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Fracture of Wedge-Shaped Body Under Compression

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Abstract. The aim is to study the fracture process of a wedge-shaped body during compression. A large number of researchers have turned to the classical solution of the elasticity theory problem of the loaded wedge-shaped body, but the problem of a supported wedge of a finite shape still has no analytical solution. The authors conducted a study of the failure mechanism of such bodies by both computational and experimental methods. To carry out the numerical analysis, the implementation of the progressive limit state method at critical levels of strain energy in the form of the force method was used, in combination with the method of approximation of continuum by an equivalent truss. The equivalent truss model of the wedge used here clearly demonstrates the process of removing structural members due to them reaching limit states. The technique of progressive limit states, based on the consecutive identification of “weak links” in the structure, in which the limit state occurs first, made it possible to construct fracture models of the considered body. The results of the performed analysis are presented in the form of fracture models of wedge-shaped bodies. The failure mechanism of wedge-shaped bodies was also investigated by experimental methods. Wedge-shaped gypsum specimens were compressed at the tip of the wedge and brought to fracture. The differences between the obtained fracture patterns and the classical results known from the theory of elasticity obtained for infinite wedge-shaped bodies are shown. A comparison of experimental and numerical results is performed, and a conclusion is made about the real fracture patterns of wedge-shaped bodies with a supported part.

Keywords: limit states, critical energy levels, fracture pattern, experimental studies, wedge

Conflicts of interest. The authors declare that there is no conflict of interest.

Authors’ contribution: *Stupishin L.Yu.* — supervision, conceptualization, methodology, conclusions; *Nikitin K.E.* — numerical implementation, formal analysis; *Masalov A.V.* — experimental investigation.

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Разрушение тел клиновидной формы при сжатии

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Аннотация. Цель — исследование процесса разрушения тела клиновидной формы при сжатии. К решению классической задачи теории упругости о нагружении тела клиновидной формы возвращалось большое количество исследователей, однако задача об опертом клине конечной формы аналитического решения не имеет до сих пор. Авторами проведено исследование процесса разрушения таких тел как расчетными, так и экспериментальным методами. Для проведения численного анализа была использована реализация метода прогрессирующего предельного состояния на критических уровнях энергии деформации в форме метода сил, в сочетании с методом стержневой аппроксимации континуума. Использованная здесь плоская стержневая модель клина наглядно демонстрирует процесс удаления связей конструкции вследствие наступления в них предельного состояния. Методика прогрессирующего предельного состояния, основанная на последовательном выявлении «слабых» связей в конструкции, в которых в первую очередь наступает предельное состояние, позволила построить схемы разрушения рассматриваемого тела. Представлены результаты выполненного анализа, в виде схем разрушения тел клиновидной формы. Характер разрушения тел клиновидной формы так же исследовался экспериментальными методами. Образцы из гипса клиновидной формы были подвергнуты сжатию в вершине клина, и доведены до разрушения. Показаны отличия получаемых форм разрушения от классических результатов, известных из теории упругости, полученных для бесконечных клиновидных тел. Выполнено сравнение экспериментальных и численных результатов, и сделано заключение о реальных формах разрушения клиновидных тел с опорной частью.

Ключевые слова: предельные состояния, критические уровни энергии, форма разрушения, экспериментальные исследования, клин

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1. Introduction

Analytical solutions to the problem of determining the stress-strain state of an infinite wedge compressed by a force applied at its apex are known [1]. However, the classical solution to the elasticity theory problem of a wedge-shaped body does not allow to take into account the support conditions of a finite-length wedge in analytical form. Solving the problem, for example, by the finite element method leads to a solution, which refers to the known results for an infinite wedge. On the basis of this solution, using one of the strength hypotheses, it is possible to estimate the strength of the wedge in contact problems, when the fracture of the top of the wedge is the only option. Experimental results showing that this is not always unambiguous will be given below.

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Commonly used structural design problem settings are mainly based on the Lagrangian formulation. Research approaches to nonlinear problems of large strains in solid mechanics in thermomechanical formulation [2], nonlinearity of materials with voids [3], and brittle materials [4–6] are being actively generalized. New three-dimensional models and variational formulations of behavior in the elastoplastic stage of deformation are proposed [7; 8]. The stress-strain state of structures with cracks is investigated [9], and on the basis of numerical implementation of variational problem settings [10; 11] methods of analysis of various models of material behavior are constructed [12–14]. There is a great interest in the development of energy-based approaches for the calculation of brittle media with imperfections [15; 16], strength evaluation of orthotropic and anisotropic materials [17; 18]. Identification methods based on information on the entropy of strain energy are being developed [19].

Despite the scope of problems, the classical approach has a number of limitations. The model under consideration is always limited to the specific given load, and a reaction response is obtained, which does not cover possible actions on the structure. As a consequence, it is impossible to determine its full (maximum) and residual bearing capacity. An attempt to cover all possible load cases of structures within the framework of the Lagrangian approach leads to the necessity to solve an infinite chain of problems, which is infeasible even taking into account the capabilities of modern computer software.

