Vestnik KRAUNC. Fiz.-Mat. nauki. 2024. vol. 48. no. 3. P. 33 - 42. ISSN 2079-6641

MATHEMATICS

https://doi.org/10.26117/2079-6641-2024-48-3-33-42

Research Article

Full text in English

MSC 35M10, 35R11

# Bitsadze-Samarskii Type Problem for the Diffusion Equation and Degenerate Hyperbolic Equation

***M. Kh. Ruziev**[*1], **R. T. Zunnunov**[2], **N. T. Yuldasheva**[1], **G. B. Rakhimova**[3]*

[1] V. I. Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences, 100174, Tashkent, University street, 9, Uzbekistan

[2] Branch of the Russian State University of Oil and Gas (NRU) named after I.M. Gubkin in Tashkent, 100125, Tashkent, Durmon yuli street, 34, Uzbekistan

[3] Fergana State University, 150100, Fergana, Murabbiylar street, 19, Uzbekistan

**Abstract.** A boundary value problem of the Bitsadze-Samarskii type is studied in the article for a fractional-order diffusion equation and a degenerate hyperbolic equation with singular coefficients at lower terms in an unbounded domain. The article considers a mixed domain where the parabolic part of the domain under consideration coincides with the upper half-plane and the hyperbolic part is bounded by two characteristics of the equation under consideration and a segment of the abscissa axis. The uniqueness of the solution to the problem under consideration is proven by the method of energy integrals. The existence of a solution to the problem under consideration is reduced to the concept of solvability of a fractional-order differential equation. An explicit form of the solution to the modified Cauchy problem is given in the hyperbolic part of the mixed domain under consideration. Using this solution, due to the boundary condition of the problem, the main functional relationship between the traces of the unknown function brought to the interval of the degeneracy line of the equation is obtained. Further, using the representation of the solution of the diffusion equation of fractional order, the second main functional relationship between the traces of the sought-for function on the interval of the abscissa axis from the parabolic part of the considered mixed domain is obtained. Through the conjugation condition of the problem under study, an equation with fractional derivatives is obtained from two functional relationships by eliminating one unknown function; its solution is written out in explicit form. In the study of the boundary value problem, generalized fractional integro-differentiation operators with the Gauss hypergeometric function are employed. The properties of the Wright and Mittag-Leffler type functions are extensively utilized in the study.

*Key words: boundary value problem, diffusion equation, degenerate hyperbolic equation, Gauss hypergeometric function, Wright function, uniqueness of the solution to the problem, existence of a solution to the problem.*

Received: 16.09.2024; Revised: 23.09.2024; Accepted: 27.10.2024; First online: 20.11.2024

**For citation.** Ruziev M. Kh., Zunnunov R. T., Yuldasheva N. T., Rakhimova G. B. Bitsadze-Samarskii type problem for the diffusion equation and degenerate hyperbolic equation. *Vestnik KRAUNC. Fiz.-mat. nauki.* 2024, **48**: 3, 33-42. EDN: GCWUEC. https://doi.org/10.26117/2079-6641-2024-48-3-33-42

**Funding.** The first author is supported by the Grant of the Ministry of Higher Education, Science and Innovation of the Republic of Uzbekistan No. F-FA-2021-424.

**Competing interests.** There are no conflicts of interest regarding authorship and publication.

**Contribution and Responsibility.** All authors contributed to this article. Authors are solely responsible for providing the final version of the article in print. The final version of the manuscript was approved by all authors.

***Correspondence:** E-mail: mruziev@mail.ru

МАТЕМАТИКА
https://doi.org/10.26117/2079-6641-2024-48-3-33-42
Научная статья
Полный текст на английском языке
УДК 517.958

# Задача типа Бицадзе-Самарского для уравнения диффузии и вырождающегося гиперболического уравнения

***М. Х. Рузиев*[1], Р. Т. Зуннунов[2], Н. Т. Юлдашева[1], Г. Б. Рахимова[3]***

[1] Институт математики им. В. И. Романовского Академии наук Узбекистана, 100174, г. Ташкент, ул. Университетская, 9, Узбекистан
[2] Филиал Российского государственного университета нефти и газа (НИУ) имени И. М. Губкина в г. Ташкенте, 100125, г. Ташкент, улица Дурмон йули, 34, Узбекистан
[3] Ферганский государственный университет, 150100, г. Фергана, улица Мураббийлар, 19, Узбекистан

