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# Two-Field Prismatic Finite Element Under Elasto-Plastic Deformation

**Rumia Z. Kiseleva[1]✉, Vitaliy V. Ryabukha[1], Natalia A. Kirsanova[2], Yuriy V. Klochkov[1], Anatoliy P. Nikolaev[1]**

[1] Volgograd State Agrarian University, *Volgograd, Russian Federation*
[2] Financial University under the Government of the Russian Federation, *Moscow, Russian Federation*

✉ rumia1970@yandex.ru

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**Abstract.** For elasto-plastic analysis of structures at a particular load step, a mixed finite element in the form of a prism with triangular bases was obtained. Displacement increments and stress increments were taken as nodal unknowns. The target quantities were approximated using linear functions. Two versions of physical equations were used to describe elasto-plastic deformation. The first version used the constitutive equations of the theory of plastic flow. In the second version, the physical equations were obtained based on the hypothesis of proportionality of the components of the deviators of deformation increments to the components of the deviators of stress increments. To obtain the stiffness matrix of the prismatic finite element, a nonlinear mixed functional was used, as a result of the minimization of which two systems of algebraic equations with respect to nodal unknowns were obtained. As a result of solving these systems, the stiffness matrix of the finite element was determined, using which the stiffness matrix of the analysed structure was formed. After determining the displacements at a load step, the values of the nodal stress increments were determined. A specific example shows the agreement of the calculation results using the two versions of the constitutive equations of elasto-plastic deformation.

**Keywords:** elastic deformation, plastic deformation, mixed functional, mixed finite element, constitutive equations, flow theory

**Conflicts of interest.** The authors declare that there is no conflict of interest.

**Authors' contribution:** *Ryabukha V.V.* — obtaining the stiffness matrix of the finite element, writing the text; *Kirsanova N.A.*, *Kiseleva R.Z.* — implementation of the mixed functional and a hypothesis for obtaining plasticity equations; *Klochkov Yu.V., Nikolaev A.P., Kirsanova N.A.* — discussion of the concept of obtaining the constitutive equations.

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***Rumia Z. Kiseleva,*** Candidate of Technical Sciences, Associate Professor of the Department of Applied Geodesy, Environmental Management and Water Use, Volgograd State Agrarian University, 26 Universitetskiy Prospekt, Volgograd, 400002, Russian Federation; eLIBRARY SPIN-code: 1948-5390, ORCID: 0000-0002-3047-5256; e-mail: rumia1970@yandex.ru

***Vitaliy V. Ryabukha,*** Postgraduate student of the Department of Mechanics, Volgograd State Agrarian University, 26 Universitetskiy Prospekt, Volgograd, 400002, Russian Federation; eLIBRARY SPIN-code: 9596-2597, ORCID: 0000-0002-7394-8885; e-mail: vitalik30090@mail.ru

***Natalia A. Kirsanova,*** Doctor of Physical and Mathematical Sciences, Professor of the Department of Mathematics, Financial University under the Government of the Russian Federation, 49 Leningradsky Prospekt, GSP-3, Moscow, 125993, Russian Federation; eLIBRARY SPIN-code: 8393-5900, ORCID: 0000-0003-3496-2008; e-mail: nagureeve@fa.ru

***Yuri V. Klochkov,*** Doctor of Technical Sciences, Professor, Head of the Department of Higher Mathematics, Volgograd State Agrarian University, 26 Universitetskiy Prospekt, Volgograd, 400002, Russian Federation; eLIBRARY SPIN-code: 9436-3693, ORCID: 0000-0002-1027-1811; e-mail: klotchkov@bk.ru

***Anatoliy P. Nikolaev,*** Doctor of Technical Sciences, Professor of the Department of Mechanics, Volgograd State Agrarian University, 26 Universitetskiy Prospekt, Volgograd, 400002, Russian Federation; eLIBRARY SPIN-code: 2653-5484, ORCID: 0000-0002-7098-5998; e-mail: anpetr40@yandex.ru

