СТРОИТЕЛЬНАЯ МЕХАНИКА ИНЖЕНЕРНЫХ КОНСТРУКЦИЙ И СООРУЖЕНИЙ
STRUCTURAL MECHANICS OF ENGINEERING CONSTRUCTIONS AND BUILDINGS

2024. 20(6). 567–592
ISSN 1815-5235 (Print), 2587-8700 (Online)
HTTP://JOURNALS.RUDN.RU/STRUCTURAL-MECHANICS

РАСЧЕТ ТОНКИХ УПРУГИХ ОБОЛОЧЕК
ANALYSIS OF THIN ELASTIC SHELLS

DOI: 10.22363/1815-5235-2024-20-6-567-592
UDC 624.04’074.4:539.41:513.73:514.74:514.8
EDN: CYGRSH

Research article / Научная статья

Analytical Calculation of Cylindrical Shells in the Form of Second-Order Algebraic Surfaces

Sergey N. Krivoshapko

RUDN University, *Moscow, Russia*
✉ sn_krivoshapko@mail.ru

Received: July 30, 2024
Accepted: September 27, 2024

Abstract. When choosing the shape of a shell, one should strive for the boundary conditions to ensure momentless behavior of the shell. Second-order algebraic surfaces include three degenerate surfaces: parabolic, elliptic, and hyperbolic cylindrical surfaces, and two surfaces derived from them: circular cylindrical surface and cylindrical surface with incomplete ellipse in cross-section. These five surfaces are the objects of this research. For the first time, comparative static analysis of the five shells under a load of self-weight type is performed using the momentless shell theory. The explicit formulae for the determination of three internal membrane forces are obtained. It is shown that the parabolic cylindrical shell and the cylindrical shell with incomplete ellipse in cross-section perform better within the momentless shell theory. The constraints for the application of the momentless theory obtained earlier by other authors are confirmed. For the first time, a system of three partial differential equations with respect to the displacements of middle surfaces of the five cylindrical shells given in previously unused curvilinear coordinates is derived. It is established that no studies dealt with the calculation of hyberbolic cylindrical shells so far. A brief review of publications on the analysis of strength, stability, dynamics, and application of the five considered cylindrical shells is given to clarify the directions of investigation of these five cylindrical shells.

Keywords: thin shell, hyperbolic cylindrical shell, parabolic cylindrical shell, circular cylindrical shell, elliptic cylindrical shell, linear shell theory in lines of curvature, momentless shell theory

Conflicts of interest. The author declares that there is no conflict of interest.

For citation: Krivoshapko S.N. Analytical calculation of cylindrical shells in the form of second-order algebraic surfaces. *Structural Mechanics of Engineering Constructions and Buildings.* 2024;20(6):567–592. http://doi.org/10.22363/1815-5235-2024-20-6-567-592

Sergey N. Krivoshapko, Doctor of Technical Sciences, Professor of the Department of Construction Technology and Structural Materials, Academy of Engineering, RUDN University, Moscow, Russia; eLIBRARY SPIN-code: 2021-6966, ORCID: 0000-0002-9385-3699; e-mail: sn_krivoshapko@mail.ru

Аналитический расчет цилиндрических оболочек в форме алгебраических поверхностей второго порядка

С.Н. Кривошапко

Российский университет дружбы народов, *Москва, Россия*
✉ sn_krivoshapko@mail.ru

Поступила в редакцию: 30 июля 2024 г.
Принята к публикации: 27 сентября 2024 г.

Аннотация. При выборе формы оболочек нужно стремиться, чтобы граничные условия обеспечивали работу оболочек в безмоментном состоянии. В состав алгебраических поверхностей второго порядка входят три вырожденные поверхности: параболическая, эллиптическая и гиперболическая цилиндрические поверхности, а также две производные от них поверхности: круговая цилиндрическая поверхность и цилиндрическая поверхность с неполным эллипсом в поперечном сечении. Эти пять цилиндрических поверхностей стали объектами исследования в статье. Впервые произведен сравнительный расчет по безмоментной теории пяти оболочек на действие статической нагрузки типа собственного веса, для чего получены в явном виде формулы для определения трех тангенциальных внутренних усилий. Показано, что в рамках безмоментной теории оболочек лучше работает параболическая цилиндрическая оболочка и цилиндрическая оболочка с неполным эллипсом в поперечном сечении. Подтверждены полученные ранее другими авторами ограничения на применение безмоментной теории. Впервые выведена система трех дифференциальных уравнений в частных производных относительно перемещений срединной поверхности пяти цилиндрических оболочек, заданных в ранее не применявшихся криволинейных координатах. Установлено, что до настоящего времени никто не занимался расчетом гиперболической цилиндрической оболочки. Приведен краткий обзор опубликованных работ по расчету на прочность, устойчивость, колебания и применение пяти рассматриваемых цилиндрических оболочек для выяснения направлений исследований этих пяти цилиндрических оболочек.

