
$$

is considered. Here ∂S is a piecewise smooth curvilinear boundary of the not canonical region S, $u = u(x,y,t)$ – a function, continuous $\forall t \geq 0$ in a closed region $\overline{S} = S + \partial S$, $a^2 = const > 0$.

The not canonical region $\overline{S}$ is, for the example, the circular region of radius $R = 1$ with the hole cut out by the elliptical line

$$\frac{(x-0.6)^2}{0.1^2} + \frac{y^2}{0.2^2} = 1.$$

The boundary ∂S of the not canonical region S in this case consists of a circle of radius $R = 1$ and a given ellipse.

In the general case, a not canonical region $\overline{S}$ with a curvilinear boundary ∂S fits into a rectangular region $\overline{Q}$

$$\overline{Q} = \{a \leq x \leq b; c \leq y \leq d\}.$$

An auxiliary initial boundary value problem

$$
\begin{aligned}
& L[u] = \frac{\partial u}{\partial t} \qquad \forall\ (x,y) \in S,\ \forall\ t \geq 0; \\
& u|_{t=0} = \varphi(x,y) \quad \forall\ (x,y) \in S; \\
& u|_{\partial Q} = 0, \quad u = 0\ \ \forall\ (x,y) \in (Q \backslash S),\ \forall\ t \geq 0;
\end{aligned} \tag{2.2}
$$

is also considered. This task is equivalent to task (2.1) in the region $\overline{S}$.

3. Description of Method

According to the Fourier method, the solutions of the problems (2.1)–(2.2) is sought as a product of two functions

$$u(x,y,t) = U(x,y) \cdot V(t).$$

Substitution of this product in (2.1)–(2.2) and separation of variables U, V leads to the equation with the parameter λ

$$\frac{dV}{dt} + \lambda V = 0 \quad \forall\ t \geq 0 \tag{3.1}$$

and to the Sturm-Liouville problem

$$L[U] + \lambda U = 0 \quad (S), \quad U|_{\partial S} = 0, \tag{3.2}$$

which connected with the problem (2.1), and also leads to the modified Sturm-Liouville problem

$$\begin{aligned} & L[U] + \lambda U = 0 \quad (S), \\ & U = 0 \quad (Q \backslash S); \quad U|_{\partial Q} = 0, \end{aligned} \tag{3.3}$$

which connected with the problem (2.2). The problems (3.2)–(3.3) are equivalent in the region $\overline{S}$.

The B-spline of first degree

$$b_i(\mu) = \begin{cases} \dfrac{\mu - \mu_{i-1}}{h}, \ \mu \in [\mu_{i-1}, \mu_i]; \\ \dfrac{\mu_{i+1} - \mu}{h}, \ \mu \in [\mu_i, \mu_{i+1}]; \\ 0, \ \mu \in (-\infty, \mu_{i-1}) \bigcup (\mu_{i+1}, \ +\infty); \end{cases}$$

corresponds to node μ_i of the grid. Coordinates of nodes

$$\mu_i = -1 + ih; \ 0 \le i \le N; \ h = 2/N;$$

are defined here for constant step h and, for example, on $[-1, 1]$. The compact support of the spline $b_i(\mu)$ is $[\mu_{i-1}, \mu_{i+1}]$. The B-spline of first degree

$$b_i^{(-)}(\mu) = \begin{cases} \dfrac{\sqrt{2}(\mu - \mu_{i-1})}{h}, \ \mu \in [\mu_{i-1}, \mu_{i-1} + h/2]; \\ \dfrac{\sqrt{2}[h - (\mu - \mu_{i-1})]}{h}, \ \mu \in [\mu_{i-1} + h/2, \mu_i]; \\ 0, \ \mu \in (-\infty, \mu_{i-1}) \bigcup (\mu_i, \infty); \end{cases}$$