# Двупольный конечный элемент при упругопластическом деформировании твердого тела

**Р.З. Киселева**[1] ✉, **В.В. Рябуха**[1], **Н.А. Кирсанова**[2], **Ю.В. Клочков**[1], **А.П. Николаев**[1]

[1] Волгоградский государственный аграрный университет, *Волгоград, Российская Федерация*
[2] Финансовый университет при Правительстве Российской Федерации, *Москва, Российская Федерация*
✉ c216@ya.ru

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**Аннотация.** Для упругопластического расчета конструкций на шаге нагружения получен смешанный конечный элемент в форме призмы с треугольными основаниями. В качестве узловых неизвестных приняты приращения деформаций и приращения напряжений. Искомые величины аппроксимировались с использованием линейных функций. Для описания упругопластического деформирования использовались два варианта физических уравнений. В первом варианте применялись определяющие уравнения теории пластического течения. Во втором варианте физические уравнения получены на основе гипотезы о пропорциональности компонент девиаторов приращений деформаций компонентам девиаторов приращений напряжений. Для получения матрицы жесткости призматического конечного элемента использовался нелинейный смешанный функционал, в результате минимизации которого получены две системы алгебраических уравнений относительно узловых неизвестных. В результате решения этих систем определена матрица жесткости конечного элемента, с использованием которой формировалась матрица жесткости рассчитываемой структуры. После определения перемещений на шаге нагружения определены значения узловых величин приращений напряжений. На конкретном примере показано совпадение результатов расчета с использованием вариантов определяющих уравнений упругопластического деформирования.

**Ключевые слова:** упругая деформация, пластическая деформация, смешанный функционал, смешанный конечный элемент, определяющие уравнения, теория течения

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## 1. Introduction

When real structures are loaded, stress concentration zones always appear, in which local stresses exceed the yield strength of the material and the physical relationships between stress and strain are nonlinear. Analysis of the stress-strain state of structures in the zones of elasto-plastic deformation of the material is an important engineering problem. The solution of such problems is performed using numerical methods for determining design values. Among numerical methods for determining the strength parameters of structures, the finite element method (FEM) in various formulations has become widespread. The dis-

***Киселева Румия Зайдуллаевна***, кандидат технических наук, доцент кафедры прикладной геодезии, природообустройства и водопользования, Волгоградский государственный аграрный университет, Российская Федерация, 400002, г. Волгоград, Университетский проспект, д. 26; eLIBRARY SPIN-код: 1948-5390, ORCID: 0000-0002-3047-5256; e-mail: rumia1970@yandex.ru

***Рябуха Виталий Васильевич***, аспирант кафедры механики, Волгоградский государственный аграрный университет, Российская Федерация, 400002, г. Волгоград, Университетский проспект, д. 26; eLIBRARY SPIN-код: 9596-2597, ORCID: 0000-0002-7394-8885; e-mail: vitalik30090@mail.ru

***Кирсанова Наталья Анатольевна,*** доктор физико-математических наук, профессор департамента математики, Финансовый университет при Правительстве Российской Федерации, Российская Федерация, г. Москва, 125993, ГСП-3, Ленинградский проспект, д. 49; eLIBRARY SPIN-код: 8393-5900, ORCID: 0000-0003-3496-2008; e-mail: nagureeve@fa.ru;

***Клочков Юрий Васильевич,*** доктор технических наук, профессор, заведующий кафедрой высшей математики, Волгоградский государственный аграрный университет, Российская Федерация, 400002, г. Волгоград, Университетский проспект, д. 26; eLIBRARY SPIN-код: 9436-3693, ORCID: 0000-0002-1027-1811; e-mail: klotchkov@bk.ru;

***Николаев Анатолий Петрович***, доктор технических наук, профессор кафедры механики, Волгоградский государственный аграрный университет, Российская Федерация, 400002, г. Волгоград, Университетский проспект, д. 26; eLIBRARY SPIN-код: 2653-5484, ORCID: 0000-0002-7098-5998; e-mail: anpetr40@yandex.ru

placement-based FEM has been widely used to analyze elasto-plastic deformation of structures [1–4]. In this formulation, the FEM was used in the analysis of thermoplastic and contact problems of continuum mechanics [5–9]. The finite element method has also been used in cases of finite strains in elasto-plastic deformation of bodies of various configurations [10–13].

