Известия Саратовского университета. Новая серия. Серия: Математика. Механика. Информатика. 2025. Т. 25, вып. 3. С. 316–324
Izvestiya of Saratov University. Mathematics. Mechanics. Informatics, 2025, vol. 25, iss. 3, pp. 316–324
https://mmi.sgu.ru DOI: https://doi.org/10.18500/1816-9791-2025-25-3-316-324
EDN: https://elibrary.ru/AXCYPW

Article

Asymptotics of optimal investment behavior under a risk process with two-sided jumps

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Abstract. We study an optimal investment control problem for an insurance company having two business branches, life annuity insurance and non-life insurance. The company can invest its surplus into a risk-free asset and a risky asset with the price dynamics given by a geometric Brownian motion. The optimization objective is to maximize the survival probability of the total portfolio over the infinite time interval. In the absence of investments, the portfolio surplus is described by a stochastic process involving two-sided jumps and a continuous drift. Downward jumps correspond to the claim sizes, and upward jumps are interpreted as random gains that arise at the final moments of the life annuity contracts’ realizations (i.e., at the moments of death of policyholders) as a result of the release of unspent funds. The drift is determined by the difference between premiums in the non-life insurance contracts and the annuity payments. The solving to the optimization problem that yields the maximal survival probability, as well as the optimal strategy, is related to the classical solution of the corresponding Hamilton – Jacobi – Bellman (HJB) equation, if this solution exists. In the considered risk model, HJB includes integral operators of two types: Volterra and non-Volterra ones. The presence of the latter makes the asymptotic analysis of the solution quite complicated. However, for the case of small jumps (when the jumps have exponential distributions), we obtained asymptotic representations of solutions for both small and large values of the initial surplus.

Keywords: insurance, two-sided jumps, optimal investments, risky asset, survival probability, Hamilton – Jacobi – Bellman equation

For citation: Belkina T. A., Kurochkin S. V., Tarkhova A. E. Asymptotics of optimal investment behavior under a risk process with two-sided jumps. *Izvestiya of Saratov University. Mathematics. Mechanics. Informatics*, 2025, vol. 25, iss. 3, pp. 316–324. DOI: 10.18500/1816-9791-2025-25-3-316-324, EDN: AXCYPW

Научная статья
УДК 519.624,519.86

Асимптотики оптимального инвестиционного поведения в модели риска с двусторонними скачками

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Аннотация. Исследуется проблема оптимального управления инвестициями для страховой компании, имеющей два направления бизнеса: страхование пожизненной ренты и рисковое страхование (не связанное со страхованием жизни). Компания может инвестировать свой излишек в безрисковый актив и рисковый актив с динамикой цен, заданной геометрическим броуновским движением. Целью оптимизации является максимизация вероятности неразорения по суммарному портфелю на бесконечном интервале времени. При отсутствии инвестиций излишек портфеля описывается стохастическим процессом, включающим двусторонние скачки и непрерывный детерминированный снос. Скачки вниз соответствуют размерам требований по рисковому страхованию, а скачки вверх интерпретируются как случайные доходы, возникающие в конечные моменты реализации договоров пожизненной ренты (т. е. в моменты смерти страхователей) в результате высвобождения неизрасходованных средств. Непрерывный снос определяется разностью между премиями по договорам рискового страхования и аннуитетными платежами. Решение задачи оптимизации, которое дает максимальную вероятность неразорения, а также оптимальную стратегию, связано с классическим решением соответствующего уравнения Гамильтона – Якоби – Беллмана (HJB), если это решение существует. В рассматриваемой модели риска HJB включает интегральные операторы двух типов: вольтерровские и невольтерровские. Наличие последних делает асимптотический анализ решения достаточно сложным. Однако для случая малых скачков (когда скачки имеют показательное распределение) получены асимптотические представления решений как для малых, так и для больших значений начального резерва.