**Аннотация.** В статье изучается краевая задача типа Бицадзе-Самарского для дробного уравнения диффузии и вырождающегося гиперболического уравнения с сингулярными коэффициентами при младших членах в неограниченной области. В статье рассматривается смешанная область, в которой параболическая часть рассматриваемой области совпадает с верхней полуплоскостью, а гиперболическая часть ограничена двумя характеристиками рассматриваемого уравнения и отрезком оси абсцисс. Единственность решения рассматриваемой задачи доказывается методом интегралов энергии. Существование решения рассматриваемой задачи сводится к понятию разрешимости дробного дифференциального уравнения. Приводится явный вид решения модифицированной задачи Коши в гиперболической части рассматриваемой смешанной области. С помощью этого решения в силу граничного условия задачи получена основная функциональная связь между следами неизвестной функции, приведенными на интервал линии вырождения уравнения. Далее, используя представление решения уравнения диффузии дробного порядка, получено второе основное функциональное соотношение между следами искомой функции на отрезке оси абсцисс из параболической части рассматриваемой смешанной области. Через условие сопряжения исследуемой задачи из двух функциональных соотношений путем исключения одной неизвестной функции получено уравнение с дробными производными, решение которого выписано в явном виде. При исследовании краевой задачи используются обобщенные операторы дробного интегро-дифференцирования с гипергеометрической функцией Гаусса. При исследовании широко используются свойства функций типа Райта и Миттаг-Леффлера.

*Ключевые слова: краевая задача, уравнение диффузии, вырожденное гиперболическое уравнение, гипергеометрическая функция Гаусса, функция Райта, единственность решения задачи, существование решения задачи*

Получение: 16.09.2024; Исправление: 23.09.2024; Принятие: 27.10.2024; Публикация онлайн: 20.11.2024

**Для цитирования.** Ruziev M. Kh., Zunnunov R. T., Yuldasheva N. T., Rakhimova G. B. Bitsadze-Samarskii type problem for the diffusion equation and degenerate hyperbolic equation // *Вестник КРАУНЦ. Физ.-мат. науки.* 2024. Т. 48. № 3. C. 33-42. EDN: GCWUEC. https://doi.org/10.26117/2079-6641-2024-48-3-33-42.

**Финансирование.** Первый автор поддержан грантом Министерства высшего образования, науки и инноваций Республики Узбекистан № Ф-ФА-2021-424.

**Конкурирующие интересы.** Конфликтов интересов в отношении авторства и публикации нет.

**Авторский вклад и ответственность.** Авторы участвовали в написании статьи и полностью несут ответственность за предоставление окончательной версии статьи в печать.

***Корреспонденция:** E-mail: mruziev@mail.ru

## Introduction and problem statement

Fractal theory explains the structure of disordered media, like porous materials, and the processes that take place within them. A fractional-order differential equation is used to describe the movement of a substance in a uniform fluid flow [1]. The topic of fractional-order diffusion is covered in [2]. Fractional-order differential equations come up in various areas, such as classical mechanics (inverse problems), heat conduction (heat flow dynamics), diffusion (electrochemical analysis of electrode surfaces), and in the study of stochastic transport processes. Problems involving fluid filtration in highly porous (fractal) media often require studying boundary value problems for fractional-order partial differential equations. Boundary value problems for the fractional-order diffusion equation were explored in [3–6]. A certain family of generalized derivatives of the Riemann–Liouville operators $D^{\alpha,\beta}_{a+}$ of orders $\alpha$ and $\beta$ was studied in [7]. Applications of this operator are given in [8]. The unique solvability of the problem for a partial fractional derivative equation of the Riemann–Liouville type with a boundary condition containing a generalized fractional integro-differentiation operator is investigated in [9–11]. Reference [12] studies an analog of the Bitsadze–Samarskii type problem for a mixed-type partial fractional derivative equation in an unbounded domain.

Let us consider a partial differential equation of the second order

$$\begin{cases} u_{xx} - D^{\gamma}_{0+,y} u = 0, \;\; y > 0, 0 < \gamma < 1, \\ -(-y)^m u_{xx} + u_{yy} + \dfrac{\alpha_0}{(-y)^{1-\frac{m}{2}}} u_x + \dfrac{\beta_0}{y} u_y = 0, \;\; y < 0, \end{cases} \tag{1}$$

where $D^{\gamma}_{0+,y}$ is the partial fractional Riemann-Liouville derivative of order $\gamma (0 < \gamma < 1)$ of function $u(x,y)$ for the second variable [13] in domain $D = D^+ \cup D^- \cup I$, where $D^+$ is the half-plane $y > 0$, $D^-$ is a finite domain of the half-plane $y < 0$, bounded by characteristics $OC$ and $BC$ of equation (1) emanating from point $O(0,0)$ and $B(1,0)$ and segment $OB$ of the straight line $y = 0$, $I = \{(x,y) : 0 < x < 1, \; y = 0\}$. In equation (1) $m, \; \alpha_0, \; \beta_0$ are some real numbers satisfying conditions $m > 0, \;\; |\alpha_0| < \frac{m+2}{2}, \;\; -\frac{m}{2} < \beta_0 < 1$.