Elasto-plastic deformation processes of plates and shells have also been investigated using the mixed FEM [14–18].

In this paper, a prismatic finite element in mixed formulation is developed, with strains and stresses as the nodal unknowns. A nonlinear mixed functional was used to obtain the stiffness matrix of the finite element. As physical equations, two versions of the constitutive equations of the theory of plastic flow were used. In the second version of the physical equations, the relations between the strain increments and stress increments obtained based on the hypothesis of proportionality of the components of the incremental strain deviator and the components of the incremental stress deviator were used.

## 2. Methods

### *2.1. Linear Geometric Relationships at a Load Step*

In Cartesian coordinate system *x*, *y*, *z* the components of the incremental strain tensor $\Delta\varepsilon_{ij}$ are related to the displacement increments $\Delta u_i$ by relationships[1]

$$\Delta\varepsilon_{ij} = \frac{1}{2}\left(\Delta u_{ij} + \Delta U_{ij}\right);\ (i,j = 1,2,3), \tag{1}$$

or in matrix form

$$\underset{6\times1}{\{\Delta\varepsilon\}} = \underset{6\times3}{[L]}\underset{3\times1}{\{\Delta v\}}, \tag{2}$$

where $\underset{1\times6}{\{\Delta\varepsilon\}^T} = \{\Delta\varepsilon_{11}\ \Delta\varepsilon_{22}\ \Delta\varepsilon_{33}\ 2\Delta\varepsilon_{12}\ 2\Delta\varepsilon_{13}\ 2\Delta\varepsilon_{23}\}$ is the row of strain increments; $\underset{1\times3}{\{\Delta v\}^T} = \{\Delta u_1\ \Delta u_2\ \Delta u_3\}$ is the row of displacement increments; $\underset{6\times3}{[L]}$ is the matrix of differential operators.

### *2.2. Physical Equations at a Load Step*

#### *2.2.1. Physical Equations of Plastic Flow Theory*

In the theory of plastic flow, it is assumed that the total strain increments are equal to the sum of elastic strain increments $\Delta\varepsilon_{ij}^e$ and plastic strain increments $\Delta\varepsilon_{ij}^p$.

The increments of elastic strain are determined by the relations of the Hooke's law[1]

$$\Delta\varepsilon_{11}^e = \frac{1}{E}\left(\Delta\sigma_{11} - \nu\Delta\sigma_{22} - \nu\Delta\sigma_{33}\right);$$

$$\Delta\varepsilon_{22}^e = \frac{1}{E}\left(\Delta\sigma_{22} - \nu\Delta\sigma_{11} - \nu\Delta\sigma_{33}\right);$$

$$\Delta\varepsilon_{33}^e = \frac{1}{E}\left(\Delta\sigma_{33} - \nu\Delta\sigma_{11} - \nu\Delta\sigma_{22}\right);$$

$$2\Delta\varepsilon_{12}^e = \frac{2(1+\nu)}{E}\Delta\sigma_{12};\quad 2\Delta\varepsilon_{13}^e = \frac{2(1+\nu)}{E}\Delta\sigma_{13};\quad 2\Delta\varepsilon_{23}^e = \frac{2(1+\nu)}{E}\Delta\sigma_{23}, \tag{3}$$

where $E$ is the elastic modulus of the material in tension; $\nu$ is the Poisson's ratio.

[1] Demidov S.P. *Theory of Elasticity*. Moscow: Vysshaya Shkola Publ.; 1979. (In Russ.)

To determine the plastic strain increments, the flow theory uses the hypothesis of proportionality of the components of the incremental plastic strain tensor to the components of the total stress deviator, occurring before the considered load step. According to this hypothesis,

$$\Delta\varepsilon_{ij}^{p} = \frac{\Delta\varepsilon_i^p}{\sigma_i}\left(\sigma_{ij} - \delta_{ij}\sigma_c\right), \qquad (4)$$

where $\sigma_i$ is the stress intensity; $\delta_{ij}$ is the Kronecker delta symbol; $\sigma_c = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33})$ is the first invariant of the stress tensor.