According to the current design codes, limit states are not allowed to occur in the elements of the structure. But in reality, after reaching the limit state in one of the cross-sections or in the element as a whole, the load-bearing capacity of the structure, generally, is not lost. Thus, in statically indeterminate systems the forces are redistributed and the structural model changes. Such systems lose bearing capacity only when they become unstable. Most of the calculation methods used are oriented to the normative approach and do not allow to take into account the effects of force redistribution.

The use of an approach to the problems of analysis and design, which is based on the theory of critical strain energy levels of structures [20–23] proposed by one of the authors of [24], allows to overcome the above difficulties.

Limitation in the results obtained using classical methods of elasticity theory and structural mechanics, based on the Lagrangian formulation for the problem of wedge-shaped body failure, force to use new approaches to solving this problem.

In this paper, the failure mechanism of a wedge-shaped body was investigated from the standpoint of progressive limit states. The calculation methodology was based on the variational principle of critical strain energy levels of the structural system. The results obtained using this technique allow to estimate the full load-bearing capacity of the wedge and reveal its fracture pattern.

Experimental studies of brittle wedge-shaped specimens were performed to validate the results obtained by the proposed calculation method. For these specimens, the first approximation results obtained according to the plane wedge model were used. The experiment was carried out by bringing the specimens to fracture, when they were separated into parts. The fracture patterns were recorded at different stiffness of the loading plate of the testing machine.

The results of numerical and experimental investigations were compared, and a conclusion is made about the real fracture patterns of wedge-shaped bodies with a supported part.

2. Materials and Methods

2.1. Computational Methods

The condition of the critical state of the strain energy of the structure, according to the variational criterion of critical energy levels, is written in the form:

$$\delta U(\chi)=0;\ \sum_j U_j(\chi)=1;\ \Gamma(\chi)=0. \tag{1}$$

Here, U is the strain potential energy of the structure; χ is the extrema of generalized displacements and forces (parameters of the problem). The first equality in (1) represents the condition of minimum internal strain potential energy of the structure in the state of self-equilibrium (self-stress). The second one is the condition of orthonormalization of the problem parameters. The third describes the boundary conditions.

The critical state condition of the structure in the form of the force method [22] is expressed as:

$$[L]\{\delta\Phi\}=\left[\lambda^{L}\right]\{\delta\Phi\}, \tag{2}$$

where $[L]$ is the flexibility matrix of the structure; $\{\delta\Phi\}$ is the vector of variation of amplitudes of generalized reaction forces in all directions of the degrees of freedom of the structure for self-stress states, represented by a set of orthonormal functions; $\left[\lambda^{L}\right]$ is the eigenvalue matrix, which signifies unit nodal displacements of the structure.

According to the solution of the eigenvalue problem (2), the vector of maximal nodal displacements of the structure is calculated:

$$\{Z_{\max}\}=\left[\lambda^{L}_{\max}\right]\{\delta\Phi_{\max}\}, \tag{3}$$

Further, the values of vector (3) of maximum displacements can be used to find strains or internal forces in the elements:

$$\{\varepsilon_{\max}\}=-[A]^{T}\{Z_{\max}\}; \tag{4}$$

$$\{N\}=-[C]\{\varepsilon_{\max}\}. \tag{5}$$

Here, $[C]$ is the internal flexibility matrix; $[A]$ is the static matrix of the structure.

In order to investigate the deformation process of a structure up to its fracture, the methodology of progressive limit states proposed in [21; 22] is used. At each stage of this technique, problem (2) is solved by the method of critical energy levels to determine the extreme parameters of the deformed structure. This allows to identify the elements in which the limit state is reached first (“weak links”). These elements are eliminated from the model, and the procedure is repeated for a new, corrected model. The investigation of the process of elimination of the “weak links” is completed after formation of an unstable system. This moment is considered to be the moment of structural failure, when the load-bearing capacity is exhausted.

In this study, the continuum is modeled by truss elements, so reaching the limit state in weak links is treated in the traditional way as in tension-compression truss members. The ultimate forces are found at each stage of the analysis by the maximum values of the internal forces in vector $\{N\}$.

The methodology described above was implemented in the “CLE” computational software for structural analysis developed by the authors of [23]. This software is currently being actively developed and tested by solving various design problems (including [20–22]). It includes: a module for the preparation of initial data; modules for solving problems on the basis of the method of progressive limit states, on the basis of the method of critical energy levels; a module for processing and output of calculation results. The results presented below were obtained using this software.

A structure in the form of a plane triangular wedge fixed along the entire length of the base is considered further. The thickness of the wedge in the model was assumed to be constant.

The model of this wedge-shaped body was created using the equivalent truss method [25]. According to the adopted approach, the continuum of the solid body is replaced by an equivalent regular hinged truss, the parameters of which were adjusted such that the strain of the cells of this structure corresponds to the strains of the corresponding regions of the original continuum system. A fragment of this truss system is shown in Figure 1.

The stiffness values of the members of the structure were determined using the formulas:

$$EA_1 = \frac{3}{4} btE; \quad (6)$$

$$EA_2 = \frac{3}{4\sqrt{2}} btE. \quad (7)$$

Here, b is the size of the truss cell; t is the thickness of the modelled body; E is the elastic modulus of the modelled body.