Let us introduce the following notation: $\Theta(x) = \left(\frac{x}{2}; -\left(\frac{m+2}{4}x\right)^{\frac{m+2}{2}}\right)$ is the intersection point of the characteristic of equation (1) emanating from point $(x,0)$ $(x \in I)$, with characteristic $OC$, $I^{\sigma,\delta,\eta}_{0+}$ is the operator of generalized fractional integro-differentiation with the Gauss hypergeometric function $F(a,b,c;z)$ introduced by M.A. Saigo [14] and having the following form for real $\sigma, \delta, \eta$ and $x > 0$

$$(I^{\sigma,\delta,\eta}_{0+} f)(x) = \begin{cases} \dfrac{x^{-\sigma-\delta}}{\Gamma(\sigma)} \displaystyle\int_0^x (x-t)^{\sigma-1} F\left(\sigma+\delta, -\eta, \sigma; 1 - \frac{t}{x}\right) f(t)dt, \; (\sigma > 0), \\ \dfrac{d^n}{dx^n} (I^{\sigma+n,\delta-n,\eta-n}_{0+} f)(x), (\sigma \le 0, n = [-\alpha] + 1). \end{cases} \tag{2}$$

In particular [14],

$$(I^{0,0,\eta}_{0+} f)(x) = f(x), \;\; (I^{\sigma,-\sigma,\eta}_{0+} f)(x) = (I^{\sigma}_{0+} f)(x),$$

$$(I_{0+}^{-\sigma,\sigma,\eta}f)(x) = (D_{0+}^{\sigma}f)(x), \tag{3}$$

where $(I_{0+}^{\sigma}f)(x)$ and $(D_{0+}^{\sigma}f)(x)$ are the Riemann–Liouville fractional integration and differentiation operators of order $\sigma > 0$.

**Problem 1.** *Find in domain* $D$ *solution* $u = u(x, y)$ *to equation* (1) *satisfying the following conditions:*

$$y^{1-\gamma}u|_{y=0} = 0, \ (-\infty < x \le 0, 1 \le x < \infty), \tag{4}$$

$$A_1\left(I_{0+}^{a,b,\beta-1-a}u[\Theta_0(t)]\right)(x) + A_2\left(I_{0+}^{a+1,b-1+\alpha,\beta-1-a}\lim_{y\to 0-}(-y)^{\beta_0}u_y(t,y)\right)(x)+$$
$$+A_3\left(I_{0+}^{\gamma+a+\alpha,b-\gamma,\beta-1-a}\lim_{y\to 0-}(-y)^{\beta_0}u_y(t,y)\right)(x) = g(x), \tag{5}$$

*and the conjugation conditions*

$$\lim_{y\to 0+} y^{1-\gamma}u(x,y) = \lim_{y\to 0-} u(x,y), \ \forall x \in \bar{I}, \tag{6}$$

$$\lim_{y\to 0+} y^{1-\gamma}(y^{1-\gamma}u(x,y))_y = \lim_{y\to 0-}(-y)^{\beta_0}u_y(x,y), \ \forall x \in I. \tag{7}$$

*Here* $\alpha = \frac{m+2(\beta_0+\alpha_0)}{2(m+2)}$, $\beta = \frac{m+2(\beta_0-\alpha_0)}{2(m+2)}$, $0 < \alpha, \beta < \frac{1}{2}$, $A_1, A_2, A_3$ *are real constants such that* $A_1 > 0, A_2 \le 0, A_3 \le 0$ *or* $A_1 < 0, A_2 \ge 0, A_3 \ge 0$, $a, b, \gamma$ *are real numbers,* $g(x)$ *is a given function such that* $g(x) \in C^1(\bar{I}) \cap C^2(I)$. We will seek solution $u(x, y)$ to the problem the class of twice differentiable functions in domain $D$ such that $u(x, y)$ tends to zero as $(x^2 + y^2) \to \infty$,

$$y^{1-\gamma}u(x,y) \in C(\overline{D^+}), \ u(x,y) \in C(\overline{D^-}),$$

$$y^{1-\gamma}(y^{1-\gamma}u(x,y))_y \in C(D^+ \cup \ I),$$

$$u_{xx} \in C(D^+ \cup D^-), \ u_{yy} \in C(D^-).$$

Note that nonlocal boundary value problems for equation (1) in unbounded and bounded domains were studied in [15], [16], [17], and for equation (1) for $\alpha_0 = 0$, $\beta_0 = 0$ problems were considered in [9, 11]. In works [18], [19] nonlocal problems with shift on conjugation of two hyperbolic equations of the second order, consisting of a wave equation in one part of the domain and a degenerate hyperbolic equation of the first kind in the other part, are studied.

