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The Control Problem for a Heat Conduction Equation with Neumann Boundary Condition

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Abstract. Previously, boundary control problems for a heat conduction equation with Dirichlet boundary condition were studied in a bounded domain. In this paper, we consider the boundary control problem for the heat conduction equation with Neumann boundary condition in a bounded one-dimensional domain. On the part of the border of the considered domain, the value of the solution with control parameter is given. Restrictions on the control are given in such a way that the average value of the solution in some part of the considered domain gets a given value. The studied initial boundary value problem is reduced to the Volterra integral equation of the first type using the method of separation of variables. It is known that the solution of Volterra's integral equation of the first kind cannot always be shown to exist. In our work, the existence of a solution to the Volterra integral equation of the first kind is shown using the method of Laplace transform. For this, the necessary estimates for the kernel of the integral equation were found. Finally, the admissibility of the control function is proved.

Key words: parabolic equation, integral equation, initial-boundary problem, admissible control, Laplace transform.

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Задача управления для уравнения теплопроводности с граничным условием Неймана

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Аннотация. Ранее были исследованы задачи граничного управления для уравнения теплопроводности с граничным условием Дирихле в ограниченной области. В данной работе рассматривается задача граничного управления для уравнения теплопроводности с граничными условиями Неймана в ограниченной одномерной области. На части границы рассматриваемой области задано значение решения с управляющим параметром. Ограничения на управление задаются таким образом, чтобы среднее значение решения в некоторой части рассматриваемой области получало заданное значение. Исследуемая начально-краевая задача сводится к интегральному уравнению Вольтерра первого типа с использованием метода разделения переменных. Известно, что не всегда можно доказать существование решения интегрального уравнения Вольтерра первого рода. В нашей работе существование решения интегрального уравнения Вольтерра первого рода показано с помощью метода преобразования Лапласа. Для этого были найдены необходимые оценки ядра интегрального уравнения. Наконец, допустимость функции управления доказана.

Ключевые слова: параболическое уравнение, интегральное уравнение, начально-краевая задача, допустимое управление, преобразование Лапласа.

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1 Introduction

In this article, we consider the following heat conduction equation in the bounded domain $\Omega = \{(x,t) : 0 < x < l, \quad t > 0\}$:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(k(x)\frac{\partial u(x,t)}{\partial x}\right), \quad (x,t) \in \Omega, \tag{1}$$

with Neumann boundary conditions

$$u_x(0,t) = \ -\mu(t), \quad u_x(l,t) = 0, \quad t > 0, \tag{2}$$

and initial condition

$$u(x,0) = 0, \quad 0 \le x \le l, \tag{3}$$

where $\mu(t)$ is control function.

Assume that the function $k(x) \in C^2\big([0,l]\big)$ satisfies conditions

$$k(x) \ge k_0 > 0, \quad k'(x) \le 0, \quad 0 \le x \le l. \tag{4}$$

Definition 1. It is called that the function $\mu(t) \in W_2^1(\mathbb{R}_+)$ is admissible control, if it fulfills the conditions $\mu(0) = 0$ and $|\mu(t)| \le 1$ for all $t \ge 0$.

Control Problem. For the given function $\theta(t)$ Problem consists looking for the admissible control $\mu(t)$ such that the solution $u(x,t)$ of the initial-boundary problem (1)-(3) exists and for all $t \ge 0$ satisfies the equation

$$\int_0^l u(x,t)\,dx = \theta(t). \tag{5}$$

Control problems for parabolic equations were first studied in [1, 2]. Control problems for the infinite-dimensional case were studied by Egorov [3], who generalized Pontryagin's maximum principle to a class of equations in Banach space, and the proof of a bang-bang principle was shown in the particular conditions.

The optimal time problem for second-order parabolic type equation in the bounded $n-$dimensional domain was studied in [4,5] and the optimal time estimate for achieving a given average temperature was found. The control problem for the heat equation associated with the Neumann boundary condition in a bounded three-dimensional domain is studied in [6]. In this work, an estimate of the optimal time was found when the average temperature is close to the critical value.