Ключевые слова: страхование, двусторонние скачки, инвестиции, рисковый актив, вероятность неразорения, уравнение Гамильтона – Якоби – Беллмана

Для цитирования: *Belkina T. A., Kurochkin S. V., Tarkhova A. E.* Asymptotics of optimal investment behavior under a risk process with two-sided jumps [*Белкина Т. А., Курочкин С. В., Тархова А. Е.* Асимптотики оптимального инвестиционного поведения в модели риска с двусторонними скачками] // Известия Саратовского университета. Новая серия. Серия: Математика. Механика. Информатика. 2025. Т. 25, вып. 3. С. 316–324. DOI: 10.18500/1816-9791-2025-25-3-316-324, EDN: AXCYPW

Introduction

The optimal investor behavior whose objective is to minimize the ruin probability in the presence of uncontrollable stochastic cash flow, or a risk process, was first studied in [1]. It was assumed that the risk process (for example, the surplus of an insurer) is described by Brownian motion with drift and the risky stock price following a geometric Brownian motion. Such a model of the surplus process for the insurer can be obtained as a result of the diffusion approximation of the net claims process in the classical Cramér-Lundberg (CL) model [2]. For the case when there is no risk-free interest rate, it was shown in [1] that the non-constrained optimal policy is to always invest a constant amount of money into a risky asset (regardless of the level of surplus). If the interest rate is positive, the optimal control is a state-dependent function. In particular, for the case of zero correlation between the processes of insurance risk and of asset price, it was shown that the optimal amount tends to zero as the surplus tends to infinity.

For the CL model (a compound Poisson risk model with negative jumps and positive deterministic drift), the structure of the optimal strategy changes crucially at least, at low level of surplus:

a constant amount (CA) strategy, which is similar to the one described above, cannot be optimal, because it leads to immediate ruin at zero initial surplus. This is contrary with to a non-zero survival probability at zero initial surplus in the absence of investments. Moreover, as shown in [3,4], the optimal amount tends to zero as the surplus tends to zero. Nevertheless, the optimal strategy requires borrowing money when the surplus level is close to zero.

Of course, borrowing restrictions can also lead to significant changes in the optimal investment behaviors; moreover, unexpected effects can arise in the zone of low surplus levels. For the CL model, the optimal investment problem with limited leveraging level and allowed shortselling was studied in [5]: it was shown that some unusual strategy (short-selling the high return stock to earn interest) can be optimal when a strong investment constraint on borrowing (money) and buying (stock) is imposed. As for the optimal investment strategy in the CL model at large levels of surplus, it turns out to be asymptotically close to some CA strategy in the case of small jumps in the risk process (i.e., in the case when exponential moments of jumps are finite, see, e.g., [6,7]).

In the case of zero interest rate, some (non-zero) CA strategy again becomes optimal for arbitrary initial surplus (not only asymptotically) if we consider a compound Poisson risk model with small positive jumps and negative deterministic drift (see [8]), which is interpreted as a *life annuity insurance model* [2]. In the general case of a non-negative interest rate, a limit value of the optimal amount at zero surplus is also not zero for this model, as well as for the diffusion risk model [1] and for the CL model perturbed by diffusion [9].

Let us recall that, as stated above, in the CL model, the corresponding limit (at zero surplus) is equal to zero. As will be shown below for a more general model with *two-sided jumps*, this fact can be established a priori (i.e., before solving the optimization problem), and it is important for formulating the correct condition on the solution to Hamilton – Jacobi – Bellman equation (HJB equation) as the optimal survival probability function (in particular, this condition allows us to reject the solution corresponding to the CA strategy). We show in this paper that in the case of exponential distributions of jumps, the optimal strategy, the value function, and some additionally introduced function satisfy a system of nonlinear ordinary differential equations (ODE) of the first order with initial conditions depending on an unknown parameter. This parameter can be calculated after solving the problem for the ODE with a parameter using the normalization condition. As a result of the asymptotic study of the ODE system, we obtain asymptotic representations for the optimal strategy and the value function.