The value of the plastic strain intensity increment included in (4) is determined by the expression [19]

$$\Delta\varepsilon_i^p = \Delta e_i - \Delta\varepsilon_i^e = \frac{\Delta\sigma_i}{E_x} - \frac{\Delta\sigma_i}{E_н}, \qquad (5)$$

where $\Delta\varepsilon_i$ is the strain intensity increment; $\Delta\varepsilon_{ij}^e$ is the elastic strain intensity increment; $E_x$ is the chord modulus of the stress-strain diagram; $E_н$ is the modulus of the initial region of the stress-strain diagram; $\Delta\sigma_i$ is the stress intensity increment.

Considering (5), relationships (4) can be expressed as

$$\Delta\varepsilon_{mn}^{p} = \Delta\sigma_i\left(\frac{1}{E_c} - \frac{1}{E_0}\right)\frac{1}{\sigma_i}\left(\sigma_{mn} - \delta_{mn}\sigma_c\right). \qquad (6)$$

In order to express $\Delta\varepsilon_{mn}^p$ according to (6) by stress increment functions $\Delta\sigma_{ij}$, $\Delta\sigma_i$ needs to be written in such general form

$$\Delta\sigma_i = \frac{\partial\sigma_i}{\partial\sigma_{mn}}\Delta\sigma_{mn}, \qquad (7)$$

where $\sigma_i = \frac{1}{\sqrt{2}}\left[\left(\sigma_{11} - \sigma_{22}\right)^2 + \left(\sigma_{22} - \sigma_{33}\right)^2 + \left(\sigma_{33} - v_{11}\right)^2 + 6\left(\sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2\right)\right]^{1/2}$.

By summing (3) and (6) and taking into account (7), the matrix relationship of the constitutive equations of the flow theory is formed

$$\underset{6\times1}{\{\Delta\varepsilon\}} = \underset{6\times6}{\{C_1^P\}}\underset{6\times1}{\{\Delta\sigma\}}, \qquad (8)$$

where $\underset{1\times6}{\{\Delta\varepsilon\}^T} = \{\Delta\varepsilon_{11}\,\Delta\varepsilon_{22}\;\Delta\varepsilon_{33}\;2\Delta\varepsilon_{12}\;2\Delta\varepsilon_{13}\;2\Delta\varepsilon_{23}\}$; $\underset{1\times6}{\{\Delta\sigma\}^T} = \{\Delta\sigma_{11}\,\Delta\sigma_{22}\;\Delta\sigma_{33}\;\Delta\sigma_{12}\;\Delta\sigma_{13}\;\Delta\sigma_{23}\}$.

### *2.2.2. Version of Physical Equations without Separation of Strain Increments into Elastic and Plastic Parts*

The hypothesis of proportionality of the components of the incremental strain deviator to the components of the incremental stress deviator was used in obtaining the constitutive equations of the specified version, leading to the following relations

$$\Delta\varepsilon_{mn} - \delta_{mn}\Delta\varepsilon_c = \frac{3}{2}\frac{\Delta\varepsilon_i}{\Delta\sigma_i}\left(\Delta\sigma_{mn} - \delta_{mn}\Delta\sigma_c\right), \qquad (9)$$

where $\frac{\Delta\varepsilon_i}{\Delta\sigma_i} = \frac{1}{E_x}$; $\Delta\varepsilon_c = \frac{1}{3}(\Delta\varepsilon_{11} + \Delta\varepsilon_{22} + \Delta\varepsilon_{33})$; $\Delta\sigma_c = \frac{1}{3}(\Delta\sigma_{11} + \Delta\sigma_{22} + \Delta\sigma_{33})$.