In [7, 8], the control problems of the heat equation associated with the Dirichlet boundary condition in the two-dimensional domain are studied. In these articles, an estimate of the minimum time for achieving a given average temperature was found, and the existence of a control function is proved by the Laplace transform method. The boundary control problem related to the fast heating of the thin rod for the inhomogeneous heat conduction equation was studied in works [9] and the existence of the admissible control function was proved.

The minimal time problem for the heat conduction equation with the Neumann boundary condition in a one-dimensional domain is studied in [10]. The difference of this work from the previous works is that the required estimate for the minimum time is found with a non-negative definite weight function under the integral condition. In [11], the control problem for a second-order parabolic type equation with two control functions was studied and the existence of admissible control functions was proved by the Laplace transform method.

Boundary control problems for parabolic type equations are also studied in works [12–14].

A lot of information on the optimal control problems was given in detail in the monographs of Lions and Fursikov [15,16]. General numerical optimization and optimal boundary control have been studied in a great number of publications such as [17]. The practical approaches to optimal control of the heat equation are described in publications like [18].

In this work, the boundary control problem for the heat transfer equation is considered. The difference of this work from the previous works is that in this problem, the control problem for the heat conduction equation related to the Neumann boundary condition is studied. In Section 2, the boundary control problem studied in this work is reduced to the Volterra integral equation of the first kind by the Fourier method. In Section 3, the solution of Volterra's integral equation is proved using the Laplace transform method.

2 Main integral equation

Consider the following spectral problem

$$\frac{d}{dx}\left(k(x)\frac{dv_k(x)}{dx}\right) + \lambda_k v_k(x) = 0, \quad 0 < x < l, \tag{6}$$

with boundary condition

$$v_k'(0) = v_k'(l) = 0, \quad 0 \le x \le l. \tag{7}$$

It is well-know that this problem is self-adjoint in $L_2(\Omega)$ and there exists a sequence of eigenvalues $\{\lambda_k\}$ so that $\lambda_1 \le \lambda_2 \le ... \le \lambda_k \to \infty, \quad k \to \infty$.

The corresponding eigenfuction v_k form a complete orthonormal system $\{v_k\}$ in $L_2(\Omega)$ and these function belong to $C(\bar{\Omega})$, where $\bar{\Omega} = \Omega \cup \partial\Omega$ (see [19,20]).

Definition 2. By the solution of the problem (1)–(3) we understand the function $u(x,t)$ represented in the form

$$u(x,t) = \frac{(l-x)^2}{2l}\,\mu(t) - w(x,t), \tag{8}$$

where the function $w(x,t) \in C^{2,1}_{x,t}(\Omega) \cap C(\bar{\Omega})$, $w_x \in C(\bar{\Omega})$ is the solution to the problem:

$$w_t = \frac{\partial}{\partial x}\left(k(x)\,\frac{\partial w}{\partial x}\right) + \frac{d}{dx}\left(k(x)\,\frac{l-x}{l}\right)\mu(t) + \frac{(l-x)^2}{2l}\,\mu'(t), \tag{9}$$

with boundary value conditions

$$w_x(0,t)=0, \quad w_x(l,t)=0, \tag{10}$$

and initial value condition

$$w(x,0)=0. \tag{11}$$

We set

$$\beta_k=\left(\lambda_k b_k-a_k\right)c_k, \quad k=1,2,..., \tag{12}$$

where coefficients a_k, b_k and c_k are as follows

$$a_k=\int_0^l \frac{d}{dx}\left(k(x)\,\frac{l-x}{l}\right)\nu_k(x)dx, \quad b_k=\int_0^l \frac{(l-x)^2}{2l}\,\nu_k(x)dx, \tag{13}$$

and

$$c_k=\int_0^l \nu_k(x)dx. \tag{14}$$

We understand the coefficients a_0 and b_0 as follows

$$a_0=\int_0^l \frac{d}{dx}\left(k(x)\,\frac{l-x}{l}\right)dx=-\frac{k(0)}{2},$$

and

$$b_0=\int_0^l \frac{(l-x)^2}{2l}\,dx=\frac{l^2}{6}.$$

Thus, we have

$$w(x,t)=\frac{l^2}{12}\,\mu(t)-\frac{k(0)}{2}\int_0^t \mu(s)\,ds+$$

$$+\sum_{k=1}^{\infty}\left(\int_0^t e^{-\lambda_k(t-s)}\left(\mu(s)\,a_k+\mu'(s)\,b_k\right)ds\right)\nu_k(x), \tag{15}$$

where a_k and b_k defined by (13).