1. The model description and optimization problem

We will consider below the optimal investment problem in the presence of an uncontrollable risk process with two-sided jumps. This process can be considered as a surplus process of an insurance portfolio that combines surpluses for two types of insurance business: life and non-life insurance (see [10,11] and references there). The total portfolio surplus is of the form

$$R_t = u + ct + \sum_{i=1}^{N_1(t)} Y_i - \sum_{j=1}^{N(t)} Z_j, \quad t \geqslant 0. \tag{1}$$

Here R_t is the total portfolio surplus at time $t \geqslant 0$; u is the initial surplus, c is the difference between the premium rate in non-life insurance and the life annuity rate (or the pension payments per unit of time), assumed to be deterministic and fixed. $N_1(t)$ is a homogeneous Poisson process with intensity $\lambda_1 > 0$ that, for any $t > 0$, determines the number of random revenues up to the time t; Y_k $(k = 1, 2, ...)$ are independent identically distributed (i.i.d.) random variables (r.v.) with a distribution function $G(z)$ $(G(0) = 0,\ \mathbf{E}Y_1 = n < \infty,\ n > 0)$ that determine the revenue sizes and are assumed to be independent of $N_1(t)$. These random revenues arise at the final moments of the life annuity contracts' realizations. Further, $N(t)$ is a homogeneous Poisson process with intensity $\lambda > 0$ that, for any $t > 0$, determines the number of claims up to the time t; Z_k $(k = 1, 2, ...)$ are i.i.d. r.v. with a distribution function $F(z)$ $(F(0) = 0,\ \mathbf{E}Z_1 = m < \infty,$

$m > 0$) that determine the the claim sizes and are assumed to be independent of $N(t)$. In addition, we assume that the processes of total premiums and total payments are independent. If $\lambda_1 = 0$, $\lambda > 0$, we have a CL model (only with the second sum in (1)); if $\lambda_1 > 0$, $\lambda = 0$ (only with the first sum in (1)), this is the life annuity insurance model.

Suppose that at time t, the insurance company invests a fraction α_t of the surplus to a risky asset whose price follows a geometric Brownian motion $dS_t = \mu S_t dt + \sigma S_t dw_t$, where μ is the stock return rate, σ is the volatility, and $w := \{w_t\}_{t\geqslant 0}$ is a standard Brownian motion independent of $\{N(t)\}_{t\geqslant 0}$, $\{N_1(t)\}_{t\geqslant 0}$ Y_i's and Z_i. The rest fraction $(1-\alpha_t)$ of the surplus is invested in a risk-free asset which evolves as $dP_t = rP_t dt$, where r is the interest rate.

With dynamic investment control, denoted by $\pi := \{\alpha_s\}_{s\geqslant 0}$, the surplus process is governed by

$$dX_t^\pi = (\mu - r)\alpha_t X_t^\pi dt + rX_t^\pi dt + \sigma\alpha_t X_t^\pi dw_t + dR_t, \quad t \geqslant 0, \quad X_0 = u, \tag{2}$$

where R_t is defined by (1).

Definition 1. A control policy $\pi := \{\alpha_s\}_{s\geqslant 0}$ is said to be *admissible* if α_t satisfies the following conditions:

(i) α_t is $\mathcal{F}_t$ predictable where $\{\mathcal{F}_t\}_{t\geqslant 0}$ is a filtration generated by processes $\{w_t\}_{t\geqslant 0}$ and $\{R_t\}_{t\geqslant 0}$;

(ii) $\alpha_t X_t^\pi$ is square integrable over any finite time interval almost surely.

Note that we do not impose assumption $0 \leqslant \alpha_t \leqslant 1$ for $t \geqslant 0$. This means that we allow both borrowing and shortselling, and $\alpha_t \in \mathbb{R}$, $t \geqslant 0$.