has the compact support $[\mu_{i-1}, \mu_i]$. The B-spline of first degree

$$b_i^{(+)}(\mu) = \begin{cases} \dfrac{\sqrt{2}(\mu - \mu_i)}{h}, \ \mu \in [\mu_i, \mu_i + h/2]; \\ \dfrac{\sqrt{2}[h - (\mu - \mu_i)]}{h}, \ \mu \in [\mu_i + h/2, \mu_{i+1}]; \\ 0, \ \mu \in (-\infty, \mu_i) \bigcup (\mu_{i+1}, \ +\infty); \end{cases}$$

has the compact support $[\mu_i, \mu_{i+1}]$. The compactly supported functions

$$\hat{\varphi}_i(\mu) = b_i(\mu) + b_i^{(-)}(\mu) + b_i^{(+)}(\mu)$$

have properties

$$(\hat{\varphi}_i, \hat{\varphi}_j) = \|\hat{\varphi}_i\|^2 \delta_{ij}$$

and are orthogonal differentiable continuous splines [15] on each specific grid. It are piecewise linear finite differentiable contituous functions. Here δ_{ij} – the Kronecker symbols.

The splines

$$\gamma_i(x) = \varphi_i(x), \ \ \delta_j(y) = \varphi_j(y),$$

where

$$\varphi_i = \frac{\hat{\varphi}_i}{\|\hat{\varphi}_i\|^2}, \quad (\varphi_i, \varphi_j) = \delta_{ij},$$

are used in the approximation

$$U_N(x,y) = \sum_{i=0}^{N}\sum_{j=0}^{M} d_{ij}\gamma_i(x)\delta_j(y) \tag{3.4}$$

of a solution of the problem (3.3) on a region $\overline{Q}$. Here N, M are numbers of a grid respectively for axes $Ox, Oy; d_{ij}$are unknown constant coefficients.

A system of splines $\varphi_i\,(x)$ on an infinite sequence of grids formed by a discrete decrease of step $h \to 0$ is complete and this system of splines approximates in Sobolev space W_2^0 any function $U(x)$ of the Sobolev space W_2^1 [15]:

$$\left\| U - \sum_{i=1}^{N} c_i\varphi_i \right\|_{W_2^0} \leq Ch\, \|U\|_{W_2^1}\,,$$

where C is a constant which not depends on $U, h; c_i-$some constant coefficients.

The coefficients d_{ij}, corresponding to nodes placed in the region $(Q\backslash S)$ and on the boundary ∂Q are equal to zero, thus the sum (3.4) takes into account how the conditions $U = 0 (Q\backslash S)$, so and the boundary conditions $U|_{\partial Q} = 0$. It is used that in accordance with properties [15] of used splines each coefficient d_{ij} is equal to a value of $U_N(x,y)$ at the node (x_i, y_j) of a grid in a region $(Q\backslash S)$.

The stationary condition

$$\delta F = \int_S \left\{ \left[\left(\frac{\partial U_1}{\partial x} + \frac{\partial U_2}{\partial y} \right) + \frac{\lambda U}{a^2} \right] \delta U - \left(\frac{\partial U}{\partial x} - U_1 \right) \delta U_1 - \left(\frac{\partial U}{\partial y} - U_2 \right) \delta U_2 \right\} dS + \\ + \int_{Q\backslash S} U\delta U dQ + \int_{\partial S} U(n_x\delta U_1 + n_y\delta U_2)dl + \int_{\partial Q} U\delta U dl = 0 \tag{3.5}$$

of the functional

$$2F(U, U_1, U_2) = \\ = \int_S \left\{ \left[\left(\frac{\partial U_1}{\partial x} + \frac{\partial U_2}{\partial y} \right) + \frac{\lambda U}{a^2} \right] U - \left(\frac{\partial U}{\partial x} - U_1 \right) U_1 - \left(\frac{\partial U}{\partial y} - U_2 \right) U_2 \right\} dS + \\ + \int_{Q\backslash S} U^2 dQ + \int_{\partial S} U(U_1 n_x + U_2 n_y)dl + \int_{\partial Q} U^2 dl, \tag{3.6}$$

is used for determination of the coefficients d_{ij} of the approximate analytical solution (3.4). Here

$$\frac{\partial U}{\partial x} = U_1, \quad \frac{\partial U}{\partial y} = U_2 \qquad (S);$$

n_x, n_y are components of external normal to a boundary ∂S.