Using the condition of incompressibility under plastic deformation assumed in the theory of plastic flow, according to which the following relation holds

$$\Delta\varepsilon_c = \frac{1-2\nu}{E}\Delta\sigma, \tag{10}$$

constitutive equations (9) can be expressed in matrix form

$$\underset{6\times1}{\{\Delta\varepsilon\}} = \underset{6\times6}{\left[C_2^P\right]}\underset{6\times1}{\{\Delta\sigma\}}. \tag{11}$$

### 3. Stiffness Matrix of Prismatic Finite Element

A prism with triangular bases is taken as the finite element. Nodal unknowns are displacement increments and stress increments. For integration over the finite element volume, rectangular prism is assumed, the height of which is determined by coordinate $-1 \le \zeta \le 1$, and the bases are right triangles with coordinates $0 \le \xi,\ \eta \le 1$.

Approximation of Cartesian coordinates $x_i$ of displacement increments $\Delta v_i$ and stress increments $\Delta\sigma_{ij}$ is performed in terms of the corresponding nodal quantities by linear functions

$$\lambda = \left\{(1-\xi-\eta)\frac{1-\zeta}{2} \vdots\ \xi\frac{1-\zeta}{2} \vdots\ \eta\frac{1-\zeta}{2} \vdots \{1-\xi-\eta\}\frac{1+\zeta}{2} \vdots\ \xi\frac{1+\zeta}{2} \vdots\ \eta\frac{1+\zeta}{2}\right\}\{\lambda\} = \underset{1\times6}{\{f(\xi,\eta)\}^T}\underset{6\times1}{\{\lambda_y\}}, \tag{12}$$

where $\underset{1\times6}{\{\lambda_y\}^T} = \left\{\lambda^i\ \lambda^j\ \lambda^k\lambda^m\ \lambda^n\ \lambda^p\right\}$ is the row of nodal values of $\lambda$; symbol $\lambda$ denotes the values of $x_i, \Delta v_i, \Delta\sigma_{ij}$.

On the basis of (12), the necessary approximating expressions are written in matrix form

$$\underset{3\times1}{\{\Delta v\}} = \underset{3\times18}{[A]}\underset{18\times1}{\{\Delta v_y\}};\ \underset{6\times1}{\{\Delta\varepsilon\}} = \underset{6\times3}{[L]}\ \underset{3\times18}{[A]}\ \underset{18\times1}{\{\Delta v_y\}} = \underset{6\times18}{[B]}\ \underset{18\times1}{\{\Delta v_y\}};\ \underset{6\times1}{\{\Delta\sigma\}} = \underset{6\times36}{[S]}\underset{36\times1}{\{\Delta\sigma_y\}}, \tag{13}$$

where

$$\underset{1\times18}{\{\Delta v_y\}^T} = \left\{\Delta v^{1i}\Delta v^{1j}\Delta v^{1k}\Delta v^{1m}\Delta v^{1n}\Delta v^{1p}\ldots v^{3i}\Delta v^{3j}\Delta v^{3k}\Delta v^{3m}\Delta v^{3n}\Delta v^{3p}\right\};$$

$$\underset{1\times36}{\{\Delta\sigma_y\}^T} = \{\Delta\sigma_{11}^i\Delta\sigma_{22}^i\Delta\sigma_{33}^i\Delta\sigma_{12}^i\Delta\sigma_{13}^i\Delta\sigma_{23}^i\ldots\ \Delta\sigma_{11}^p\Delta\sigma_{22}^p\Delta\sigma_{33}^p\Delta\sigma_{12}^p\Delta\sigma_{13}^p\Delta\sigma_{23}^p\}.$$

To form the stiffness matrix of the prismatic finite element, a mixed functional for the considered load step is used [16]

$$\Phi = \int_V \left[\underset{1\times6}{\{\sigma\}^T} + \frac{1}{2}\underset{1\times6}{\{\Delta\sigma\}^T}\right]\underset{6\times3}{[L]}\underset{3\times1}{\{\Delta v\}}dV - \frac{1}{2}\int_V \underset{1\times6}{\{\Delta\sigma\}^T}\underset{6\times6}{\left[G_\mu^p\right]^{-1}}\underset{6\times1}{\{\Delta\sigma\}}dV -$$

$$-\int_S \underset{1\times3}{\{\Delta v\}^T}\left(\underset{3\times1}{\{q\}} + \frac{1}{2}\underset{3\times1}{\{\Delta q\}}\right)dS; \quad (\mu = 1.2), \tag{14}$$

where $\{\Delta q\}, \{q\}$ is the load applied at the considered load step and before the considered step; $V$ is the volume of the finite element; $S$ is the area of load application.