We denote by Π the set of all admissible controls. The survival probability of the process X_t^π defined in (2) under policy $\pi \in \Pi$ is $\varphi^\pi(u) = \mathbf{P}\left(X_t^\pi \geqslant 0,\ t \geqslant 0\right)$, and the maximal survival probability (the value function) is

$$V(u) = \sup_{\pi\in\Pi} \varphi^\pi(u). \tag{3}$$

It is clear that $V(u)$ is a non-decreasing function by its definition. If we assume that V is twice continuously differentiable, then it solves the following HJB equation:

$$\sup_{\alpha\in\mathbb{R}}\Big\{\frac{1}{2}\sigma^2\alpha^2u^2V''(u) + V'(u)[(\alpha(\mu-r)+r)u + c] - (\lambda+\lambda_1)V(u) + \\ +\lambda\int_0^u V(u-z)\,dF(z) + \lambda_1\int_0^\infty V(u+z)\,dG(z)\Big\} = 0, \quad u > 0. \tag{4}$$

2. Preliminary results for the case r=0: Lundberg bounds for ruin probabilities under CA strategies and lower bound for the value function

Let us return to the controlled process of the form (2) and denote $A_t := \alpha_t X_t^\pi$. Then A_t is an amount of money invested in a risky asset at the moment t, and equation (2) can be rewritten as

$$dX_t^\pi = (\mu - r)A_t dt + rX_t^\pi dt + \sigma A_t dw_t + dR_t, \quad t \geqslant 0, \quad X_0 = u.$$

If $r = 0$ and $A_t \equiv A$ for some constant A, then we have a CA strategy and the corresponding process (for brevity, we will denote it as X_t^A) satisfies the equation

$$dX_t^A = \mu A dt + \sigma A dw_t + dR_t, \quad t \geqslant 0, \quad X_0 = u. \tag{5}$$

This process can be considered as a process R_t perturbed by diffusion with drift (in the case $A \neq 0$). Assuming finiteness of exponential moments of Y_1 and Z_1, we can write the equation

$$\lambda(M_{Z_1}(\gamma) - 1) + \lambda_1(M_{-Y_1}(\gamma) - 1) - (c + \mu A)\gamma + \frac{1}{2}\sigma^2A^2\gamma^2 = 0,$$

where $M_Z(\gamma)$ is the moment generating function of r.v. Z. It can be checked that if there exists a positive $\gamma(A)$ satisfying the last equation (so-called adjustment coefficient), then the process $\exp(-\gamma X_t^A)$ is a *martingale* with mean equal to $\exp\{-\gamma(A)u\}$. This property allows us to obtain the Lundberg bounds for the ruin probability $\Psi^A(u) = 1-\varphi^A(u)$: $\Psi^A(u) \leqslant K\exp\{-\gamma(A)u\}$ for some constant K, $0 < K \leqslant 1$ (for the CL model perturbed by diffusion see, e.g., [12]). It is easy to understand that, for $\gamma = \sup\limits_A \gamma(A)$, the maximizer A^* is also the minimizer for the equation

$$\inf_A \left[\lambda(M_{Z_1}(\gamma)-1) + \lambda_1(M_{-Y_1}(\gamma)-1) - (c+\mu A)\gamma + \frac{1}{2}\sigma^2 A^2\gamma^2\right] = 0,$$

therefore, we get $A^* = \dfrac{\mu}{\sigma^2\gamma}$, where γ is a positive solution of the equation

$$\lambda M_{Z_1}(\gamma) + \lambda_1 M_{-Y_1}(\gamma) = \lambda + \lambda_1 + c\gamma + \frac{\mu^2}{2\sigma^2} \tag{6}$$

(for the case $\lambda_1 = 0$, i.e., in CL model with investments, the corresponding equation see in [7]). Let us introduce the following assumption:

(A) $c > 0$; the *safety loading* is positive, i.e., $c + \lambda_1 n - \lambda m > 0$.

If the assumption **(A)** holds, then the positive solution of the equation (6) is unique (we will see it below for the case of exponential distributions of jumps). Thus, we get the best estimate in the set of all CA strategies: $\Psi^{A^*}(u) \leqslant K\exp\{-\gamma u\}$, hence, for the value function (3) the following inequality holds: $V(u) \geqslant 1 - K\exp\{-\gamma u\}$ for some constant K, $0 < K \leqslant 1$ (recall that γ is a positive solution to equation (6)). This is in contrast with the power asymptotic representation at infinity of the survival probability in the case when the whole of the surplus or some of its constant proportion is invested in a risky asset; see [10] and comments in [11].