## Uniqueness of the solution to the problem

**Theorem 1.** *Let condition* $g(x) \equiv 0$ *be satisfied. Then the problem cannot have more than one solution.*

**Proof.** Let there be a solution to the problem. We introduce the following notation

$$\lim_{y\to 0+} y^{1-\gamma}u(x,y) = \tau_1(x), \ \lim_{y\to 0-} u(x,y) = \tau_2(x), \tag{8}$$

$$\lim_{y\to 0+} y^{1-\gamma}(y^{1-\gamma}u(x,y))_y = \nu_1(x), \ \lim_{y\to 0-}(-y)^{\beta_0}u_y(x,y) = \nu_2(x). \tag{9}$$

It is known [5] that the solution to equation (1) in domain $D^+$ satisfying condition (4) and condition

$$\lim_{y\to 0+} y^{1-\gamma}u(x,y) = \tau_1(x), \ \forall x \in \bar{I}$$

is given by the following formula

$$u(x,y) = \int_0^1 G(x,y,t)\tau_1(t)dt, \tag{10}$$

where

$$G(x,y,t) = \frac{\Gamma(\gamma)}{2} y^{\frac{\gamma}{2}-1} e^{1,\frac{\gamma}{2}}_{1,\frac{\gamma}{2}}\left(-|x-t|y^{-\frac{\gamma}{2}}\right),$$

$$e^{\zeta,\xi}_{\mu,\delta} = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\zeta n+\mu)\Gamma(\delta-\xi n)}, \ \zeta > \xi, \ \zeta > 0, \ z \in C$$

is the Wright-type function [6].

By (10), the functional relationship between $\tau_1(x)$ and $\nu_1(x)$ brought the parabolic part $D^+$ to the line $y = 0$, following form [20]

$$\nu_1(x) = \frac{1}{\Gamma(1+\gamma)}\tau_1''(x). \tag{11}$$

Let us find the functional relationship between $\tau_2(x)$ and $\nu_2(x)$ brought to the line $y = 0$ from the hyperbolic part $D^-$ of domain D.

The solution to the modified Chauchy problem (8)-(9) in domain $D^-$ has the following form [15], [21]

$$u(x,y) = \gamma_1\int_0^1 \tau_2\left(x + \frac{2}{m+2}(2t-1)(-y)^{\frac{m+2}{2}}\right)t^{\beta-1}(1-t)^{\alpha-1}dt+$$
$$+\gamma_2(-y)^{1-\beta_0}\int_0^1 \nu_2\left(x+\frac{2}{m+2}(2t-1)(-y)^{\frac{m+2}{2}}\right)t^{-\alpha}(1-t)^{-\beta}dt, \tag{12}$$

где $\gamma_1 = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$, $\gamma_2 = -\frac{2\Gamma(1-\alpha-\beta)}{(m+2)\Gamma(1-\alpha)\Gamma(1-\beta)}$.

From formula (12) and relation (2) we obtain

$$u[\theta(x)] = \gamma_1\Gamma(\alpha)(I^{\alpha,0,\beta-1}_{0+}\tau_2)(x) + \gamma_2\left(\frac{m+2}{4}\right)^{1-\alpha-\beta}\Gamma(1-\beta)(I^{1-\beta,\alpha+\beta-1,\beta-1}_{0+}\nu_2)(x). \tag{13}$$