Taking into account approximating relations (13), functional (14) will be expressed as

$$\Phi \equiv \underset{1\times36}{\left\{\Delta v_y\right\}^T} \int\limits_V \underset{36\times6}{\left[B\right]^T} \underset{6\times1}{\left\{\sigma\right\}} dV + \underset{1\times36}{\left\{\Delta\sigma_y\right\}^T} \int\limits_V \underset{36\times6}{\left[S\right]^T} \underset{6\times18}{\left[B\right]} dV \underset{18\times1}{\left\{\Delta v_y\right\}} -$$

$$-\frac{1}{2}\underset{1\times36}{\left\{\Delta\sigma_y\right\}^T} \int\limits_V \underset{36\times6}{\left[S\right]^T} \underset{6\times6}{\left[G_\mu^p\right]} \underset{6\times36}{\left[S\right]} dV \underset{36\times1}{\left\{\Delta\sigma_y\right\}} - \frac{1}{2}\underset{1\times18}{\left\{\Delta v_y\right\}^T} \int\limits_S \underset{18\times3}{\left[A\right]^T} \underset{3\times1}{\left\{\Delta q\right\}} dS - \underset{1\times18}{\left\{\Delta v_y\right\}^T} \int\limits_S \underset{18\times3}{\left[A\right]^T} \underset{3\times1}{\left\{q\right\}} dS . \quad (15)$$

As a result of minimization of functional (15) with respect to nodal unknowns, two systems of equations are obtained

$$\frac{\partial\Phi}{\partial\left\{\Delta\sigma_y\right\}^T} \equiv \underset{36\times18}{\left[Q\right]} \underset{18\times1}{\left\{\Delta v_y\right\}} - \underset{36\times36}{\left[H\right]} \underset{36\times1}{\left\{\Delta\sigma_y\right\}} = 0, \quad (16)$$

$$\frac{\partial\Phi}{\partial\left\{\Delta v_y\right\}^T} \equiv \underset{18\times36}{\left[Q\right]^T} \underset{36\times1}{\left\{\Delta\sigma_y\right\}} - \underset{18\times1}{\left\{\Delta f_q\right\}} - \underset{18\times1}{\left\{R\right\}} = 0, \quad (17)$$

where $\underset{36\times18}{\left[Q\right]} = \int\limits_V \underset{36\times6}{\left[S\right]^T} \underset{6\times18}{\left[B\right]} dV$; $\underset{36\times36}{\left[H\right]} = \int\limits_V \underset{36\times6}{\left[S\right]^T} \underset{6\times6}{\left[G_\mu^p\right]^{-1}} \underset{6\times36}{\left[S\right]} dV$; $\underset{18\times1}{\left[\Delta f_q\right]} = \int\limits_S \underset{18\times3}{\left[A\right]^T} \underset{3\times1}{\left\{\Delta q\right\}} dS$;

$\underset{18\times1}{\left[R\right]} = \int\limits_S \underset{18\times3}{\left[A\right]^T} \underset{3\times1}{\left\{q\right\}} dS - \int\limits_V \underset{18\times6}{\left[B\right]^T} \underset{6\times1}{\left\{\sigma\right\}} dV$ is the Raphson residual error.