The condition (3.5) is equivalent to

$$a^2 \left(\frac{\partial U_1}{\partial x} + \frac{\partial U_2}{\partial y} \right) + \lambda U = 0 \quad (S),$$

$$\frac{\partial U}{\partial x} = U_1, \; \frac{\partial U}{\partial y} = U_2 \quad (S);$$

$$U = 0 \quad (Q \backslash S); \;\; U|_{\partial S} = 0, \;\; U|_{\partial Q} = 0; \tag{3.7}$$

because variations δU and $\delta U_1, \delta U_2$ are independent and arbitrary in $S, \partial S, Q\backslash S$ and on ∂Q. The condition $U|_{\partial S} = 0$ follows from the condition $U = 0 \quad (Q\backslash S)$, therefore, equations and conditions (3.7) after exclusion U_1, U_2 are written as (3.3).

To obtain an approximate analytical solution of the problem (2.2) on a region $\overline{Q}$, into which the region $\overline{S}$ fits, a uniform grid with steps h_1, h_2 is constructed.

Substitution (3.4) into the condition (3.5) leads to a system of finite difference equations

$$a^2 \left[\frac{U_{n+1,m} - 2U_{nm} + U_{n-1,m}}{h_1^2} + \frac{U_{n,m+1} - 2U_{nm} + U_{n,m-1}}{h_2^2} \right] + \lambda_p U_{nm} = 0, \tag{3.8}$$

$$U_{nm} = 0 \quad (Q\backslash S); \;\; U_{nm} = 0 \quad (\partial Q). \tag{3.9}$$

The equation (3.1) is presented in next finite difference form

$$\frac{V^{l+1} - V^l}{\Delta t} + \lambda_p V^{l+1} = 0. \tag{3.10}$$

The exception λ_p from finite difference equations (3.8), (3.10), taken in pairs for each internal node of the region S, gives for the node (x_n, y_m), taking into account that $u^l_{nm} = U_{nm} V^l$, the finite difference equation

$$a^2 (\Lambda^{l+1}_{nm,x} u + \Lambda^{l+1}_{nm,y} u) = \Lambda^l_{nm,t} u \quad (S), \tag{3.11}$$

where

$$\Lambda^{l+1}_{nm,x} u = \frac{u^{l+1}_{n+1,m} - 2u^{l+1}_{nm} + u^{l+1}_{n-1,m}}{h_1^2},$$

$$\Lambda^{l+1}_{nm,y} u = \frac{u^{l+1}_{n,m+1} - 2u^{l+1}_{nm} + u^{l+1}_{n,m-1}}{h_2^2},$$

$$\Lambda^l_{nm,t} u = \frac{u^{l+1}_{nm} - u^l_{nm}}{\Delta t}$$

– finite-difference operators on a uniform grid with steps h_1, h_2 for the directions Ox, Oy respectively and with a step Δt of a grid of time.

Nodes of the grid, for example, on

$$\overline{Q} = \{-1 \le x \le 1; -1 \le y \le 1\}$$

have coordinates

$$(x_i = -1 + ih, y_j = -1 + jh) \in \overline{Q}, \;\; 0 \le i, j \le N.$$

The conditions

$$u^l_{nm} = 0 \quad (Q\backslash S), \; u^l_{nm} = 0 \quad (\partial Q) \tag{3.12}$$

containing a boundary condition

$$u^l_{nm} = 0 \quad (\partial S),$$

are added to the system of equations (3.11).

The equations and the conditions (3.11)–(3.12) are considered together with the initial condition (2.2), written in nodes of a grid

$$u^l_{nm}\big|_{t_l=0} = \varphi(x_n, y_m) \qquad \forall\ (x_n, y_m) \in S. \tag{3.13}$$

Solving a system of grid equations and conditions (3.11)–(3.13) for a given sequence of time values t_l gives the values u^l_{nm} of a numerical solution in all nodes of the grid in a region $\overline{Q}$ for given points of time.