Substituting (13) into the boundary condition (5), in view of (8) and (9) and applying the following relation [14]

$$(I^{\alpha,\beta,\eta}_{0+}I^{\gamma,\delta,\alpha+\eta}_{0+}f)(x) = (I^{\alpha+\gamma,\beta+\delta,\eta}_{0+}f)(x), (\gamma > 0), \tag{14}$$

we obtain

$$A_1\gamma_1\Gamma(\alpha)\left(I_{0+}^{\alpha+a,b,\beta-1-a}\tau_2\right)(x)+$$
$$+A_1\gamma_2\Big(\frac{m+2}{4}\Big)^{1-\alpha-\beta}\Gamma(1-\beta)\left(I_{0+}^{a+1-\beta,b+\alpha+\beta-1,\beta-1-a}\nu_2\right)(x)+$$
$$+A_2\left(I_{0+}^{a+1,b+\alpha-1,\beta-1-a}\nu_2\right)(x)+A_3\left(I_{0+}^{\gamma+a-\alpha,b-\gamma,\beta-1-a}\nu_2\right)(x)=g(x). \quad (15)$$

where $k_1=A\gamma_1\Gamma(\alpha)$, $k_2=A\gamma_2(\frac{m+2}{2})^{1-\alpha-\beta}\Gamma(1-\beta)$. We apply operator $I_{0+}^{-\alpha-a,-b,\alpha+\beta-1}$ to both sides of equality (15). Direct calculations using formulas (14) and (3) show that

$$\tau_2(x)=-k_1\left(I_{0+}^{1-\alpha-\beta}\nu_2\right)(x)-k_2\left(I_{0+}^{1-\alpha}\nu_2\right)(x)-k_3\left(I_{0+}^{\gamma}\nu_2\right)(x)+g_1(x), \quad (16)$$

where

$$k_1=\frac{\gamma_2(\frac{m+2}{4})^{1-\alpha-\beta}\Gamma(1-\beta)}{\gamma_1\Gamma(\alpha)},\ k_2=\frac{A_2}{A_1\gamma_1\Gamma(\alpha)},$$
$$k_3=\frac{A_3}{A_1\gamma_1\Gamma(\alpha)}, g_1(x)=\frac{1}{A_1\gamma_1\Gamma(\alpha)}\left(I_{0+}^{-a-\alpha,-b,\alpha+\beta-1}g\right)(x).$$

Let us estimate the integral

$$K=\int_0^1\tau_2(x)\nu_2(x)dx.$$

By virtue of the conjugation conditions (6), (7) and relation (11), we have

$$K=\frac{1}{\Gamma(1+\gamma)}\int_0^1\tau_1(x)\tau_1{}''(x)dx.$$

Integrating by parts and assuming that $\tau_1(0)=\tau_1(1)=0$, we obtain

$$K=-\frac{1}{\Gamma(1+\gamma)}\int_0^1[\tau_1{}'(x)]^2dx\le 0. \quad (17)$$

Now we find a lower bound for the integral $K$. For $g(x)=0$, equality (16) takes the following form

$$\tau_2(x)=-k_1\left(I_{0+}^{1-\alpha-\beta}\nu_2\right)(x)-k_2\left(I_{0+}^{1-\alpha}\nu_2\right)(x)-k_3\left(I_{0+}^{\gamma}\nu_2\right)(x)=$$
$$=-\frac{k_1}{\Gamma(1-\alpha-\beta)}\int_0^x\nu_2(t)(x-t)^{-\alpha-\beta}dt-\frac{k_2}{\Gamma(1-\alpha)}\int_0^x\nu_2(t)(x-t)^{-\alpha}dt-$$
$$-\frac{k_3}{\Gamma(\gamma)}\int_0^x\nu_2(t)(x-t)^{\gamma-1}dt,$$

and, therefore,

$$K=-\frac{k_1}{\Gamma(1-\alpha-\beta)}\int\limits_0^1 \nu_2(x)dx\int\limits_0^x (x-t)^{-\alpha-\beta}\nu_2(t)dt-$$

$$-\frac{k_2}{\Gamma(1-\alpha)}\int\limits_0^1 \nu_2(x)dx\int\limits_0^x (x-t)^{-\alpha}\nu_2(t)dt-\frac{k_3}{\Gamma(\gamma)}\int\limits_0^1 \nu_2(x)dx\int\limits_0^x (x-t)^{\gamma-1}\nu_2(t)dt.$$

Next, we use the well-known formula for the gamma function $\Gamma(\sigma)$ [22]

$$\int\limits_0^\infty s^{\sigma-1}\cos(ks)ds=\frac{\Gamma(\sigma)}{k^\sigma}\cos\Big(\frac{\sigma\pi}{2}\Big),\ \ (k>0,\ 0<\sigma<1).$$