The case of exponential distributions of jumps

Let $F(z) = 1-e^{-z/m}$, $G(z) = 1-e^{-z/n}$, $m,n > 0$. Then the equation (6) can be rewritten in the form

$$\frac{\lambda}{1-\gamma m} = \lambda + c\gamma + \frac{\lambda_1\gamma n}{1+\gamma n} + \frac{\mu^2}{2\sigma^2}. \tag{7}$$

For simplicity, we will assume here that assumption **(A)** is fulfilled. In the case $\mu = 0$ (without investments), it is easy to see that there exists a unique positive solution $\gamma_0 < 1/m$ of (7) (at the point $1/m$ we have a vertical asymptote: $1/(1-\gamma m) \to \infty$, $\gamma \uparrow 1/m$). The Figure shows the graph of a convex function of γ defined on the left side of the equation (7) and two graphs of concave functions defined on the right side of the equation (7) for $\mu = 0$ (lower curve) and for $\mu > 0$ (upper curve). The intersection points correspond to the solutions of the equation (7): it is obvious that $\gamma_0 < \gamma < 1/m$, where γ satisfies (7) for $\mu > 0$.

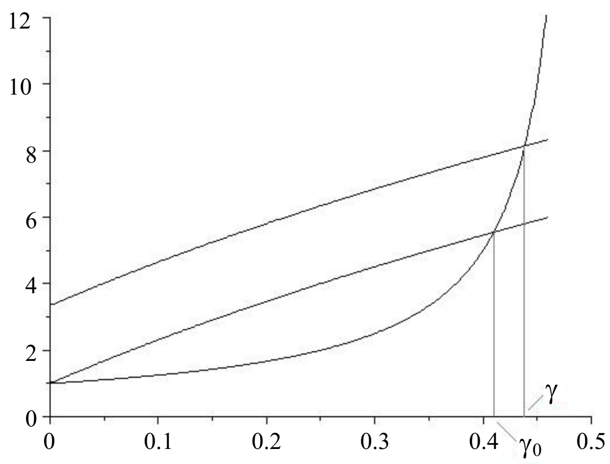

Figure. Adjustment coefficients

Thus, CA strategy $A_t \equiv A^*$ gives the best Lundberg bound among all CA strategies, including zero strategy $A_t \equiv 0$ (without investment). However, let us show that any CA strategy $A_t \equiv A$, $A \neq 0$ is worse than zero strategy for small levels of the surplus. Thus, any CA strategy $A_t \equiv A$, $A \neq 0$ cannot be optimal for the optimization problem (3). Indeed, for the survival probability $\varphi^A(u)$ of the process (5), where $A \neq 0$, we have the condition

$\varphi^A(0) = 0$ due to Brownian perturbation. Moreover, it can be shown that $\lim_{u\to+0} \varphi^A(u) = 0$. At the same time, in the case of zero CA strategy and exponential distributions of jumps it is easy to obtain (using the method of integro-differential equations (IDE); in a more general case, see, for example, [11]) the exact formula for survival probability $\varphi^0(u)$:

$$\varphi^0(u) = 1 - (1 - \gamma_0 m)\exp(-\gamma_0 u), \tag{8}$$

where γ_0 is defined above. Therefore, $\varphi^0(0) = \gamma_0 m > 0$, and we see that CA strategies cannot be optimal. But in [7] it was shown that, for the CL model with small claim sizes, the optimal strategy as a function of the surplus converges, as the surplus tends to infinity, to the value $A^* = \dfrac{\mu}{\sigma^2\gamma_1}$ that maximizes the corresponding adjustment coefficient, where γ_1 is the unique positive solution to equation (6) (or (7) in the case of exponential distributions of claim sizes) with $\lambda_1 = 0$. In addition, the Cramér-Lundberg approximation for the minimal ruin probability was obtained: there exists a constant $\varsigma \in (0,\infty)$ such that $\lim_{u\to\infty}(1 - V(u))\exp(\gamma_1 u) = \varsigma$.