Solutions of problems (3.1), (3.2) or (3.3) have next form within the framework of the classical Fourier method

$$u^{(K)}(x, y, t) = \sum_{k=1}^{K} V_k(t) U_N^{(k)}(x, y), \tag{3.14}$$

where the functions V_k, $U_N^{(k)}$ correspond to eigenvalues $\lambda_k (k = 1, ..., K)$ of the Sturm-Liouville problems (3.2)–(3.3). Functions V_k, $U_N^{(k)}$ are formed as result of solving of a system of finite-difference equations (3.10)–(3.11) together with (3.12)– (3.13). The finite series (3.14) are Fourier series generated by orthogonal splines for all nodal values t_l of time, but Fourier series (3.14) are formed here immediately for used grids and for nodal values of time in a final form without first determining the eigenvalues λ_k and functions $U_N^{(k)}$, what is possible thanks to the use of orthogonal splines. Solutions of the systems of equations (3.10)–(3.13) gives values of solutions in nodes of a grid in regions $\overline{S}$, $\overline{Q}$ at defined moments t_l of time and these nodal values are equal to coefficients d_{ij} of finite series (3.4). Such solutions in form of finite Fourier series are followed from (3.10)–(3.13) for every set moments t_l of time.

General solutions of differential equations (3.1) within the framework of the classical Fourier method have the form

$$V_k(t) = A_k \exp(-\lambda_k t), \qquad (k = 1, ..., K); \tag{3.15}$$

where A_k are unknown constant coefficients.

The sum (3.14), taking into account (3.4), (3.15), has the form

$$u^{(K)}(x, y, t) = \sum_{k=1}^{K} \left[A_k \exp(-\lambda_k t) \sum_{i=0}^{N} \sum_{j=0}^{M} d_{ij}^{(k)} \gamma_i(x) \delta_j(y) \right] \tag{3.16}$$

on each given grid in regions $\overline{S}$, $\overline{Q}$.

Substituting (3.16) into the initial condition (2.1)–(2.2) gives an equation, after multiplying both parts of which by $U_N^{(k)}(x, y)$ for a fixed value k and integrating it over the region S, taking into account the orthogonality of the eigenfunctions, the formula appears

$$A_k = \frac{\iint_S \varphi(x, y) U_N^{(k)}(x, y) dS}{\left\| U_N^{(k)} \right\|^2}. \tag{3.17}$$

Thus, the sum (3.16), whose coefficients A_k are determined by formula (3.17), satisfies equation (3.3) in variational form, equation (3.1), as well as boundary condition (3.3) and initial condition (2.1)–(2.2). The sums (3.16) for different values of N, M are approximate analytical solutions of equivalent parabolic problems (2.1)–(2.2) in regions $\overline{S}$, $\overline{Q}$.

For given fixed values of time, sums (3.16) represent finite generalized Fourier series which are approximate analytical solutions of boundary value problems (2.1)–(2.2) in regions $\overline{S}$, $\overline{Q}$, and also are finite generalized Fourier series generated by orthogonal splines.

The solution (3.16) is written at moments t_l of time in the form

$$\begin{aligned} & u^{(K)}(x,y,t_l) = \textstyle\sum_{i=0}^{N} \sum_{j=0}^{M} u_{ij}^{l} \gamma_i(x) \delta_j(y), \\ & u_{ij}^{l} = \textstyle\sum_{k=1}^{K} \left[A_k \exp(-\lambda_k t_l) d_{ij}^{(k)} \right]. \end{aligned} \tag{3.18}$$

4. Convergence of Method

The convergence investigation of the proposed method uses here the theory of finite difference schemes [16–17].