Assuming that $k=|x-t|,\ \sigma=\alpha+\beta$, we obtain

$$|x-t|^{-\alpha-\beta}=$$

$$=\frac{1}{\Gamma(\alpha+\beta)\cos(\pi\frac{\alpha+\beta}{2})}\int\limits_0^\infty s^{\alpha+\beta-1}\cos(s\,|x-t|)ds,\ \ (0<\alpha+\beta<1),$$

for $k=|x-t|,\ \sigma=\alpha$ we obtain

$$|x-t|^{-\alpha}=\frac{1}{\Gamma(\alpha)\cos(\frac{\pi\alpha}{2})}\int\limits_0^\infty s^{\alpha-1}\cos(s\,|x-t|)ds,$$

for $k=|x-t|,\ \sigma=1-\gamma$ we obtain

$$|x-t|^{\gamma-1}=\frac{1}{\Gamma(1-\gamma)\cos(\frac{\pi(1-\gamma)}{2})}\int\limits_0^\infty s^{-\gamma}\cos(s\,|x-t|)ds.$$

Applying these formulas and the Dirichlet formula for the permutation of the order of integration in the repeated, we arrive at the following relation

$$K=-\frac{2k_1\sin(\pi\frac{\alpha+\beta}{2})}{\pi}\int\limits_0^\infty s^{\alpha+\beta-1}\Big[\Big(\int\limits_0^1\nu_2(x)\cos(sx)dx\Big)^2+\Big(\int\limits_0^1\nu_2(x)\sin(sx)dx\Big)^2\Big]ds-$$

$$-\frac{2k_2\sin\frac{\alpha\pi}{2}}{\pi}\int\limits_0^\infty s^{\alpha-1}\Big[\Big(\int\limits_0^1\nu_2(x)\cos(sx)dx\Big)^2+\Big(\int\limits_0^1\nu_2(x)\sin(sx)dx\Big)^2\Big]ds-$$

$$-\frac{2k_3\sin\frac{(1-\gamma)\pi}{2}}{\pi}\int\limits_0^\infty s^{-\gamma}\Big[\Big(\int\limits_0^1\nu_2(x)\cos(sx)dx\Big)^2+\Big(\int\limits_0^1\nu_2(x)\sin(sx)dx\Big)^2\Big]ds\geq 0. \quad (18)$$

From (17) and (18), it follows that $K=0$, and, consequently, according to (17)

$$\int\limits_0^1[\tau_1(x)]^2dx=0.$$

Hence, by virtue of equalities $\tau_1(0) = \tau_1(1) = 0$, we obtain $\tau_1(x) = 0$ for all $x \in \bar{I}$.

Thus, according to formula (10), makes it possible to assert that $u(x, y) \equiv 0$ in domain $\bar{D}^{+}$.

By virtue of the conjugation condition (6) $\tau_2(x) \equiv \tau_1(x)$ and so $\tau_2(x) \equiv 0\, \forall x \in [0, 1]$, and by virtue of (7), (9), (11) , also $\nu_1(x) \equiv 0\, \forall x \in [0, 1]$. Then $u(x, y) \equiv 0$ in the domain $\bar{D}^{-}$ as a solution to the modified Cauchy problem with zero data, which proves the uniqueness of the solution to the original problem. □

## Existence of a solution to the problem

**Theorem 2.** *Let conditions* $a > \max\{-\alpha, \beta - 1\}$, $\gamma > 1 - \beta$ *be satisfied. Then the solution to the problem exists.*

**Proof.** Differentiate both sides of relation (16) with respect to $x$ twice:

$$\frac{d^2}{dx^2}\tau_2(x) = -k_1\frac{d^2}{dx^2}\left(I_{0+}^{1-\alpha-\beta}\nu_2\right)(x) - k_2\frac{d^2}{dx^2}\left(I_{0+}^{1-\alpha}\nu_2\right)(x) - k_3\frac{d^2}{dx^2}\left(I_{0+}^{\gamma}\nu_2\right)(x)+$$

$$+\frac{d^2}{d(x)^2}g_1(x),$$

or (assuming that $\tau_1(x) = \tau_2(x) = \tau(x),\ \nu_1(x) = \nu_2(x) = \nu(x)$)

$$\left(D_{0+}^{1+\alpha+\beta}\nu\right)(x) - \lambda\left(D_{0+}^{1+\alpha}\nu\right)(x) - \delta\left(D_{0+}^{2-\gamma}\nu\right)(x) - \mu\nu(x) = g_2(x), \tag{19}$$

where $\lambda = -\frac{k_2}{k_1}$, $\delta = -\frac{k_3}{k_1}$, $\mu = -\frac{\Gamma(1+\gamma)}{k_1}$, $g_2(x) = \frac{1}{k_1}g_1''(x)$. In the monograph [23], the equation with fractional derivatives is considered