Ключевые слова: тонкая оболочка, гиперболическая цилиндрическая оболочка, параболическая цилиндрическая оболочка, круговая цилиндрическая оболочка, эллиптическая цилиндрическая оболочка, линейная теория оболочек в линиях кривизны, безмоментная теория оболочек

Заявление о конфликте интересов. Автор заявляет об отсутствии конфликта интересов.

Для цитирования: *Krivoshapko S.N.* Analytical calculation of cylindrical shells in the form of second-order algebraic surfaces // Строительная механика инженерных конструкций и сооружений. 2024. Т. 20. № 6. С. 567–592. http://doi.org/10.22363/1815-5235-2024-20-6-567-592

1. Introduction

Various analytical, semi-analytical and numerical methods for the calculation of thin- and thick-walled, single- and multi-layer shells made of physically linear and nonlinear structural materials have been developed in response to the demands of practice. To assist the design engineer, the corresponding computational software systems have been created. Various optimality criteria have been developed for the design of thin shells.

A number of review papers [1–4] show that shells of rotation, translation, cylindrical and conical shells are among the most popular thin and thick shells that can satisfy a variety of practical and scientific demands arising in different time periods.

The greatest attention has been given to shells of rotation. Twenty three optimality criteria were proposed for them [5]. V.V. Novozhilov et al. [6] showed that, in terms of stress magnitude, a parabolic dome is the most efficient, but it requires the largest area of the supporting ring and is the least efficient from this standpoint. In [7], the stress-strain state of the rotation shells with holes in the apex and the same overall dimensions, but different meridians, was compared. Comparative calculations of domes were performed to determine the strength parameters under external static [6], dynamic and explosive [8] loads. The choice of an optimal velaroidal shell on a rhombic flat base, the middle surface of which is formed by the motion of different plane curves with variable curvature, was studied in [9].

Кривошапко Сергей Николаевич, доктор технических наук, профессор, профессор кафедры технологий строительства и конструкционных материалов, инженерная академия, Российский университет дружбы народов, Москва, Россия; eLIBRARY SPIN-код: 2021-6966, ORCID: 0000-0002-9385-3699; e-mail: sn_krivoshapko@mail.ru

However, such comparative calculations have not been performed for hyperbolic, parabolic, elliptic and circular cylindrical roofs, although architects have tried to expand the list of surfaces of zero Gaussian curvature on a rectangular base by including developable surfaces [10].

Materials for analytical calculation of cylindrical shells defined by algebraic surfaces are considered primarily in this paper. At present time, virtually all problems of structural mechanics are solved using numerical methods. But there is another standpoint, such as that of the authors of monograph [6]: “In a reasonable combination of analytical and numerical methods with an understanding of the mechanical side of the problem under consideration”.

Purpose of the study. To determine the strength parameters of two elliptic, circular, hyperbolic and parabolic cylindrical thin shells, having the same span and height, constant thickness, physical and mechanical characteristics of the structural material and subjected to the same static external load using the analytical calculation technique of the momentless shell theory.

By comparing the obtained approximate calculation results, conclusions need to be drawn regarding the choice of an optimal cylindrical shell in the form of a second-order algebraic surface and regarding the applicability of the momentless shell theory to the calculation of the considered shells.

It is also necessary to determine the position of the shells in the form of second-order cylindrical surfaces in the modern construction practice and in the solution of new problems, arising in the design of these shells for various needs of society.

2. Explicit and Parametric Equations of Second-Order Algebraic Cylindrical Surfaces and Their Coefficients of Fundamental Quadratic Forms

Second-order algebraic surfaces include 3 degenerate surfaces: parabolic (Figure 1), elliptic (Figure 2) and hyperbolic cylindrical surfaces. These three surfaces are well known to geometers, architects [11], and engineers. Based on these surfaces, two more surfaces can be derived: circular cylindrical surface (Figure 3) and cylindrical surface with incomplete ellipse in cross-section (Figure 4).