Theorem 4.1. *The solutions* (3.16), (3.18) *of the parabolic initial boundary value problem* (2.2) *converge to an exact solution u of the problem* (2.2) *in a region $\overline{Q}$ and to an exact solution of the problem* (2.1) *in a region $\overline{S}$, if*

$$h = \max(h_1, h_2)$$

and

$$\Delta t = \alpha h^2, \quad \alpha = const > 0.$$

The inequality

$$\left\| u - u^{(K)} \right\|_{W^0_{h,2}} \leq Ch^2,$$

determinates rate of convergence of numerical solutions u_{nm}^l in (3.18)*. Here, the Sobolev space $W^0_{h,2}$ on grid which is defined by the norm*

$$\|u\|_{W^0_{h,2}} = \left(h^2 \sum_{i=0}^{N} \sum_{j=0}^{M} |u(x_i, y_j)|^2 \right)^{1/2}$$

associated with a grid in a region $\overline{Q}$.

Proof. Approximation of differential equations. The next notations are used

$$(x_n, y_m, t_l) = \chi_{n,m,l}, \quad (x_n, y_m, t_{l+1}) = \chi_{n,m,l+1}.$$

If continuous partial derivatives of a function $u(x,y,t)$ of the fourth order in coordinates and the first order in time are existed, then the Taylor formula gives

$$
\begin{aligned}
\Lambda^{l+1}_{nm,x}u &= \frac{u^{l+1}_{n+1,m} - 2u^{l+1}_{nm} + u^{l+1}_{n-1,m}}{h_1^2} = \\
&= \frac{u(x_n + h_1, y_m, t_{l+1}) - 2u(x_n, y_m, t_{l+1}) + u(x_n - h_1, y_m, t_{l+1})}{h_1^2} = \\
&= \frac{1}{h_1^2}\Bigg\{\Bigg[u^{l+1}_{nm} + h_1 \frac{\partial u}{\partial x}\Big|_{\chi_{n,m,l+1}} + \frac{h_1^2}{2!}\frac{\partial^2 u}{\partial x^2}\Big|_{\chi_{n,m,l+1}} + \frac{h_1^3}{3!}\frac{\partial^3 u}{\partial x^3}\Big|_{\chi_{n,m,l+1}} + \frac{h_1^4}{4!}\frac{\partial^4 u}{\partial x^4}\Big|_{\chi_{n,m,l+1}} + \\
&+ o(h_1^4)\Bigg] - 2u^{l+1}_{nm} + \Bigg[u^{l+1}_{nm} - h_1 \frac{\partial u}{\partial x}\Big|_{\chi_{n,m,l+1}} + \frac{h_1^2}{2!}\frac{\partial^2 u}{\partial x^2}\Big|_{\chi_{n,m,l+1}} - \frac{h_1^3}{3!}\frac{\partial^3 u}{\partial x^3}\Big|_{\chi_{n,m,l+1}} + \\
&+ \frac{h_1^4}{4!}\frac{\partial^4 u}{\partial x^4}\Big|_{\chi_{n,m,l+1}} + o(h_1^4)\Bigg]\Bigg\} = \frac{\partial^2 u}{\partial x^2}\Big|_{\chi_{n,m,l+1}} + \frac{h_1^2}{12}\frac{\partial^4 u}{\partial x^4}\Big|_{\chi_{n,m,l+1}} + o(h_1^2) = \\
&= \frac{\partial^2 u}{\partial x^2}\Big|_{\chi_{n,m,l+1}} + O(h_1^2) \qquad (4.1)
\end{aligned}
$$