$$\left(D_{0+}^{\alpha}y\right)(x) - \lambda\left(D_{0+}^{\beta}y\right)(x) - \delta\left(D_{0+}^{\gamma}y\right)(x) - \mu y(x) = f(x),$$

where $x > 0$, $\alpha > \beta > \gamma > 0$, $\lambda, \mu, \delta \in R$, $l-1 < \alpha \le l$, $l \in R$, and its solution is written out in the following form

$$y(x) = \int_0^x (x-t)^{\alpha-1}G_{\gamma,\beta,\alpha;\lambda}(x-t)f(t)dt.$$

Here

$$G_{\gamma,\beta,\alpha;\lambda}(z) = \sum_{n=0}^{\infty}\Big(\sum_{i+\nu=n}\Big)\frac{\mu^i\delta^\nu}{i!\nu!}z^{(\alpha-\beta)n+\beta i-(\beta-\gamma)\nu}\times$$

$$\times{}_1\Psi_1\left[\begin{array}{c}(n+1,1)\\ \Big((\alpha-\beta)n+\beta i+\alpha+(\beta-\gamma)\nu,\alpha-\beta\Big)\end{array}\Bigg|\lambda z^{\alpha-\beta}\right].$$

$${}_p\Psi_q(z) = \sum_{k=0}^{\infty}c_k z^k,\ c_k = \frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i k)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j k)}\frac{1}{k!},\ (k \in N_0 = \{0, 1, ...\}),$$

$$z, a_i, b_j \in C,\ \alpha_i, \beta_j \in R,\ i = \overline{1,p},\ j = \overline{1,q}.$$

For equation (19), this solution takes the following form

$$\nu(x) = \int_0^x (x-t)^{\alpha+\beta} G_{2-\alpha,1+\alpha,1+\alpha+\beta;\lambda,}(x-t) g_2(t)dt,$$

$$G_{2-\alpha,1+\alpha,1+\alpha+\beta;\lambda,}(x-t) = \sum_{n=0}^{\infty}\Big(\sum_{i+\nu=n}\Big)\frac{\mu^i\delta^\nu}{i!\nu!}(x-t)^{\beta n+(1+\alpha)i-(\alpha+\gamma-1)\nu}\times$$
$$\times {}_1\Psi_1\left[\begin{matrix}(n+1,1)\\ (\beta n+(1+\alpha)i+(1+\alpha+\beta)+(\alpha+\gamma-1)\nu,\beta)\end{matrix}\Big|\lambda(x-t)^\beta\right].$$

This completes the proof of the existence of a solution to the original problem. □

## Conclusion

The article investigated a nonlocal boundary value problem for a fractional diffusion equation and a degenerate hyperbolic equation in an unbounded domain. The main results obtained are new. We can use these results to investigate various boundary value problems for differential equations with a partial fractional derivative.

## References


1. Nigmatullin R. R. The realization of generalized transfer equation in a medium with fractal geometry, *Phys. Status solidi*, 1986. vol. 133, pp. 425–430 (In Russian).
2. Kochubey A. N. Fractional order diffusion, *Differential equations*, 1990. vol. 26, no. 4, pp. 660–670 (In Russian).
3. Gekkieva S. Kh. On one analog of the Tricomi problem for a mixed-type equation with a fractional derivative, *Reports of the AMAN*, 2001. vol. 5, no. 2, pp. 18–22 (In Russian).
4. Gekkieva S. Kh. The Cauchy problem for a generalized transport equation with a fractional time derivative, *Reports of the AMAN*, 2000. vol. 5, no. 1, pp. 16–19 (In Russian).
5. Kilbas A. A., Repin O. A. Analog of the Bitsadze-Samarskii problem for a mixed-type equation with a fractional derivative, *Differential Equations*, 2003. vol. 39, no. 5, pp. 638–644 (In Russian).
6. Pskhu A. V. Solution of boundary value problems for a fractional-order diffusion equation by the Green's function method, *Differential Equations*, 2003. vol. 39, no. 10, pp. 1430–1433 (In Russian).
7. Tomovski Z., Hilfer R., Srivastava H. M. Fractional and operational calculus with generalized fractional derivative operators and Mittag-Leffler type functions, *Trans. and Special functions*, 2010. vol. 21, no. 11, pp. 797–814 DOI:10.1080/10652461003675737.
8. Hilfer R. Experimental evidence for fractional time evolution in glass forming materials, *Chemical Phys.*, 2002. vol. 284, no. 1-2, pp. 399–408.
9. Repin O. A., Frolov A. A. On a boundary value problem for an equation of mixed type with a Riemann–Liouville fractional partial derivative, *Differential Equations*, 2016. vol. 52, no. 10, pp. 1384–1388 DOI: 10.1134/S0012266116100165.
10. Kilbas A. A., Repin O. A. An analogue of the Bitsadze-Samarskiy problem for a mixed-type equation with a fractional derivative, *Differential Equations*, 2003. vol. 39, no. 10, pp. 1430–1433.
11. Repin O. A. Boundary value problem for a differential equation with a partial fractional Riemann-Liouville derivative, *Ufa Mathematical Journal*, 2015. vol. 7, no. 3, pp. 70–75 (In Russian).
12. Zunnunov R. T. Analog of Bitsadze-Samarskii problem for a mixed-type equation with a fractional derivative an unbounded domain, *Uzbek Mathematical Journal*, 2023. vol. 67, no. 3, pp. 189–195.
13. Samko S. G., Kilbas A. A., Repin O. A. *Integrals and derivatives of fractional order and some of their applications.* Minsk: Science and Technology, 1987. 688 pp. (In Russian)
14. Saigo M. A remark on integral operators involving the Gauss hypergeometric function, *Math. Rep. Kyushu Univ.*, 1978. vol. 11, no. 2, pp. 135–143.