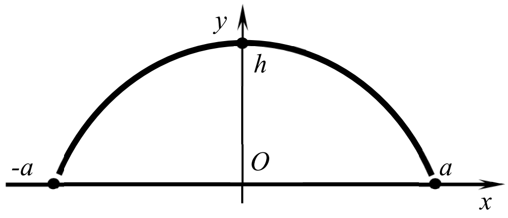

Figure 1. Parabolic cylindrical shell
S o u r c e: made by S.N. Krivoshapko

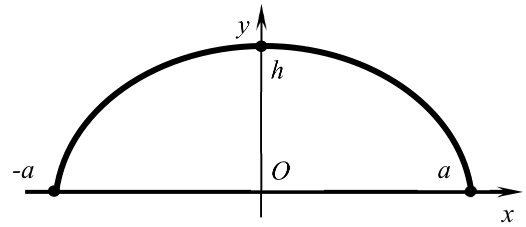

Figure 2. Semielliptic cylindrical shell
S o u r c e: made by S.N. Krivoshapko

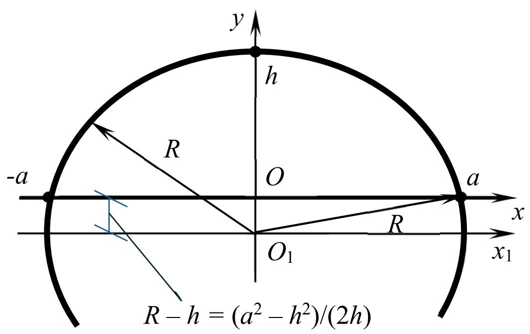

Figure 3. Circular cylindrical shell
S o u r c e: made by S.N. Krivoshapko

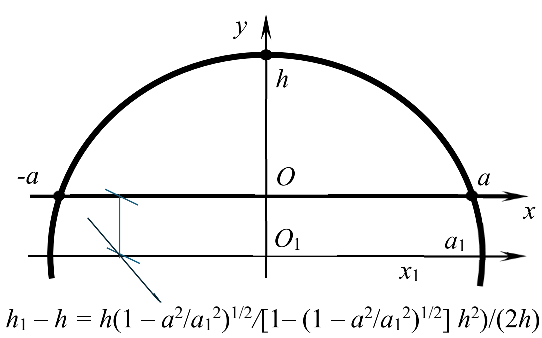

Figure 4. Elliptic cylindrical shell
S o u r c e: made by S.N. Krivoshapko

2.1. Parabolic Cylindrical Thin Shell

The canonical explicit equation of parabolic cylindrical surface (see Figure 1):

$$y = y(x) = h\left(1 - \frac{x^2}{a^2}\right), \quad z = z.$$

The parametric equations of this surface may be expressed as

$$x = x(u) = au, \quad y = y(u) = h\left(1 - u^2\right), \quad z = z(v) = lv,$$

where *u, v* are the dimensionless independent parameters, $-1 \le u \le 1, \quad 0 \le v \le 1$.

Then, the coefficients of the first (*A, F, B*) and the second (*L, M, N*) quadratic forms of the surface will be the following:

$$A^2 = a^2 + 4h^2u^2,\ F = 0, \quad B = l,\ L = 2ah/A,\ \ M = 0,\ N = 0,$$

and the radius of curvature of coordinate line *u* will be

$$R_1 = A^2/L = A^3/(2ah).$$

Hence, hereinafter, a coordinate grid in the lines of curvature is used, with coordinate line *u* coinciding with the directing parabola of the cylinder, and coordinate lines *v* coinciding with the rectilinear generators of the cylinder $(R_v = R_2 = \infty)$.

2.2. Elliptic Cylindrical Thin Shell

The canonical explicit equation of elliptic cylindrical surface:

$$\frac{x^2}{a^2} + \frac{y^2}{h^2} = 1,\ z = z,$$

where *a, h* are the lengths of the semi-axes of the ellipse.

2.2.1. Cylindrical Thin Shell With Semi-Ellipse in Cross-Section (Figure 2)

The parametric equations of this surface can be represented as

$$x = x(u) = au,\ y = y(u) = h\left(1 - u^2\right)^{1/2},\ z = z(v) = lv,$$

where *u, v* are the dimensionless independent parameters, $-1 \le u \le 1, \quad 0 \le v \le 1$.

The coefficients of the first (*A, F, B*) and the second (*L, M, N*) quadratic forms of the surface will be

$$A^2 = a^2 + h^2u^2/\left(1 - u^2\right),\ F = 0,\ B = l,\ L = ah/\left[A\left(1 - u^2\right)^{3/2}\right],\ M = 0,\ N = 0,$$

and the radius of curvature of coordinate line *u* will be

$$R_1 = A^2/L = A^3(1 - u^2)^{3/2}/(ah).$$

2.2.2. Cylindrical Thin Shell with Incomplete Ellipse in Cross-Section

The canonical explicit equation of the considered cylindrical surface (see Figure 4):

$$\frac{x^2}{a_1^2}+\left(\frac{y}{h_1}+\sqrt{1-\frac{a^2}{a_1^2}}\right)^2=1, \quad z=z.$$