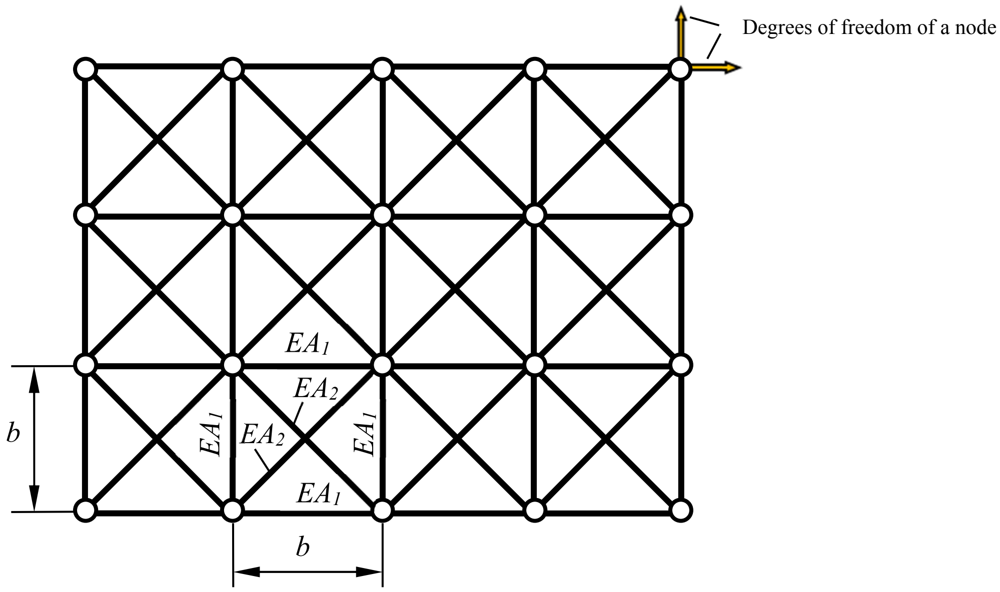

Figure 1. Regular truss structure for approximating the continuum of a deformable body
S o u r c e: made by K.E. Nikitin

2.2. Experimental Materials and Methods

Experimental studies for validation of the computational method results included preparing specimens, their testing under a load applied at the apex and recording the fracture pattern of the specimens.

Gypsum mortar was used for specimen preparation. The technology of making gypsum castings is relatively simple and allows to obtain specimens with low labor intensity and cost. Gypsum mortar consisted of G4 grade gypsum according to the GOST 125–2018 Interstate Standard[3] and water according to the GOST R 51232–98 State Standard[4], in force in the Russian Federation, in 1:1 ratio by weight. A JKD-500 scale (JADEVER, Taipei, Taiwan) with 0.1 g sensitivity and 500 g measurement limit was used for the preparation.

The formwork was made of KIM TEC SILICON 101E silicone sealant. Plastic models of specimens in the form of a cone, pyramid and prism were 3D-printed as a mold-forming base for their preparation (Figure 2).

Fracture tests of finished gypsum specimens according to GOST 125–2018 were performed on PGM-100MG4 small-size hydraulic test press (SKB Stroypribor, Chelyabinsk, Russia).

The specimens were loaded in two ways: with direct load transfer from the metal plate of the press to the top of the specimen and through a rectangular pad made of 10 mm thick drywall, 70x70 mm in size.

The loading rate was 0.3–0.5 kN/s. The loading process was videotaped during the test.

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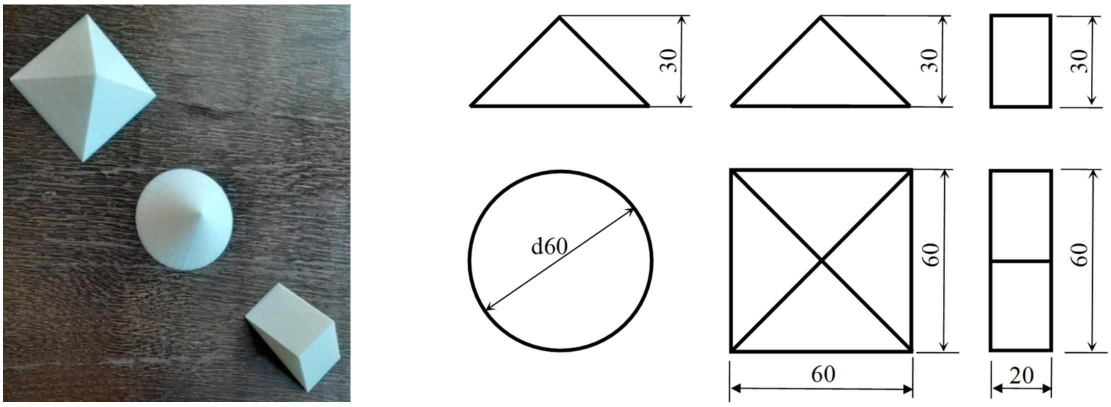
[3] GOST 125–2018. Gypsum binders. Specifications. Moscow: Standartinform Publ.; 2018. (In Russ.)