From (8) and (15), we get the solution of the mixed problem (1)–(3) (see [19]):

$$u(x,t)=\frac{(l-x)^2}{2l}\,\mu(t)-\frac{l^2}{12}\,\mu(t)+\frac{k(0)}{2}\int_0^t \mu(s)\,ds-$$

$$-\sum_{k=1}^{\infty}\left(\int_0^t e^{-\lambda_k(t-s)}\left(\mu(s)\,a_k+\mu'(s)\,b_k\right)ds\right)\nu_k(x).$$

We know that the eigenvalues λ_k of the boundary value problem (6)-(7) satisfies the following inequalities

$$\lambda_k \geq 0, \quad k = 0, 1, ... \tag{16}$$

Indeed, since

$$\frac{d}{dx}\left(k(x)\frac{d\nu_k(x)}{dx}\right) + \lambda_k \nu_k(x) = 0, \quad 0 < x < l,$$

then we get

$$\lambda_k = -\int_0^l \frac{d}{dx}\left(k(x)\frac{d\nu_k(x)}{dx}\right)\nu_k(x)dx = \int_0^l k(x)\,|\nu_k'(x)|^2 dx \geq 0.$$

According to condition (5) and the solution of the problem (1)-(3), we may write

$$\theta(t) = \int_0^l u(x,t)dx = \mu(t)\int_0^l \frac{(l-x)^2}{2l}\,dx - \frac{l^3}{12}\,\mu(t) + \frac{k(0)\,l}{2}\int_0^t \mu(s)\,ds -$$

$$-\sum_{k=1}^{\infty}\left(\int_0^t e^{-\lambda_k(t-s)}\big(\mu(s)a_k + \mu'(s)\,b_k\big)\,ds\right)\int_0^l \nu_k(x)\,dx =$$

$$= \mu(t)\int_0^l \frac{(l-x)^2}{2l}\,dx - \frac{l^3}{12}\,\mu(t) + \frac{k(0)\,l}{2}\int_0^t \mu(s)\,ds -$$

$$-\sum_{k=1}^{\infty} a_k\,c_k\int_0^t e^{-\lambda_k(t-s)}\,\mu(s)ds - \sum_{k=1}^{\infty} b_k\,c_k\int_0^t e^{-\lambda_k(t-s)}\mu'(s)ds =$$

$$= \mu(t)\int_0^l \frac{(l-x)^2}{2l}\,dx - \frac{l^3}{12}\,\mu(t) + \frac{k(0)\,l}{2}\int_0^t \mu(s)\,ds -$$

$$-\sum_{k=1}^{\infty} a_k\,c_k\int_0^t e^{-\lambda_k(t-s)}\mu(s)ds - \mu(t)\sum_{k=1}^{\infty} b_k\,c_k +$$

$$+\sum_{k=1}^{\infty}\lambda_k\,b_k\,c_k\int_0^t e^{-\lambda_k(t-s)}\mu(s)ds, \tag{17}$$

where c_k defined by (14).

Note that

$$\int_0^l \frac{(l-x)^2}{2l}\,dx = \frac{l^3}{12} + \sum_{k=1}^{\infty} b_k\,c_k, \tag{18}$$

where b_k, c_k are defined by (13) and (14).

As a result, from (17) and (18), we obtain

$$\theta(t) = \int_0^t \left(\frac{k(0)\, l}{2} + \sum_{k=1}^{\infty} (\lambda_k\, b_k - a_k)\, c_k e^{-\lambda_k (t-s)} \right) \mu(s) ds.$$

We set

$$B(t) = \frac{k(0)\, l}{2} + \sum_{k=1}^{\infty} \beta_k\, e^{-\lambda_k t}, \quad t > 0. \tag{19}$$

where β_k defined by (12).