In the considered case it is necessary to set the length of the semi-axis of the full ellipse $a_1 \geq a$, and then to determine the length of the other semi-axis of the full ellipse h_1:

$$h_1=\frac{h}{1-\sqrt{1-\frac{a^2}{a_1^2}}}.$$

The parametric equations of the considered cylindrical surface can be represented in the following form:

$$x=x(u)=au,\; z=z(v)=lv,$$

$$y=y(u)=h_1\left(\sqrt{1-u^2\frac{a^2}{a_1^2}}-\sqrt{1-\frac{a^2}{a_1^2}}\right).$$

The coefficients of the first (*A, F, B*) and the second (*L, M, N*) quadratic forms of the surface will be

$$A^2=a^2+\frac{h_1^2a^4u^2}{a_1^4\left(1-\frac{a^2}{a_1^2}u^2\right)},\; F=0,\; B=l,$$

$$L=a^3h_1a_1/\left[A\left(a_1^2-a^2u^2\right)^{3/2}\right],\; M=0,\; N=0,$$

and the radius of curvature of coordinate line u will be

$$R_u=R_1=\frac{A^3a_1^2}{a^3h_1}\left(1-\frac{a^2}{a_1^2}u^2\right)^{3/2}.$$

2.2.3. Circular Cylindrical Thin Shell (See Figure 3)

The formulas obtained in section 2.2.2 may be used for the circular cylindrical shell, but one must set $a_1 = R$, $h_1 = R$, then

$$x^2+\left(y+\frac{a^2-h^2}{2h}\right)^2=\frac{\left(h^2+a^2\right)^2}{4h^2}=R^2,$$

$$x=x(u)=au,\; z=z(v)=lv,$$

$$y=y(u)=\sqrt{R^2-a^2u^2}-\frac{a^2-h^2}{2h}.$$

The coefficients of the first (*A, F, B*) and the second (*L, M, N*) quadratic forms of the surface will be

$$A^2 = \frac{a^2R^2}{\left(R^2 - a^2u^2\right)},\ F = 0,\ B = l,$$

$$L = \frac{a^3R^2}{A\left(R^2 - a^2u^2\right)^{3/2}} = \frac{a^2R}{R^2 - a^2u^2},\quad M = 0,\ \ N = 0,$$

and the radius of curvature of coordinate line u will be

$$R_u = R_1 = R = \frac{a^2 + h^2}{2h}.$$

The radius of curvature of coordinate line v will be $R_v = R_2 = \infty$.

2.3. Hyperbolic Cylindrical Thin Shell

The implicit equation of one branch of a hyperbolic cylindrical surface can be written in the following form (Figure 5):

$$\frac{y^2}{c^2} - \frac{x^2}{b^2} = 1,$$

where for a single considered branch

$$b^2 = \frac{a^2c^2}{h(2c+h)}.$$

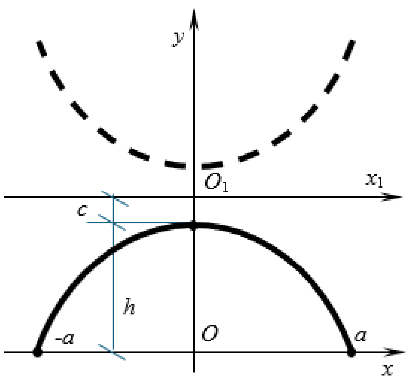

Figure 5. One branch of the hyperbolic cylindrical shell
S o u r c e: made by S.N. Krivoshapko

In this case, the explicit equation of the hyperbolic cylindrical surface defined in the *xOy* axes is written as

$$y = h + c - \left[c^2 + x^2h\cdot(2c+h)/a^2\right]^{1/2},$$

taking into account the introduced geometric notation shown in Figure 5.

The parametric equations of the considered cylindrical surface can be represented in the following form:

$$x = x(u) = au,$$
$$z = z(v) = lv,$$
$$y = c + h - \sqrt{c^2 + hu^2\left(2c+h\right)}.$$

The coefficients of the first (*A, F, B*) and the second (*L, M, N*) quadratic forms of the surface will be

$$A^2 = a^2 + \frac{h^2\left(2c+h\right)^2u^2}{c^2 + hu^2\left(2c+h\right)},\ F = 0,\ B = l,$$

$$L = \frac{ahc^2\left(2c+h\right)}{A\left[c^2 + hu^2\left(2c+h\right)\right]^{3/2}},\ M = 0,\ N = 0.$$

The radii of curvature of coordinate lines u and v are $R_1 = A^2/L$, $R_v = R_2 = \infty$.

3. Three Groups of Governing Equations of Linear Theory of Thin Shells in Lines of Principle Curvatures

Taking into account that the coefficients of the first fundamental form for cylindrical surfaces are $B = l$, $A = A(u)$, then

$$\partial B/\partial u = \partial B/\partial v = 0,\ \partial A/\partial v = 0,\ R_v = R_2 = \infty,$$

and therefore the equations of the general linear theory of thin shells will be simplified in some way.