[4] GOST R 51232–98. Drinking water. General requirements for organization and quality control methods. Moscow: Standartinform Publ.; 2008. (In Russ.)



a *b*

Figure 2. Plastic models of test speciments used for the preparation of formwork
S o u r c e: made by A.V. Masalov

3. Results and Discussion

Using the above calculation methodology, by consecutively traversing a chain of limit states, and eliminating the “weak” elements at each stage, the wedge fracture pattern was obtained, shown in Figure 3.

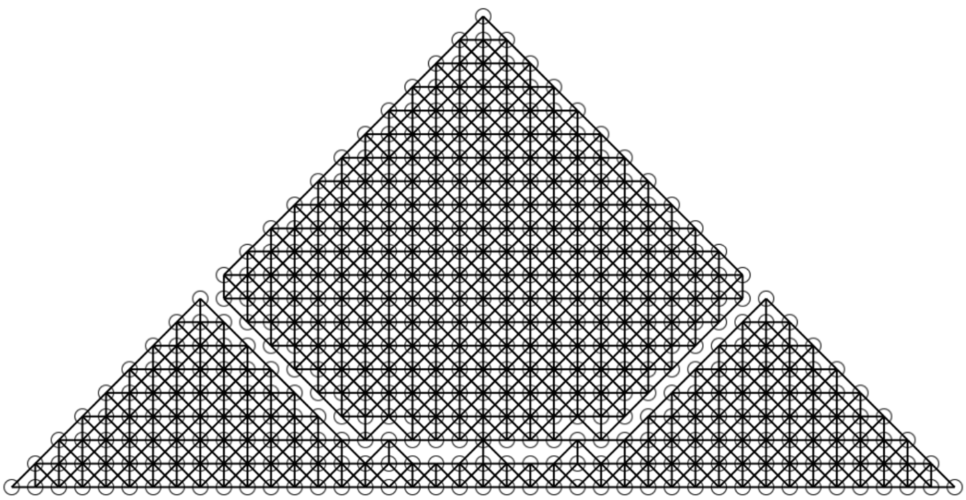


Figure 3. Unstable truss structure, which models a wedge with a sharp top at the moment of its fracture
S o u r c e: made by K.E. Nikitin

The fracture of the model started with the elements located on the inclined faces of the wedge, below the centerline of the triangle closer to the supported part of the wedge. It can be argued that the propagation of the link elimination band resembles the classical analytical solution for an infinite wedge. The truss model was formed in such a way that the mechanical and geometrical characteristics of the members are the same. Therefore, the results of the analysis did not show the crushing process of the top of the wedge, which occurs in the experiment. This was done intentionally, because in the first formulation of the problem, it was necessary to reveal the aspects of the propagation of extreme values of strain energy under self-stress conditions, which are dictated by the geometry of the body and support conditions.

In contrast to the classical solution of the elasticity theory problem of the plane semi-infinite wedge, the obtained solution takes into account the conditions of the wedge supported at the base. At the same time, the qualitative pattern of stress distribution along the radius of the circle centered at the point of load application is preserved. But the solution for an infinite wedge implies a decrease in stress with increasing distance from the apex. For a supported wedge of finite length, this pattern changes. In [1], the possibility of obtaining a solution for the case of a restrained wedge base is discussed, but the solution to this problem is not given. The exact analytical solutions of such a problem with supports is not known by the authors.

If the load is considered as a force applied at the top of the wedge, then in reality the load is transferred to the wedge not at a point, but through a surface of a particular area. At the beginning of loading, the load transfer area is small, but as the magnitude of the load increases, this area increases due to crushing of the wedge top. This is due to the fact that the hardness of the metal plate of the testing machine is much higher than the hardness of the wedge material. The result of this process is clearly shown in the photographs of the tested wedge-shaped specimens (Figure 4).

Figure 4. Test results of the specimens of the first group, brought to fracture
S o u r c e: made by A.V. Masalov

Without changing the mechanical and geometric characteristics of the model, the model of a wedge with a truncated top was considered (Figure 5). As a result of the analysis, fracture patterns that qualitatively repeat the pattern for the sharp-top wedge were obtained. In general, the fracture pattern of the truncated wedge does not differ much from the fracture pattern of the wedge with a sharp top, but the fracture band extends to the lower supported part. The results of the analysis for one of the versions of the truncated wedge are shown in Figure 5.