and also

$$
\begin{aligned}
\Lambda^{l+1}_{nm,y}u &= \frac{u^{l+1}_{n,m+1} - 2u^{l+1}_{nm} + u^{l+1}_{n,m-1}}{h_2^2} = \\
&= \frac{u(x_n, y_m + h_2, t_{l+1}) - 2u(x_n, y_m, t_{l+1}) + u(x_n, y_m - h_2, t_{l+1})}{h_2^2} = \\
&= \frac{1}{h_1^2}\Bigg\{\Bigg[u^{l+1}_{nm} + h_2 \frac{\partial u}{\partial y}\Big|_{\chi_{n,m,l+1}} + \frac{h_2^2}{2!}\frac{\partial^2 u}{\partial y^2}\Big|_{\chi_{n,m,l+1}} + \frac{h_2^3}{3!}\frac{\partial^3 u}{\partial y^3}\Big|_{\chi_{n,m,l+1}} + \frac{h_2^4}{4!}\frac{\partial^4 u}{\partial y^4}\Big|_{\chi_{n,m,l+1}} + \\
&+ o(h_2^4)\Bigg] - 2u^{l+1}_{nm} + \Bigg[u^{l+1}_{nm} - h_2 \frac{\partial u}{\partial y}\Big|_{\chi_{n,m,l+1}} + \frac{h_2^2}{2!}\frac{\partial^2 u}{\partial y^2}\Big|_{\chi_{n,m,l+1}} - \frac{h_2^3}{3!}\frac{\partial^3 u}{\partial y^3}\Big|_{\chi_{n,m,l+1}} + \\
&+ \frac{h_2^4}{4!}\frac{\partial^4 u}{\partial y^4}\Big|_{\chi_{n,m,l+1}} + o(h_2^4)\Bigg]\Bigg\} = \frac{\partial^2 u}{\partial y^2}\Big|_{\chi_{n,m,l+1}} + \frac{h_2^2}{12}\frac{\partial^4 u}{\partial y^4}\Big|_{\chi_{n,m,l+1}} + o(h_2^2) = \\
&= \frac{\partial^2 u}{\partial y^2}\Big|_{\chi_{n,m,l+1}} + O(h_2^2) \qquad (4.2)
\end{aligned}
$$

and

$$
\begin{aligned}
\Lambda^{l}_{nm,t}u &= \frac{u^{l+1}_{nm} - u^{l}_{nm}}{\Delta t} = \frac{u(x_n, y_m, t_l + \Delta t) - u(x_n, y_m, t_l)}{\Delta t} = \\
&= \frac{1}{\Delta t}\left[u^{l}_{nm} + \Delta t \frac{\partial u}{\partial t}\Big|_{\chi_{n,m,l}} + \frac{(\Delta t)^2}{2!}\frac{\partial^2 u}{\partial t^2}\Big|_{\chi_{n,m,l}} + o((\Delta t)^2)\right] - \frac{u^{l}_{nm}}{\Delta t} = \\
&= \frac{\partial u}{\partial t}\Big|_{\chi_{n,m,l}} + O(\Delta t). \qquad (4.3)
\end{aligned}
$$

The formulas (4.1)– (4.3) show that the finite difference equations

$$a^2(\Lambda^{l+1}_{nm,x}u + \Lambda^{l+1}_{nm,y}u) = \Lambda^{l}_{nm,t}u \ (S),$$

approximate the equation

$$L[u] = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = \frac{\partial u}{\partial t} \qquad \forall\ (x,y) \in S, \quad \forall\ t \geq 0; \tag{4.4}$$

in nodes of a grid with errors, magnitudes of which have the order

$$a^2[O(h_1^2) + O(h_2^2)] + O(\Delta t).$$

If $h = \max(h_1, h_2)$ and $\Delta t = \alpha h^2$, $\alpha = const > 0$, then the order of approximation of the differential equation (4.4) in all nodes of the region S is determined by $O(h^2)$.

The differential equation (4.4) will be considered in an entire region Q, which is consistent with the equation

$$u = 0 \quad (Q \backslash S),$$

since taking this condition into account in the equation (4.4) turn it into an identity within the region $(Q \backslash S)$. Also the finite difference equations (3.11) are extended to an entire region Q. The finite difference equations (3.11) turn into identities in all nodes of the grid located in a region $(Q \backslash S)$. Thus, the finite difference equations (3.11) approximate the differential equation (4.4) in the region $(Q \backslash S)$ without errors. The boundary and initial conditions (2.2) are approximated by (3.12) and (3.13) in the grid nodes without errors. Consequently, the finite difference equations (3.11) together with a boundary condition (3.12) and together with an initial condition (3.13) approximate the problem (2.2) in a region Q with an error of an order $O(h^2)$.

Stability of solutions of finite difference equations. It is widely known [16, Chapter 6; 17, Chapter 8] that the system of the finite difference equations

$$a^2(\Lambda^{l+1}_{nm,x} u + \Lambda^{l+1}_{nm,y} u) = \Lambda^{l}_{nm,t} u$$

is characterized by absolute stability in the rectangular region Q.