Equilibrium equations:

$$l\frac{\partial N_u}{\partial u} + A\frac{\partial S}{\partial v} - Al\frac{Q_u}{R_u} + AlX = 0,$$

$$\frac{\partial S}{\partial u} + A\frac{\partial N_v}{\partial v} + AlY = 0,$$

$$Al\frac{N_u}{R_u} + l\frac{\partial Q_u}{\partial u} + A\frac{\partial Q_v}{\partial v} - AlZ = 0,$$

$$l\frac{\partial H}{\partial u} - A\frac{\partial M_v}{\partial v} - AlQ_v = 0,$$

$$l\frac{\partial M_u}{\partial u} - A\frac{\partial H}{\partial v} + AlQ_u = 0, \qquad (1)$$

where the positive directions of bending and twisting moments, internal normal and shear forces, external surface load are shown in Figure 6.

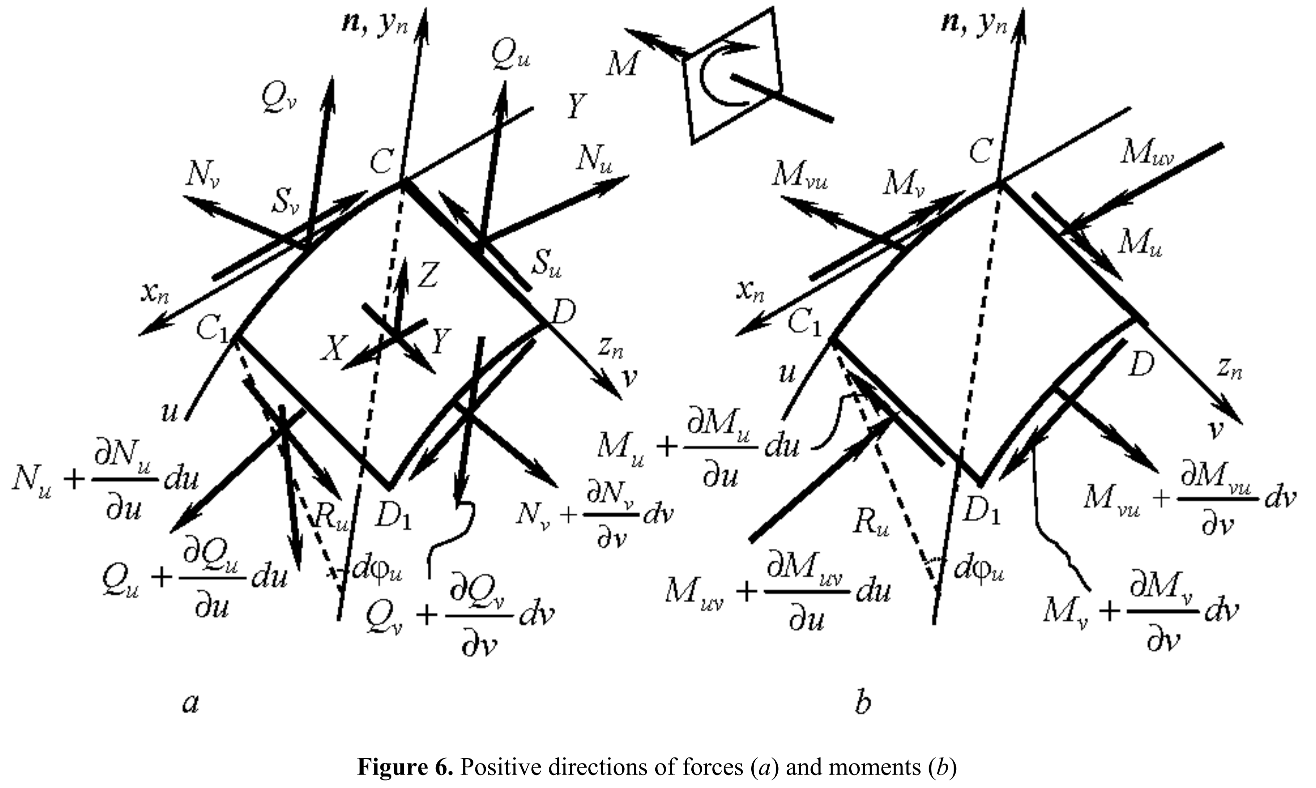

Figure 6. Positive directions of forces (*a*) and moments (*b*)
S o u r c e: made by S.N. Krivoshapko