Then we get the main integral equation

$$\int_0^t B(t-s)\, \mu(s) ds = \theta(t), \quad t > 0. \tag{20}$$

Lemma 1. *For the cofficients* $\{\beta_k\}_{k=1}^{\infty}$ *the following estimate is valid:*

$$0 \le \beta_k \le C, \quad k = 1, 2, ...,$$

where C *is a positive constant.*

Proof. First we calculate the following equality using (13)

$$\lambda_k\, b_k = \int_0^l \frac{(l-x)^2}{2l} \lambda_k\, \nu_k(x) dx = -\int_0^l \frac{(l-x)^2}{2l} \frac{d}{dx}\left(k(x) \frac{d\nu_k(x)}{dx} \right) dx =$$

$$= -\left(\frac{(l-x)^2}{2l} k(x)\, \nu_k'(x) \Big|_{x=0}^{x=l} + \int_0^l \frac{(l-x)}{l} k(x)\, \nu_k'(x) dx \right) =$$

$$= -\int_0^l \frac{(l-x)}{l} k(x)\, \nu_k'(x) dx = k(0)\nu_k(0) + \int_0^l \frac{d}{dx}\left(k(x) \frac{l-x}{l} \right) \nu_k(x) dx =$$

$$= k(0)\nu_k(0) + a_k.$$

Then we have

$$\lambda_k\, b_k - a_k = k(0)\nu_k(0), \quad k = 1, 2, ... \tag{21}$$

We know that the following inequality is true (see [20])

$$\nu_k(0) \int_0^l \nu_k(x)\, dx \ge 0, \quad k = 1, 2, ... \tag{22}$$

Thus, by (21) and (22), we have

$$\beta_k = (\lambda_k\, b_k - a_k)\, c_k = k(0)\, \nu_k(0) \cdot \int_0^l \nu_k(x) dx \ge 0.$$

It is clear that if $k(x) \in C^1([0,l])$, we may write the estimate (see [21,22])

$$\max_{0\le x\le l} |\nu_k(x)| \le C.$$

From this we can obtain the following estimates

$$\beta_k \le k(0)\left|\nu_k(0)\, c_k\right| \le C.$$

□

Lemma 2. *Let $1/2 < \alpha < 1$. Then for the function $B(t)$ defined by (19) the following estimate is valid:*

$$0 < B(t) \le \frac{C_\alpha}{t^\alpha}, \quad 0 < t \le 1,$$

where C_α is a constant depending only on α.

Proof. It is known from the general theory that if $k(x)$ is a smooth function, the following estimate is valid (see [22]):

$$\lambda_k = \frac{k^2\pi^2}{p^2} + O(k^{-2}), \quad p = \int_0^l \frac{dx}{\sqrt{k(x)}}.$$

Let $1/2 < \alpha < 1$ and $\lambda > 0$. Then the maximum value of the function $h(t,\lambda) = t^\alpha e^{-\lambda t}$ is reached at the point $t = \frac{\alpha}{\lambda}$ and this value is equal to $\frac{\alpha^\alpha}{\lambda^\alpha} e^{-\alpha}$.

As a result, for any $1/2 < \alpha < 1$, we get the estimate

$$B(t) \le \mathrm{const}\, \frac{1}{t^\alpha} \sum_{k=1}^{\infty} \beta_k t^\alpha e^{-\lambda_k t} \le \frac{C}{t^\alpha} \sum_{k=1}^{\infty} \frac{\alpha^\alpha}{\lambda_k^\alpha} e^{-\alpha} \le \frac{C_\alpha}{t^\alpha},$$

where

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k^\alpha} < +\infty.$$

□

3 Main result

In this section, we prove the existence of the control function.

Denote by $W(M)$ the set of function $\theta \in W_2^2(-\infty, +\infty)$, which satisfies the condition

$$\|\theta\|_{W_2^2(\mathbb{R}_+)} \le M \quad \theta(t) = 0 \quad \text{for} \quad t \le 0.$$

Theorem 1. *There exists $M > 0$ such that for any function $\theta \in W(M)$ the solution $\mu(t)$ of the equation (20) exists, belongs to $C(\overline{\mathbb{R}_+})$ and satisfies condition*

$$|\mu(t)| \le 1.$$

We use the Laplace transform method to solve the integral equation (20). It is known

$$\widetilde{\mu}(p)=\int_0^\infty e^{-pt}\,\mu(t)\,dt.$$

Then we use Laplace transform obtain the following equation

$$\widetilde{\theta}(p)=\int_0^\infty e^{-pt}\,dt\int_0^t B(t-s)\mu(s)ds=\widetilde{B}(p)\,\widetilde{\mu}(p).$$