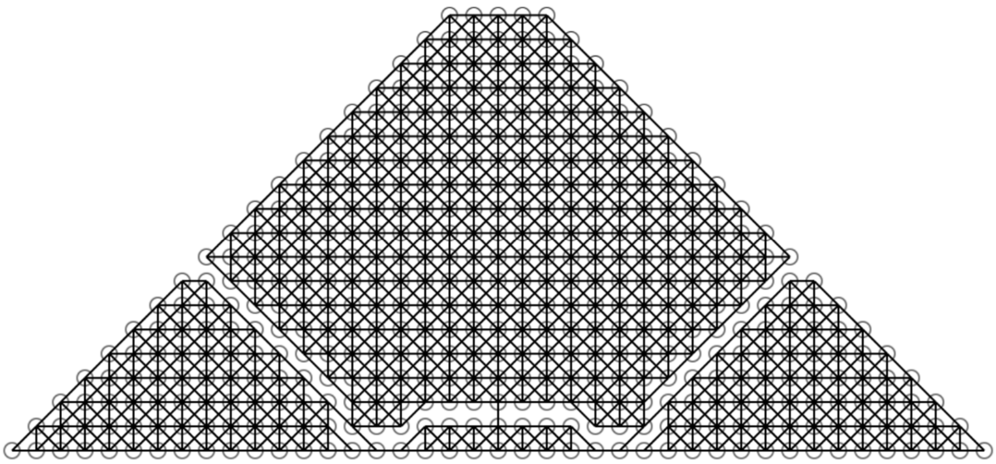


Figure 5. Unstable truss structure, which models a truncated wedge at the moment of its fracture
S o u r c e: made by K.E. Nikitin

For validation of the results computed for self-equilibrated wedge-shaped bodies by the method of critical energy levels, experimental studies of wedge-shaped specimens were carried out. Specimens of conical, pyramidal and prismatic shape were tested. All specimens were compressed by a force applied to the apex. Part of the specimens (the first group) was tested with load transfer directly from the steel plate of the press, which has significant hardness. The other part of the specimens (second group) was tested with load transfer through a drywall pad, which brought the hardness of the specimen and the loading plate to the same order of magnitude. This decreased the stress concentration and reduced crushing of the material near the apex. The loading was carried out up to the moment of fracture of the specimen, its separation into disconnected parts, after which the fracture pattern was recorded.

The experimental results of the specimens of the first group are presented in Figure 5. The second group — in Figure 6.

Figure 6. Test results of the specimens of the second group, brought to fracture

S o u r c e: made by A.V. Masalov

The tests of the specimens of the first group showed that, firstly, the sharp tip of the specimens is crushed. After this process stops, longitudinal vertical cracks appear, separating the specimen into parts of almost equal volume. Crushing occurs due to the concentration of stresses at the apex. This is due to the significant difference in the hardness of the testing machine plate and the material of the test specimen.

The fracture pattern of the specimens of the second group (with smoother load transfer to the specimen) has changed qualitatively. The crushing zone of the compressed top of the specimens decreased. Examination of the shapes of the fractured specimens revealed similar fracture patterns, which are summarized in Figure 7. The beginning of fracture of the specimens was characterized by a ring or semi-ring thread-like crack, at the moment of formation in the middle part of the conical bodies of a droplet-shaped rigid part, which “squeezed out” the peripheral parts. As a consequence of this process, longitudinal inclined cracks of the separating peripheral parts were opened.

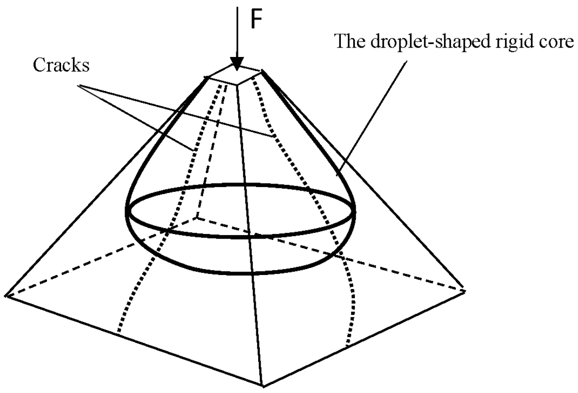


Figure 7. The shape of the rigid core forming inside the body

S o u r c e: made by L.Yu. Stupishin

As can be seen, this fracture pattern is qualitatively similar to the fracture pattern obtained by calculation using the method of critical energy levels.

The fracture does not look axisymmetric in the photos due to the uneven load transfer through the test machine plate, even more so when using the drywall pad. The non-ideal structure of the gypsum model should also be taken into account, due to difficulties related to mixing with water and uneven setting of the mixture.

4. Conclusion

1. The results obtained by the computational and experimental methods have shown that the classical solution of the elasticity theory problem of the infinite wedge is used only in contact problems, but is not applicable to the case of a supported wedge. Stress distributions having outlines of circle segments centered at the apex of the wedge, characteristic of this solution, are qualitatively true in the vicinity of the tip of the wedge. Moving away from the apex, the stresses decrease sharply, which is related to the absence of the supported part of the wedge, since an infinite model is considered.

2. Support reactions at the base of the wedge significantly affect the change in the strain distribution. The study of the model of a supported wedge under self-stress conditions shows significant differences in the stress-strain state of the wedge. The obtained results reflect the objective relationship between the strain distribution, the geometry and the wedge support conditions.

3. The conducted experimental studies confirmed that, qualitatively, the wedge fracture pattern is close to that obtained by application of the variational criterion of critical energy levels.

4. Application of the equivalent truss method within the framework of the theory of critical strain energy levels of the structure allows to obtain a more reasonable fracture pattern of the wedge. This allows to use this methodology for solving complex problems in the theory of elasticity, when classical computational methods do not always provide a correct picture of reaching limit states.