15. Ruziev M. Kh. A boundary value problem for a partial differential equation with fractional derivative, *Fractional calculus and Applied Analysis*, 2021. vol. 24, no. 2, pp. 509–517 DOI:10.1515/fca-2021-0022.
16. Ruziev M. Kh., Rakhimova G. B. On a boundary value problem for a differential equation with a partial fractional derivative, *Bulletin of the Institute of Mathematics*, 2023. vol. 6, no. 2, pp. 114–121 (In Russian).
17. Ruziev M. Kh., Zunnunov R. T. On a nonlocal problem for mixed-type equation with partial Riemann-Liouville fractional derivative, *Fractal Fractional*, 2022. vol. 6, no. 2, pp. 110 DOI:10.3390/fractalfract6020110.
18. Balkizov Zh. A. Boundary value problems with data on opposite characteristics for a second-order mixed-hyperbolic equation., *Adyghe Inst. Sci. J.*, 2023. vol. 23, no. 1, pp. 11-19 DOI: 10.47928/1726-9946-2023-23-1-11-19 (In Russian).
19. Balkizov Zh. A. Nonlocal problems with displacement for matching two second order hyperbolic equations., *Ufa Mathematical Journal*, 2023. vol. 15, no. 2, pp. 9-19 DOI: 10.13108/2023-15-2-9.
20. Gekkieva S. Kh. Analog of the Tricomi problem for a mixed-type equation with a fractional derivative, *Izv. Kabardino-Balkarian Sci. center*, 2001. vol. 2, no. 7, pp. 78–80 (In Russian).
21. Ruziev M. Kh., Yuldasheva N. T. On a boundary value problem for a mixed type equations with a partial fractional derivative, *Lobachevskii Journal of Mathematics*, 2022. vol. 43, no. 11, pp. 3264–3270 DOI: 10.1134/S1995080222140293.
22. Prudnikov A. P., Brychkov Yu. A., Marichev O. I *Integrals and series.* Moscow: Nauka, 2003. 688 pp. (In Russian)
23. Kilbas A. A., Srivastava H. M., Trujillo Y. Y. *Theory and applications of fractional differential equations.* Amsterdam-Boston. Tokio: North Holland. Math. Studies, 2006. 204 pp.


## Information about the authors

*Ruziev Menglibay Kholtozhibaevich* ✉ – D. Sci. (Phys. & Math.), Senior Researcher, Leading Researcher, V. I. Romanovsky Institite of Mathematics, Uzbekistan Academy of Sciences, Tashkent, Uzbekistan, ORCID 0000-0002-1097-0137.

*Zunnunov Rakhimjon Temirbekovich* ✉ – D. Sci. (Phys. & Math.), Senior Researcher, Senior Lecturer, Branch of the Russian State University (National Research University) named after I.M.Gubkin in Tashkent, Uzbekistan, ORCID 0000-0001-9392-5464.

*Yuldasheva Nargiza Takhirjonovna* ✉ – Basic Doctoral Candidate, V.I.Romanovsky Institite of Mathematics, Uzbekistan Academy of Sciences, Tashkent, Uzbekistan, ORCID 0000-0001-6921-5374.

*Rakhimova Gulkhayo Botirjon kizi* ✉ – Applicant, Fergana State University, Fergana, Uzbekistan, ORCID 0009-0002-3090-8442.