Constitutive equations:

$$N_u = C(\varepsilon_u + \nu\varepsilon_v), \quad N_v = C(\varepsilon_v + \nu\varepsilon_u), \quad S = C(1-\nu)\varepsilon_{uv}/2,$$

$$M_u = -D(\kappa_u + \nu\kappa_v),\ M_v = -D(\kappa_v + \nu\kappa_u),\ H = -D(1-\nu)\kappa_{uv}, \tag{2}$$

where $C = Eh/\left(1-\nu^2\right), D = Eh^3/\left[12/\left(1-\nu^2\right)\right], h$ is the shell thickness, ν is the Poisson's ratio, E is the modulus of elasticity of the shell material. ε_u, ε_v, ε_{uv}, κ_u, κ_v, κ_{uv} are the components of the membrane and flexural strains. The contradictions introduced by formulas (2) into the shell theory were first pointed out by V.Z. Vlasov. However, in the opinion of V.V. Novozhilov, all these contradictions do not exceed the errors introduced into the shell theory by the initial assumptions of the thin shell theory.

Geometrical equations:

$$\varepsilon_u = \frac{1}{A}\frac{\partial u_u}{\partial u} - \frac{u_z}{R_u},\ \varepsilon_v = \frac{1}{l}\frac{\partial u_v}{\partial v},\ \varepsilon_{uv} = \frac{1}{A}\frac{\partial u_v}{\partial u} + \frac{1}{l}\frac{\partial u_u}{\partial v},$$

$$\kappa_u = \frac{1}{A}\frac{\partial}{\partial u}\left(\frac{1}{A}\frac{\partial u_z}{\partial u} + \frac{u_u}{R_u}\right), \quad \kappa_v = \frac{1}{l^2}\frac{\partial^2 u_z}{\partial v^2}, \quad 2\kappa_{uv} = \frac{1}{l}\frac{\partial}{\partial v}\left(\frac{2}{A}\frac{\partial u_z}{\partial u} + \frac{u_u}{R_u}\right), \tag{3}$$

where u_u, u_v, u_z are the displacement components.

4. Determination of Internal Forces of Thin Cylindrical Shells in the Form of Second-Order Algebraic Surfaces Using Momentless Shell Theory

In this section, the linear theory of thin rigid shells will be considered in the approximate momentless setting, when it is assumed that internal bending and twisting moments, and hence shear forces, can be neglected. The assumption of uniform stress distribution over the thickness of the shell led to the emergence of the momentless theory of rigid shells. Those who are interested in the momentless thin shells that only accept tensile internal forces and change their shape under external forces can explore the materials of review article [12] with 80 sources used.

Assuming that a rigid shell satisfies the requirements of momentless state, then the first three general equations of equilibrium (1) of the shell can be used excluding the shear forces:

$$l\frac{\partial N_u}{\partial u} + A\frac{\partial S}{\partial v} + AlX = 0,$$

$$l\frac{\partial S}{\partial u} + A\frac{\partial N_v}{\partial v} + AlY = 0,$$

$$\frac{N_u}{R_u} - Z = 0,$$

from which it follows that

$$N_u = ZR_u,$$

$$S = -l\int\left(\frac{1}{A}\frac{\partial N_u}{\partial u} + X\right)dv + f_1(u) = -lv\left(\frac{1}{A}\frac{\partial N_u}{\partial u} + X\right) + f_1(u),$$

$$N_v = -\frac{l}{A}\int\frac{\partial S}{\partial u}dv - lYv + f_2(u). \tag{4}$$

Thus, the problem of determining the internal normal and shear forces by the approximate momentless shell theory is a statically determinate problem for all shells in the form of second-order cylindrical surfaces.

Arbitrary functions of integration $f_1(u)$, $f_2(u)$ and $df_1(u)/du$ are determined by satisfying boundary conditions according to the momentless shell theory.

In further calculations of all five cylindrical shells defined by second-order algebraic surfaces, it is assumed that a = 5 m, h = 4 m, and the external surface load components $X,\ Y,\ Z$ are

$$X = X(u) = qs\text{in}\varphi = q\frac{\text{tg}\varphi}{\left(1+\text{tg}^2\varphi\right)^{\frac{1}{2}}},\qquad Y = 0,$$

$$Z = Z(u) = -q\cos\varphi = -q\frac{1}{\left(1+\text{tg}^2\varphi\right)^{\frac{1}{2}}},$$

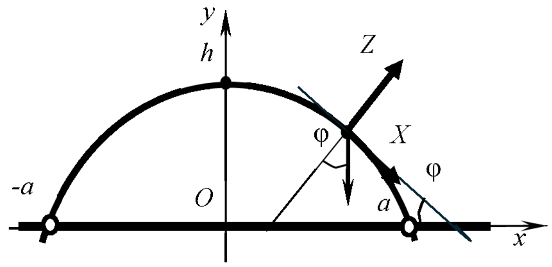

Figure 7. Geometrical parameters, external loading, and hinge supports of the considered shells
S o u r c e: made by S.N. Krivoshapko

where q is the distributed load of self-weight kind, φ is the angle between the tangent to coordinate line u and the fixed Ox axis (Figure 7).