The purpose of further research is to improve the technique of testing specimens from in terms of smoothing the load transfer to the specimen; to expand the range of tested materials. The calculation method should also be refined in terms of the possibility of taking into account stress concentration near the special points of the deformed body. Thus, it is necessary to refine the model of wedge-shaped body, representing it as a spatial model with variable mechanical and geometric characteristics. It is necessary to refine the boundary conditions on the supporting surface of the wedge-shaped body and the interaction with the base.

References

1. Zhemochkin B.N. *Theory of elasticity*. Moscow: The State Publ. House of literature on Construction and Architecture; 1957. (In Russ.) Available from: https://thelib.net/1887753-teorija-uprugosti.html (accessed: 12.08.2024)

2. Ba K., Gakwaya A. Thermomechanical total Lagrangian SPH formulation for solid mechanics in large deformation problems. *Computer Methods in Applied Mechanics and Engineering*. 2018;342:458–473. https://doi.org/10.1016/j.cma 2018.07.038

3. Bai L., Wadee M.A., Köllner A., Yang J. Variational modelling of local–global mode interaction in long rectangular hollow section struts with Ramberg–Osgood type material nonlinearity. *International Journal of Mechanical Science.* 2021; 209:106691. https://doi.org/10.1016/j.ijmecsci.2021.106691 EDN: UYQXXP

4. Jiménez F.L. Variations in the distribution of local strain energy within different realizations of a representative volume element. *Composites Part B: Engineering.* 2019;176:107111. https://doi.org/10.1016/j.compositesb.2019.107111

5. Li C.C., Zhao T, Zhang Y., WanW. A study on the energy sources and the role of the surrounding rock mass in strain burst. *International Journal of Rock Mechanics & Mining Sciences*. 2022;154:105114. https://doi.org/10.1016/j.ijrmms.2022. 105114 EDN: TTQPFW

6. Balabuši M. Virtual principle for determination initial displacements of reinforced concrete and prestressed concrete (overtop) members. *Open Journal of Civil Engineering*. 2021;11(2):235–253 https://doi.org/10.4236/ojce.2021.112015 EDN: WRIHOO

7. Portillo D., Oesterle B., Thierer R., Bischoff M., Romero I. Structural models based on 3D constitutive laws: Variational structure and numerical solution. *Computer Methods in Applied Mechanics and Engineering*. 2020;362:112872. https://doi.org/10.1016/j.cma.2020.112872 EDN: HITNXD

8. Coombsa W.M., Augardea C.E., Brennanb A.J., Brownb M.J., Charltona T.J., Knappettb J.A., Motlaghc Y.G., Wanga L. On Lagrangian mechanics and the implicit material point method for large deformation elasto-plasticity. *Computer Methods in Applied Mechanics and Engineering*. 2020;358:112622. https://doi.org/10.1016/j.cma.2019.112622

9. Branco R., Prates P., Costa J.D., Cruces A., Lopez-Crespo P., Berto F. On the applicability of the cumulative strain energy density for notch fatigue analysis under multiaxial loading. *Theoretical and Applied Fracture Mechanics*. 2022;120: 103405. https://doi.org/10.1016/j.tafmec.2022.103405 EDN: BLUUPR

10. Wang X., Xu Q., Atluri S.N. Combination of the variational iteration method and numerical algorithms for nonlinear problems. *Applied Mathematical Modelling*. 2019;79:243–259. https://doi.org/10.1016/j.apm.2019.10.034 EDN: ETTYKG

11. Ignatiev A.V., Bochkov M.I. Calculation systems with nonlinear moment connections by the finite element method in the form of the classical mixed method. *News of higher educational institutions. Construction.* 2023;11(779):5–14. (In Russ.) https://doi.org/10.32683/0536-1052-2023-779-11-5-14 EDN: OSFTEQ

12. Harlanov V.L., Harlanova S.V. Incremental methods for solving geometrically nonlinear problems. *Structural Mechanics and Analysis of Constructions*. 2023;5(310):64–69. (In Russ.) https://doi.org/10.37538/0039-2383.2023.5.64.69 EDN: EAMHYY

13. Chernov Yu.T. Integral equations of the second kind in dynamic analysis of nonlinear systems with a finite number of degrees of freedom under arbitrary dynamic loading and material dependencies. *Bulletin of Science and Research Center of Construction*. 2023;3(38):155–167. (In Russ.) https://doi.org/10.37538/2224-9494-2023-3(38)-155-167 EDN: YGYKIE

14. Yazyev S.B., Chepurnenko A.S. Buckling of rectangular plates under nonlinear creep. *Advanced Engineering Research (Rostov-on-Don).* 2023;23(3):257–268. https://doi.org/10.23947/2687-1653-2023-23-3-257-268 EDN: YXRQDK

15. Chen L., Guo L. Discussions on the complete strain energy characteristics of deep granite and assessment of rockburst tendency. *Shock and Vibration*. 2020:8825505. https://doi.org/10.1155/2020/8825505

16. Le T.C, Ho D.D., Nguyen C.T., Huynh T.C. Structural damage localization in plates using global and local modal strain energy method. *Advances in Civil Engineering*. 2022;2022:4456439. https://doi.org/10.1155/2022/4456439