Convergence of solutions of finite difference equations. Finite difference equations (3.11), together with boundary condition (3.12) and together with the initial condition (3.13), approximate the problem (2.2) in a region Q with an error of the order $O(h^2)$. The system of these finite difference equations in the rectangular region Q is characterized by absolute stability [16–17].

This means [17, Chapter 5] convergence

$$\left\| u - u^{(K)} \right\|_{W^0_{h,2}} \leq C_1 h^2$$

of values in grid nodes of a sequence of approximate solutions $u^{(K)}$ to an exact solution u when a grid step h decreases. Here C_1 is some positive constant coefficient.

End of the proof.

From the convergence of numerical solutions u^l_{ij} of finite difference equations (3.11) obtained at grid nodes at specified time points follows the convergence of a sequence of analytical approximate solutions in the form of finite Fourier series (3.18). The inequality

$$\left\| u - u^{(K)} \right\|_{W^0_2(Q)} \leq C_2 h^2 \tag{4.5}$$

defines this convergence. Here C_2 is some positive constant coefficient, $W^0_2(Q)$ is the Sobolev space.

Problems (2.1), (2.2) are equivalent in the region $\overline{S}$, so

$$\left\| u - u^{(K)} \right\|_{W_2^0(S)} \leq C_3 h^2 \quad (4.6)$$

follows from (4.5). Here C_3 is some positive constant coefficient.

5. Conclusion

The convergence of the modified Fourier method was investigated in [1] in a parabolic initial boundary value problem for a region with a noncanonical curvilinear boundary. The theory of finite difference equations was not used for investigation of convergence. Convergence conclusions in general case were obtained in [1] only with respect to eigenvalues and functions in the framework of the Sturm-Liouville problem. The modified Fourier method [1] connected with explicit difference scheme is similar to another variant of Fourier method connected with orthogonal splines proposed and investigated here and connected with implicit difference scheme.

The estimate (4.6) obtained here shows a high rate of convergence of finite Fourier series (3.16), (3.18) to an exact solution of problems (2.1)–(2.2) for a region S with a curvilinear boundary ∂S. The approximate solutions (3.16), (3.18), in contrast to the solutions obtained in such problems without using the Fourier method using the finite difference method or the finite element method, have the analytical form of Fourier series characteristic of the Fourier method. The algorithm of modified Fourier method considered in this article is associated with the use of the orthogonal splines and allows finding solutions to parabolic initial boundary value problems for non-canonical regions in the general case of their curved multi-connected boundaries.

The investigation realized here for any numbers of grid nodes demonstrated that approximate solutions (3.16), (3.18) converge to known exact solution. For example, approximate solutions (3.16), (3.18) of problems (2.1)–(2.2) for the circular region of radius $R = 1$ with the hole cut out by the elliptical line

$$\frac{(x-0.6)^2}{0.1^2} + \frac{y^2}{0.2^2} = 1$$

converge to exact solution.

The theoretical studies of convergence which were made here and in [1] show high accuracy of approximate analytical solutions in form of finite Fourier series.

Expanding of regions of application of classical analytical methods of solving initial boundary value problems is an actual problem. One of the directions of development of such methods is presented in [1] and here, where the methods of separating of variables described to solve hyperbolic and parabolic initial boundary value problems for not canonical regions. These methods give solutions in the form of finite generalized Fourier series, which converge to exact solutions. The methods pull together numerical methods for solving initial boundary value problems with an analytical method for solving them.The use of orthogonal splines brings together numerical and analytical methods - finite difference methods and the Fourier method, expanding the scope of their applications.

In the method proposed here the potential capabilities of the method of separation of variables, orthogonal splines, and the finite difference method are used together, it leads to analytical solutions in the form of finite Fourier series. The theoretical investigation of the convergence presented here shows that the modified Fourier method gives approximate analytical solutions to parabolic initial boundary value problems in the form of finite Fourier series with high accuracy.

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