It is assumed that the hinges are connected to an absolutely rigid in-plane and flexible out-of-plane support at the ends $v = 0$, $v = 1$. For open cylindrical shells it is impossible to satisfy the boundary conditions on the straight edges of the contour.

Then, formulas (4) give

$$N_v = \frac{l^2v^2}{2A}\frac{d}{du}\left(\frac{1}{A}\frac{dN_u}{du} + X\right) - \frac{l}{A}v\frac{df_1}{du} + C + f_2(u),$$

from where $C = 0$, $f_2(u) = 0$, since $N_v = 0$ at $v = 0$. Then, from the boundary condition of $N_v = 0$ at $v = 1$, the following is obtained:

$$\frac{df_1(u)}{du} = \frac{l}{2}\frac{d}{du}\left(\frac{1}{A}\frac{dN_u}{du} + X\right),$$

from where

$$f_1(u) = \frac{l}{2}\left(\frac{1}{A}\frac{dN_u}{du} + X\right).$$

By substituting the obtained expressions for $f_1(u)$, df_1/du and $f_2(u)$ into formulas (4), the final formulas for calculating the internal membrane forces under the considered boundary conditions are written:

$$N_u = ZR_u,$$

$$S = \frac{l}{2}\left(\frac{1}{A}\frac{dN_u}{du} + X\right)(1-2v),$$

$$N_v = -\frac{l^2v}{2A}\frac{d}{du}\left(\frac{1}{A}\frac{dN_u}{du} + X\right)(1-v). \tag{5}$$

4.1. Parabolic Cylindrical Thin Shell

The following was obtained previously for the cylindrical parabolic surface:

$A^2 = a^2 + 4h^2u^2$, $R_1 = R_u = A^3/(2ah)$.

In the previous section, the formuals for calculating sinφ, cosφ were presented, thus

$$X = \frac{2hu}{A}q, \quad Y = 0, \quad Z = -\frac{qa}{A}.$$

According to formulas (4):

$$N_u = -\frac{q\left(a^2 + 4h^2u^2\right)}{2h},$$

$$S = \frac{2lqhvu}{\sqrt{a^2 + 4h^2u^2}} + f_1(u),$$

$$N_v = \frac{-l^2qha^2}{\left(a^2 + 4h^2u^2\right)^2}v^2 - \frac{vl}{\sqrt{a^2 + 4h^2u^2}}\frac{df_1}{du} + f_2(u).$$

The boundary condition of $N_v = 0$ at $v = 0$ gives $f_2(u) = 0$. The boundary condition of $N_v = 0$ on the opposite end of the shell $v = 1$ gives

$$\frac{df_1(u)}{du} = -lqha^2\frac{1}{\left(a^2 + 4h^2u^2\right)^{\frac{3}{2}}},$$

from where

$$f_1(u) = -lqh\frac{u}{\sqrt{a^2 + 4h^2u^2}} + C,$$

where $C = 0$ when $S(u = 0) = 0$. Finally, for the parabolic cylindrical shell:

$$N_u = -\frac{q\left(a^2 + 4h^2u^2\right)}{2h},$$

$$S = \frac{lqhu(2v - 1)}{\sqrt{a^2 + 4h^2u^2}},$$

$$N_v = \frac{l^2qha^2v(1 - v)}{\left(a^2 + 4h^2u^2\right)^2}.$$

4.2. Elliptic Cylindrical Thin Shell

4.2.1. Cylindrical Thin Shell with Semi-Ellipse in Cross-Section

All geometric parameters of the elliptic cylindrical middle surface with a semi-ellipse in the cross-section (see Figure 2) are given in Section 2.2.1. In addition to this

$$X = q\frac{uh}{A\sqrt{1 - u^2}}, \; Y = 0, \; Z = -q\frac{a}{A}.$$

According to formulas (5):

$$N_u = -\frac{q}{h} A^2 \left(1-u^2\right)^{\frac{3}{2}},$$

$$N_v = \frac{ql^2 v(v-1)\sqrt{1-u^2}}{2h\left[a^2 - u^2\left(a^2-h^2\right)\right]^2}\left[3a^4 - 9a^2u^2\left(a^2-h^2\right) - a^2h^2 + 6u^4\left(a^2-h^2\right)^2\right].$$