17. Mirsayar M.M. Maximum principal strain criterion for fracture in orthotropic composites under combined tensile/shear loading. *Theoretical and Applied Fracture Mechanics*. 2022;118:103291. https://10.1016/j.tafmec.2022.103291 EDN: OAZJIR

18. van Dijk J N.P. Espadas-Escalante J., Isaksson P. Strain energy density decompositions in phase-field fracture theories for orthotropy and anisotropy. *International Journal of Solids and Structures*. 2020;196–197:140–153. https://doi.org/10.1016/j.ijsolstr.2020.04.022 EDN: LDKEOG

19. Xiang C.S., Li L.Y., Zhou Y., Dang C. An efficient damage identification method for simply supported beams based on strain energy information entropy. *Advances in Materials Science and Engineering*. 2020;2020:9283949. https://doi.org/10.1155/2020/9283949

20. Stupishin L.Y., Nikitin K.E., Moshkevich M.L. Methodology for determining progressing ultimate states based on the displacement method. *Structural Mechanics of Engineering Constructions and Buildings*. 2023;19(3):276–284. (In Russ.) https://doi.org/10.22363/1815-5235-2023-19-3-276-284 EDN: PQVCVH

21. Stupishin L.Y., Nikitin K.E. Determination of the residual strain energy of a structure and the method of progressive limiting state. *Advances in Transdisciplinary Engineering*. 2023;43:138–143. https://doi.org/10.3233/ATDE230711

22. Stupishin L.Y., Nikitin K.E. The process of progressive limiting state and determination of the residual strain energy of a structure based on the force method. *Lecture Notes in Civil Engineering*. 2024;400:280–289. https://doi.org/10.1007/978-3-031-47810-9_26

23. Stupishin L.Y., Nikitin K.E. Computer system for structural analysis based on the critical energy levels method. *BIM in Construction & Architecture. Proceedings of IV International Conference*. St. Petersburg: SPbGASU Publ.; 2021; 223–230. (In Russ.) https://doi.org/10.23968/BIMAC.2021.029 EDN: OHWPLB

24. Stupishin L.Y. Limit state of building structures and critical energy levels. *Industrial and civil engineering*. 2018; 10:102–106. (In Russ.) EDN: YNRBNR

25. Rzhanitsin A.R. Structural mechanics. Moscow: Higher School Publ.; 1982. (In Russ.) Available from: https://thelib.net/801450-stroitelnaja-mehanika.html (accessed: 12.08.2024).

Список литературы

1. *Жемочкин Б.Н.* Теория упругости. Москва : Гос.изд. лит. по строит. и арх., 1957. 256 с. URL: https://thelib.net/1887753-teorija-uprugosti.html (дата обращения: 12.08.2024)

2. *Ba K., Gakwaya A.* Thermomechanical total Lagrangian SPH formulation for solid mechanics in large deformation problems // Computer Methods in Applied Mechanics and Engineering. 2018. Vol. 342. P. 458–473. https://doi.org/10.1016/j.cma.2018.07.038

3. *Bai L., Wadee M.A., Köllner A., Yang J.* Variational modelling of local–global mode interaction in long rectangular hollow section struts with Ramberg–Osgood type material nonlinearity // International Journal of Mechanical Science. 2021. Vol. 9. Article no. 106691. https://doi.org/10.1016/j.ijmecsci.2021.106691 EDN: UYQXXP

4. *Jiménez F.L.* Variations in the distribution of local strain energy within different realizations of a representative volume element // Composites Part B: Engineering. 2019. Vol. 176. Article no. C107111. https://doi.org/10.1016/j.compositesb.2019.107111

5. *Li C.C., Zhao T, Zhang Y., Wan W.* A study on the energy sources and the role of the surrounding rock mass in strain burst // International Journal of Rock Mechanics & Mining Sciences. 2022. Vol. 154. Article no. 105114. https://doi.org/10.1016/j.ijrmms.2022.105114 EDN: TTQPFW

6. *Balabuši M.* Virtual principle for determination initial displacements of reinforced concrete and prestressed concrete (overtop) members // Open Journal of Civil Engineering. 2021. Vol. 11. Issue 2. P. 235–253. https://doi.org/10.4236/ojce.2021.112015 EDN: WRIHOO

7. *Portillo D., Oesterle B., Thierer R., Bischoff M., Romero I.* Structural models based on 3D constitutive laws: Variational structure and numerical solution // Computer Methods in Applied Mechanics and Engineering. 2020. Vol. 362. Article no. 112872. https://doi.org/10.1016/j.cma.2020.112872 EDN: HITNXD

8. *Coombsa W.M., Augardea C.E., Brennanb A.J., Brownb M.J., Charltona T.J., Knappettb J.A., Motlaghc Y.G., Wanga L.* On Lagrangian mechanics and the implicit material point method for large deformation elasto-plasticity // Computer Methods in Applied Mechanics and Engineering. 2020. Vol. 358. Article no. 112622. https://doi.org/10.1016/j.cma.2019.112622