From system of equations (16), the column of stress increments is determined in terms of displacement increments

$$\underset{36\times1}{\left\{\Delta\sigma_y\right\}} = \underset{36\times36}{\left[H\right]^{-1}} \underset{36\times18}{\left[Q\right]} \underset{18\times1}{\left\{\Delta v_y\right\}} . \quad (18)$$

Taking into account (18), the stiffness matrix of the finite element is determined from system (17)

$$\underset{18\times18}{\left[K\right]} \underset{18\times1}{\left\{\Delta v_y\right\}} = \underset{18\times1}{\left\{\Delta f_q\right\}} + \underset{18\times1}{\left\{R\right\}}, \quad (19)$$

where $\left[K\right] = \underset{18\times36}{\left[Q\right]^T} \underset{36\times36}{\left[H\right]^{-1}} \underset{36\times18}{\left[Q\right]}$ .

Using (19), the stiffness matrix of the considered structure is formed, by solving which the displacements of all nodes are determined.

The values of the stress increments are determined by equations (18).

## 4. Results and Discussion

### *4.1. Example of Analysis*

The stress-strain state of a simply-supported beam (Figure 1) is considered given the following data: $l$ = 200 cm; $h$ = 10 cm; $b$ = 1 cm; $P$ = 13 kN.

The tensile and flexural stress-strain diagrams of the material are shown in Figures 2 and 3.

Physical and mechanical parameters are assumed to be the following: $E = 2\cdot10^5$ MPa; $\nu = 0.3$; $\sigma_T = 200$ MPa; $\sigma_k = 300$ MPa; $\varepsilon_T = 0.001$; $\varepsilon_k = 0.01$; $\sigma_{iT} = 200$ MPa; $\sigma_{ik} = 300$ MPa; $\varepsilon_{iT} = \frac{2}{3}(1+\nu)\varepsilon_T = 0.866667\cdot10^{-3}$; $\varepsilon_{ik} = 0.866667\cdot10^{-2}$; $W = bh^2/6 = 1\times10^2/6 = 16.66667$ cm$^3$.

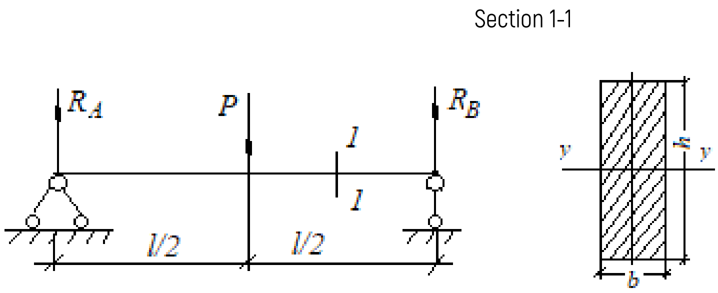


**Figure 1**. Simply-supported beam
S o u r c e: made by R.Z. Kiseleva

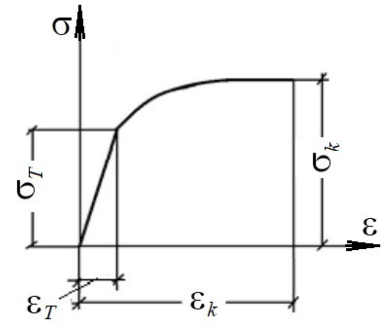


**Figure 2.** Tensile stress-strain diagram
S o u r c e: made by R.Z. Kiseleva

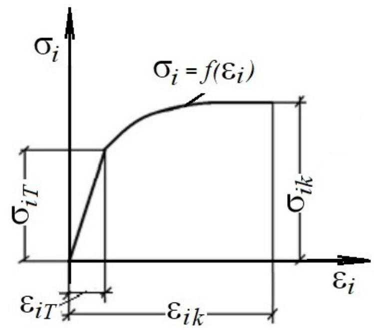


**Figure 3**. Flexural stress-strain diagram
S o u r c e: made by R.Z. Kiseleva

The strain hardening curve is described by the equation

$$\sigma_i = a\varepsilon_i^2 + b\varepsilon_i + c\,,$$

where *a, b, c* are coefficients with the following numerical values:

$a = -6612835.53$ MPa; $b = 242231.48$ MPa; $c = 1795.033$ MPa.

The model of the beam is presented in Figure 4 ($N_m = 40$; $N_t = 11$ number of nodes along the beam axis and along its height).

The results of calculation using the considered versions of the constitutive equations turned out to be virtually identical.