In what follows, we will show (for the case of exponential distributions of jumps) that similar asymptotic representations for the optimal strategy and value function can also be obtained for a more general risk model, i.e., for $\lambda_1 \geqslant 0$; the case of non-negative interest rate will also be included. For this we propose to use the asymptotic analysis of solutions to a certain singular problem for a system of nonlinear ODE, which is satisfied by an optimal strategy, a value function and a certain additionally introduced function.

3. General case (non-negative r): Conditions for the solutions of HJB equation

Let us turn to the HJB equation (4). Suppose V is a twice continuously differentiable function and V solves (4). Let us also assume that V satisfies the conditions

$$\lim_{u\to\infty} V(u) = 1, \tag{9}$$

$$V'(u) > 0, \quad V''(u) < 0, \quad u > 0. \tag{10}$$

(Note that if the second derivative of the function $V(u)$ is non-negative at some point $u > 0$, the supremum in (4) is not achieved. The condition (9) is obvious in the case of a positive safety loading taking into account (8); in the general case it can be justified by the results [10] about the asymptotic representation for the survival probability at a constant proportion of risky assets ($\alpha_t \equiv \alpha$, $0 < \alpha \leqslant 1$) in the insurer's surplus). Then the maximizer of the right side of the HJB equation has the form

$$\alpha^* = \alpha^*_V(u) := -\frac{(\mu - r)V'(u)}{\sigma^2 u V''(u)}. \tag{11}$$

Taking the expression (11) in (4), we obtain the nonlinear integro-differential equation (IDE)

$$(ru + c)V'(u) - (\lambda + \lambda_1)V(u) + \lambda\int_0^u V(u - z)\,dF(z) + \lambda_1\int_0^\infty V(u + z)\,dG(z) =$$

$$= \frac{(\mu - r)^2(V'(u))^2}{2\sigma^2 V''(u)}, \quad u > 0. \tag{12}$$

Denote $A(u) = u\alpha^*_V(u)$, then

$$A(u) = -\frac{(\mu - r)V'(u)}{\sigma^2 V''(u)}. \tag{13}$$

Note that for reasons similar to those that require to rejecting the CA strategy producing the Brownian disturbance, we must also conclude that the function $A(u)$ satisfies the condition

$$A(+0) = \lim_{u\to+0} A(u) = 0. \tag{14}$$

Then, setting $u \to 0$ in (12) in the assumption of finiteness of $V'(+0) = \lim_{u\to+0} V'(u)$ and taking to account the relations (13), (14), we obtain the following non-local condition:

$$cV'(+0) - (\lambda + \lambda_1)V(0) + \lambda_1 \int\limits_0^\infty V(z)\, dG(z) = 0. \tag{15}$$

The case of exponential distributions: Singular problem for ODE system

Let us introduce the function

$$R(u) := \frac{\exp(u/n)}{V'(u)} \int_u^\infty \exp(-z/n)V'(z)dz. \tag{16}$$

Note that, taking into account integration by parts in (15) with $G(y) = 1 - \exp(-y/n)$, the condition (15) can be rewritten in the following form:

$$cV'(+0) - \lambda V(0) + \lambda_1 V'(+0)R(+0) = 0. \tag{17}$$

From this we conclude that $V'(+0) > 0$ (recall that $V(0) > 0$ for the value function in the considered model in view of the formula (8) for the survival probability in the absence of investments; as its consequence we have $V(0) \geqslant \gamma_0 m$).