Thus, we get

$$\widetilde{\mu}(p)=\frac{\widetilde{\theta}(p)}{\widetilde{B}(p)},\quad \text{where}\;\; p=a+i\xi,\quad a>0,\quad \xi\in R,$$

and

$$\mu(t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\widetilde{\theta}(p)}{\widetilde{B}(p)}\,e^{pt}dp=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\widetilde{\theta}(a+i\xi)}{\widetilde{B}(a+i\xi)}\,e^{(a+i\xi)t}d\xi. \tag{23}$$

Then we can write

$$\widetilde{B}(p)=\int_0^\infty B(t)e^{-pt}\,dt=\frac{k(0)\,l}{2}\int_0^\infty e^{-pt}\,dt+\sum_{k=1}^\infty \beta_k\int_0^\infty e^{-(p+\lambda_k)t}\,dt=$$

$$=\frac{k(0)\,l}{2}\,\frac{1}{p}+\sum_{k=1}^\infty\frac{\beta_k}{p+\lambda_k},$$

where $B(t)$ defined by (19) and

$$\widetilde{B}(a+i\xi)=\frac{k(0)\,l}{2}\,\frac{1}{a+i\xi}+\sum_{k=1}^\infty\frac{\beta_k}{a+\lambda_k+i\xi}=$$

$$=\frac{k(0)\,l}{2}\frac{a}{a^2+\xi^2}+\sum_{k=1}^\infty\frac{\beta_k\,(a+\lambda_k)}{(a+\lambda_k)^2+\xi^2}-$$

$$-\frac{k(0)\,l}{2}\frac{i\,\xi}{a^2+\xi^2}-i\xi\sum_{k=1}^\infty\frac{\beta_k}{(a+\lambda_k)^2+\xi^2}=$$

$$=\mathrm{Re}\widetilde{B}(a+i\xi)+i\,\mathrm{Im}\widetilde{B}(a+i\xi),$$

where

$$\mathrm{Re}\widetilde{B}(a+i\xi)=\frac{k(0)\,l}{2}\frac{a}{a^2+\xi^2}+\sum_{k=1}^\infty\frac{\beta_k\,(a+\lambda_k)}{(a+\lambda_k)^2+\xi^2},$$

$$\mathrm{Im}\widetilde{B}(a+i\xi)=-\frac{k(0)\,l}{2}\frac{\xi}{a^2+\xi^2}-\xi\sum_{k=1}^\infty\frac{\beta_k}{(a+\lambda_k)^2+\xi^2}.$$

We know that

$$(a+\lambda_k)^2+\xi^2\le[(a+\lambda_k)^2+1](1+\xi^2),$$

and we get the following inequalities

$$\frac{1}{a^2+\xi^2} \geq \frac{1}{1+\xi^2}\,\frac{1}{1+a^2}, \tag{24}$$

and

$$\frac{1}{(a+\lambda_k)^2+\xi^2} \geq \frac{1}{1+\xi^2}\,\frac{1}{(a+\lambda_k)^2+1}. \tag{25}$$

Consequently, according to inequalities (24) and (25) we can obtain the following estimates

$$|\mathrm{Re}\widetilde{B}(a+i\xi)| = \frac{k(0)\,l}{2}\frac{a}{a^2+\xi^2} + \sum_{k=1}^{\infty}\frac{\beta_k\,(a+\lambda_k)}{(a+\lambda_k)^2+\xi^2} \geq$$

$$\geq \frac{1}{1+\xi^2}\left(\frac{k(0)\,l}{2}\frac{a}{1+a^2} + \sum_{k=1}^{\infty}\frac{\beta_k\,(a+\lambda_k)}{(a+\lambda_k)^2+1}\right) = \frac{C_{1a}}{1+\xi^2}, \tag{26}$$

and

$$|\mathrm{Im}\widetilde{B}(a+i\xi)| = |\xi|\left(\frac{k(0)\,l}{2}\frac{1}{a^2+\xi^2} + \sum_{k=1}^{\infty}\frac{\beta_k}{(a+\lambda_k)^2+\xi^2}\right) \geq$$