9. *Branco R., Prates P., Costa J.D., Cruces A., Lopez-Crespo P., Berto F.* On the applicability of the cumulative strain energy density for notch fatigue analysis under multiaxial loading // Theoretical and Applied Fracture Mechanics. 2022. Vol. 120. Article no. 103405. https://doi.org/10.1016/j.tafmec.2022.103405 EDN: BLUUPR

10. *Wang X., Xu Q., Atluri S.N.* Combination of the variational iteration method and numerical algorithms for nonlinear problems // Applied Mathematical Modelling. 2019. Vol. 79. P. 243–259. https://doi.org/10.1016/j.apm.2019.10.034 EDN: ETTYKG

11. *Игнатьев А.В., Бочков М.И.* Расчет систем с нелинейными моментными связями методом конечных элементов в форме классического смешанного метода // Известия высших учебных заведений. Строительство. 2023. № 11 (779). С. 5–14. https://doi.org/10.32683/0536-1052-2023-779-11-5-14 EDN: OSFTEQ

12. *Харланов В.Л., Харланова С.В*. Инкрементальные методы решения геометрически нелинейных задач // Строительная механика и расчет сооружений. 2023. № 5 (310). С. 64–69. https://doi.org/10.37538/0039-2383.2023.5.64.69 EDN: EAMHYY

13. *Чернов Ю.Т.* Интегральные уравнения второго рода в задачах расчета нелинейных систем с конечным числом степеней свободы при произвольных динамических воздействиях и характере физических зависимостей // Вестник НИЦ Строительство. 2023. № 3 (38). С. 155–167. https://doi.org/10.37538/2224-9494-2023-3(38)-155-167 EDN: YGYKIE

14. *Yazyev S.B., Chepurnenko A.S.* Buckling of rectangular plates under nonlinear creep // Advanced Engineering Research (Rostov-on-Don). 2023 Vol. 23. Issue 3. P. 257–268. https://doi.org/10.23947/2687-1653-2023-23-3-257-268 EDN: YXRQDK

15. *Chen L., Guo L.* Discussions on the complete strain energy characteristics of deep granite and assessment of rockburst tendency // Shock and Vibration. 2020. Vol. 2020. Article no. 8825505. https://doi.org/10.1155/2020/8825505

16. *Le T.C., Ho D.D., Nguyen C.T., Huynh T.C.* Structural damage localization in plates using global and local modal strain energy method // Advances in Civil Engineering. 2022. Vol. 2022. Article no. 4456439. https://doi.org/10.1155/2022/4456439

17. *Mirsayar M.M.* Maximum principal strain criterion for fracture in orthotropic composites under combined tensile/shear loading // Theoretical and Applied Fracture Mechanics. 2022. Vol. 118. Article no. 103291. https://doi.org/10.1016/j.tafmec.2022.103291 EDN: OAZJIR

18. *van DijkJ N.P. Espadas-Escalante J., Isaksson P. Strain* energy density decompositions in phase-field fracture theories for orthotropy and anisotropy // International Journal of Solids and Structures. 2020. Vol. 196–197. P. 140–153. https://doi.org/10.1016/j.ijsolstr.2020.04.022 EDN: LDKEOG

19. *Xiang C.S., Li L.Y., Zhou Y., Dang C.* An efficient damage identification method for simply supported beams based on strain energy information entropy // Advances in Materials Science and Engineering. 2020. Vol. 2020. Article no. 9283949. https://doi.org/10.1155/2020/9283949

20. *Ступишин Л.Ю., Никитин К.Е., Мошкевич М.Л.* Методика определения прогрессирующих предельных состояний на основе метода перемещений // Строительная механика инженерных конструкций и сооружений. 2023. Т. 19. № 3. С. 276–284. https://doi.org/10.22363/1815-5235-2023-19-3-276-284 EDN: PQVCVH

21. *Stupishin L.Y., Nikitin K.E.* Determination of the residual strain energy of a structure and the method of progressive limiting state // Advances in Transdisciplinary Engineering. 2023. Vol. 43. P. 138–143. https://doi.org/10.3233/ATDE230711

22. *Stupishin L.Y., Nikitin K.E.* The process of progressive limiting state and determination of the residual strain energy of a structure based on the force method // Lecture Notes in Civil Engineering. 2025. Vol. 400. P. 280–289. https://doi.org/10.1007/978-3-031-47810-9_26

23. *Ступишин Л.Ю., Никитин К.Е.* Компьютерная система анализа сооружений на основе метода критических уровней энергии // BIM-моделирование в задачах строительства и архитектуры: материалы IV Международной научно-практической конференции. Санкт-Петербург : СПбГАСУ, 2021. С. 223–230. https://doi.org/10.23968/BIMAC.2021.029 EDN: OHWPLB

24. *Ступишин Л.Ю.* Предельное состояние строительных конструкций и критические уровни энергии // Промышленное и гражданское строительство. 2018. № 10. С. 102–106. EDN: YNRBNR

25. *Ржаницын А.Р.* Строительная механика. Москва : Высшая школа, 1982. 400 с. URL: https://thelib.net/801450-stroitelnaja-mehanika.html (дата обращения: 12.08.2024).