Theorem 1. *Let $c > 0$, $F(z) = 1 - e^{-z/m}$, $G(z) = 1 - e^{-z/n}$, $m, n > 0$. If there exists a twice continuously differentiable solution V of equation (12) with the conditions (9), (10) and (15), then:*

1) *pair of functions A, R, defined in (13), (16), is a solution of the following singular initial problem for nonlinear ODE system:*

$$\begin{cases} r - \lambda_1 - \lambda - (ru + c)\left[\dfrac{(\mu - r)}{\sigma^2 A(u)} - \dfrac{1}{m}\right] + \\ + \dfrac{\mu - r}{2}\left[A'(u) + \frac{1}{m}A(u) - \dfrac{(\mu - r)}{\sigma^2}\right] + \lambda_1 \left(\dfrac{1}{n} + \dfrac{1}{m}\right) R(u) = 0, \\ R'(u) = R(u)\left(\dfrac{1}{n} + \dfrac{(\mu - r)}{\sigma^2 A(u)}\right) - 1 \end{cases} \tag{18}$$

with initial conditions (14) and (17) and unknown parameter $V(0)/V'(+0) > c/\lambda$;

2) *$V(u)$ is the maximal survival probability for the process (2) in the class of all admissible control policies, i.e. $V(u)$ is the solution of the optimization problem (3); optimal control has the form $\alpha_t^* = A(X_t^*)/X_t^*$, where X_t^*, $t \geqslant 0$, is the corresponding solution of equation (2);*

3) *the functions V, A, R have the following asymptotic representations:*

a) *for small values of the surplus*

$$A(u) \sim \frac{2\sqrt{c}}{\sigma}\sqrt{u}, \quad u \to 0,$$
$$R(u) \sim \frac{\lambda V(0)/V'(+0) - c}{\lambda_1}\left(1 + \frac{(\mu - r)}{\sigma\sqrt{c}}\sqrt{u}\right) - \frac{r}{\lambda_1}u, \quad u \to 0,$$
$$V(u) \sim V(0) + V'(+0)\left(u - \frac{2(\mu - r)}{3\sqrt{c}\sigma}u^{3/2}\right), \quad u \to 0;$$

b) *for large values of the surplus in the the case $r > 0$:*

$$A(u) \sim \frac{(\mu - r)m}{\sigma^2} + \left(\frac{\lambda}{r} - 1\right)\frac{(\mu - r)m^2}{\sigma^2}\frac{1}{u}, \quad u \to \infty,$$

$$R(u) \sim \frac{mn}{m+n} + \left(\frac{\lambda}{r} - 1\right) \frac{n^2 m^2}{(m+n)^2} \frac{1}{u}, \quad u \to \infty,$$
$$V(u) \sim 1 - K \exp\left(-\frac{u}{m}\right) u^{\left(\frac{\lambda}{r}-1\right)}, \quad u \to \infty;$$

in the case $r = 0$*:*

$$A(u) \sim \frac{\mu}{\sigma^2 \gamma}, \quad u \to \infty,$$
$$R(u) \sim \frac{n}{1+\gamma n}, \quad u \to \infty,$$
$$V(u) \sim 1 - K \exp(-\gamma u), \quad u \to \infty,$$

where γ *is the unique positive solution of equation* (7) *and* $K > 0$.

Recall that $\gamma < 1/m$; compare also the asymptotic representations at infinity for the case $r = 0$ with the results of the Section 2 for CA strategies.

The proof of the theorem is too long, we will not give it here. Note only that the equation for A can be obtained similarly to the corresponding equation in [9] (not containing the function R); the equation for R is derived by direct differentiation in (16), taking into account the relation (13). The statement 2) about the solution of the optimization problem is proved using verification arguments; asymptotic representations at infinity were found in the form of asymptotic series in inverse powers of the argument; asymptotic representations at zero were obtained by the asymptotic investigation of transformed ODE system (18) with the replacement of the function R by another function D, linearly related to A, R and ru, while $D(+0) = V(0)/V'(+0)$. It remains to be noted that this parameter can be determined numerically as a result of solving the initial ODE problem formulated above, and finally, $V(0)$ is determined using the normalization condition (9) for the solution obtained with a fixed parameter. Numerical calculations require additional studies of the ODE system.

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Поступила в редакцию / Received 12.01.2025
Принята к публикации / Accepted 19.03.2025
Опубликована / Published 29.08.2025