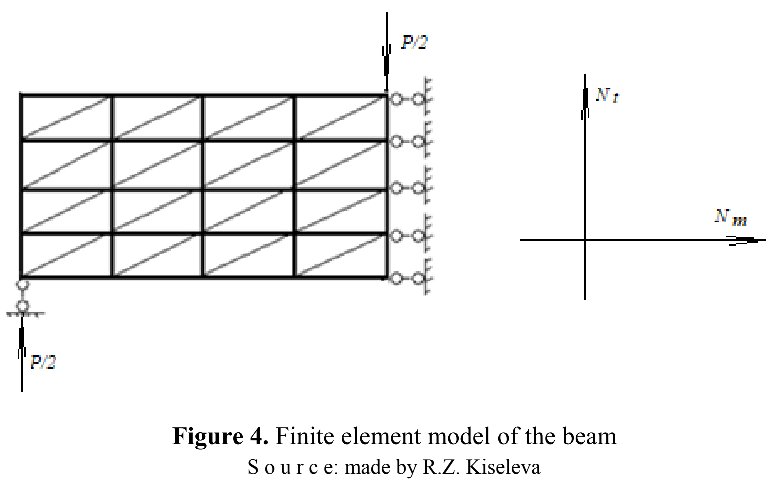


**Figure 4.** Finite element model of the beam
S o u r c e: made by R.Z. Kiseleva

According to the obtained values of normal stresses, the normal stress diagram in the beam cross-section, which is located at distance $h$ from the middle one, is plotted (Figure 5).

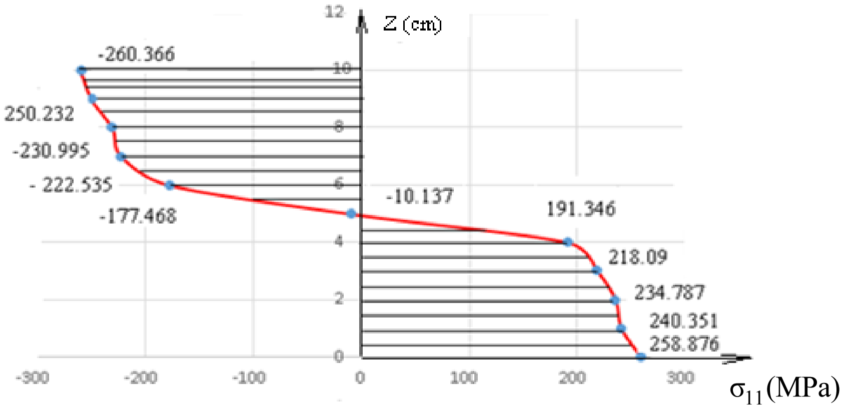


**Figure 5**. Diagram of normal stress $\sigma_{11}$ in the beam section
S o u r c e: made by R.Z. Kiseleva

Equations of statics $\Sigma x = 0$ for the diagram in Figure 5 are satisfied with a discrepancy of $\delta \approx 0.7\%$. The moment from external forces is equal to $M_p = \frac{P}{2}\left(\frac{l}{2} - h\right) = 6.5\times(100-10) = 585$ kN×cm, and the moment from the internal forces is equal to $M_{\text{int}} = 597.4$ kN×cm.

The equations of statics have a discrepancy of $\delta \approx 0.5\%$ in terms of the equality of moments from external and internal forces.

## 5. Conclusion

Based on the obtained calculation results, it can be concluded:

1. The developed version of the constitutive equations without separation of strain increments into elastic and plastic parts leads to the same results as when using the constitutive equations of the flow theory, where the separation into elastic and plastic parts is adopted.

2. The use of the hypothesis of proportionality of the components of the incremental strain and incremental stress deviators in the developed version of the constitutive equations allows to express the stress increments immediately. In flow theory, the strain increments are first expressed in terms of the stress intensity increments, and only after performing differentiation operations it is possible to obtain the required constitutive equations.

The proposed version of the constitutive equations of the plasticity theory is more preferable for use in the design practice of engineering structures.

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