$$\geq \frac{|\xi|}{1+\xi^2}\left(\frac{k(0)\,l}{2}\frac{1}{1+a^2} + \sum_{k=1}^{\infty}\frac{\beta_k}{(a+\lambda_k)^2+1}\right) = \frac{C_{2a}\,|\xi|}{1+\xi^2}, \tag{27}$$

where C_{1a}, C_{2a} as follows

$$C_{1a} = \frac{k(0)\,l}{2}\frac{a}{1+a^2} + \sum_{k=1}^{\infty}\frac{\beta_k\,(a+\lambda_k)}{(a+\lambda_k)^2+1},$$

and

$$C_{2a} = \frac{k(0)\,l}{2}\frac{1}{1+a^2} + \sum_{k=1}^{\infty}\frac{\beta_k}{(a+\lambda_k)^2+1}.$$

From (26) and (27), we have the following estimate

$$|\widetilde{B}(a+i\xi)|^2 = |\mathrm{Re}\widetilde{B}(a+i\xi)|^2 + |\mathrm{Im}\widetilde{B}(a+i\xi)|^2 \geq \frac{\min(C_{1a}^2, C_{2a}^2)}{1+\xi^2},$$

and

$$|\widetilde{B}(a+i\xi)| \geq \frac{C_a}{\sqrt{1+\xi^2}}, \quad \text{where} \quad C_a = \min(C_{1a}, C_{2a}). \tag{28}$$

Then, when $a \to 0$ from (23), we obtain

$$\mu(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\widetilde{\theta}(i\xi)}{\widetilde{B}(i\xi)}\,e^{i\xi t}d\xi. \tag{29}$$

Lemma 3. *[9] Assume that $\theta(t) \in W(M)$. Then for the image of the function $\theta(t)$ the following inequality*

$$\int_{-\infty}^{+\infty}|\widetilde{\theta}(i\xi)|\sqrt{1+\xi^2}d\xi \leq C\,\|\theta\|_{W_2^2(R_+)},$$

is valid.

Now we present the proof of the Theorem 1.

Proof.

Now, we show that $\mu \in W_2^1(\mathbb{R}_+)$. Indeed, according to (28) and (29), we obtain

$$\int_{-\infty}^{+\infty} |\widetilde{\mu}(\xi)|^2(1+|\xi|^2)\,d\xi \;=\; \int_{-\infty}^{+\infty} \left|\frac{\widetilde{\theta}(i\xi)}{\widetilde{B}(i\xi)}\right|^2 (1+|\xi|^2)\,d\xi \;\leq$$

$$\leq C \int_{-\infty}^{+\infty} |\widetilde{\theta}(i\xi)|^2(1+|\xi|^2)^2\,d\xi \;=\; C\|\theta\|^2_{W_2^2(\mathbb{R})}.$$

Further,

$$|\mu(t)-\mu(s)| \;=\; \left|\int_s^t \mu'(\tau)\,d\tau\right| \;\leq\; \|\mu'\|_{L_2}\sqrt{t-s}.$$

Hence, $\mu \in \operatorname{Lip}\alpha$, where $\alpha = 1/2$. From (28), (29) and Lemma 3, we can write

$$|\mu(t)| \leq \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{|\widetilde{\theta}(i\xi)|}{|\widetilde{B}(i\xi)|}d\xi \leq \frac{1}{2\pi C_0}\int_{-\infty}^{+\infty} |\widetilde{\theta}(i\xi)|\sqrt{1+\xi^2}d\xi \leq$$

$$\leq \frac{C}{2\pi C_0}\|\theta\|_{W_2^2(\mathbb{R}_+)} \leq \frac{CM}{2\pi C_0} = 1,$$

where

$$M = \frac{2\pi C_0}{C}.$$

□

4 Conclusion

In this paper, we have considered the boundary control problem for a parabolic-type equation in a one-dimensional bounded domain. By the method of separation of variables, the control problem was reduced to the Volterra integral equation of the first kind. Using the Laplace transform method, the existence of a solution to the integral equation was found and the admissibility of the control function was proved.

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