4.2.2. Cylindrical Thin Shell with Incomplete Ellipse in Cross-Section

All geometrical parameters of the elliptic cylindrical middle surface with an ellipse fragment in the crosssection (Figure 4) are given in Section 2.2.2. In addition to this

$$X = q\frac{\beta u h_1}{\sqrt{a^2 + \beta u^2\left(\beta h_1^2 - a^2\right)}}, \quad Y = 0, \quad Z = -q\frac{a\sqrt{1-\beta u^2}}{\sqrt{a^2 + \beta u^2\left(\beta h_1^2 - a^2\right)}},$$

$$A = \frac{\sqrt{a^2 + \beta u^2\left(\beta h_1^2 - a^2\right)}}{\sqrt{1-\beta u^2}}, \quad \beta = \frac{a^2}{a_1^2}, \quad R_u = \frac{A^3}{h_1 a\beta}\left(1-\beta u^2\right)^{3/2}.$$

According to formulas (5):

$$N_u = -\frac{qA^2\left(1-\beta u^2\right)^{\frac{3}{2}}}{\beta h_1} = -\frac{q}{\beta h_1}\sqrt{1-\beta u^2}\left[a^2 + \beta u^2\left(\beta h_1^2 - a^2\right)\right],$$

$$S = \frac{lqu(1-2v)}{2h_1 A\sqrt{1-\beta u^2}}\left[3A^2\left(1-\beta u^2\right) - \beta h_1^2\right] =$$

$$= \frac{lqu(1-2v)}{2h_1\sqrt{a^2 + \beta u^2\left(\beta h_1^2 - a^2\right)}}\left\{3\left[a^2 + \beta u^2\left(\beta h_1^2 - a^2\right) - \beta h_1^2\right]\right\},$$

$$N_v = \frac{ql^2 v(v-1)}{2h_1 A}\left\{\frac{3\left[a^2 + 2\beta u^2\left(\beta h_1^2 - a^2\right)\right]}{\sqrt{a^2 + \beta u^2\left(\beta h_1^2 - a^2\right)}} - \frac{\beta a^2 h_1^2}{\left[a^2 + \beta u^2\left(\beta h_1^2 - a^2\right)\right]^{3/2}}\right\}.$$

The obtained formulas allow to the calculate the elliptic cylindrical shell with a semi-ellipse in the cross-section (see Figure 2). For this purpose, in the formulas of Section 4.2.2, it is necessary to take

$\beta = 1$, $a_1 = a$, $h_1 = h$.

4.2.3. Circular Cylindrical Thin Shell

In this case, it is necessary to take $R = 5.125$ m,

$$\frac{dy}{dx} = \operatorname{tg}\varphi = \frac{x}{\sqrt{R^2 - x^2}}, \quad \sin\varphi = \frac{x}{R} = \frac{au}{R}, \quad \cos\varphi = \frac{\sqrt{R^2 - x^2}}{R} = \frac{\sqrt{R^2 - a^2u^2}}{R}.$$

According to formulas (4):

$$N_u = -q\sqrt{R^2 - a^2u^2},$$

$$S = -2\frac{laq}{R}uv + f_1(u),$$

$$N_v = \frac{l\sqrt{R^2 - a^2u^2}}{R}\left(\frac{lqv^2}{R} - \frac{v}{a}\frac{df_1}{du}\right) + f_2(u).$$

The boundary condition of $N_v = 0$ at $v = 0$ gives $f_2(u) = 0$. The boundary condition of $N_v = 0$ on the other end of the shell $v = 1$ gives $df_1/du = alq/R$, and $f_1(u) = alqu/(R) + C$.

Let $S = 0$ at $u = 0$, then $C = 0$. Substituting the obtained results into the equations for determining the internal forces according to the momentless theory allows to write the following:

$$N_u = -q\sqrt{R^2 - a^2u^2} = -qR\cos\varphi,$$

$$S = \frac{laq}{R}u(1 - 2v),$$

$$N_v = l^2q\frac{\sqrt{R^2 - a^2u^2}}{R^2}v(v - 1).$$

For the calculation of long circular cylindrical shells, it is advisable not to use the momentless theory. When using the momentless theory, it is necessary to fulfill the following relation [18; 12]:

$$\frac{l}{R} \ll \sqrt{\frac{R}{h}}.$$

4.3. Hyperbolic Cylindrical Thin Shell

All geometrical parameters of the hyperbolic cylindrical middle surface (see Figure 5) are given in Section 2.3. It is necessary to take into account that

$$X = q\frac{hu\gamma}{\sqrt{a^2c^2 + \gamma hu^2\left(a^2 + \gamma h\right)}},\ Y = 0,\ Z = -q\frac{a\sqrt{c^2 + \gamma hu^2}}{\sqrt{a^2c^2 + \gamma hu^2\left(a^2 + \gamma h\right)}},$$

$$A = \frac{\sqrt{a^2c^2 + \gamma hu^2\left(a^2 + \gamma h\right)}}{\sqrt{c^2 + \gamma hu^2}},\ \gamma = 2c + h,\ R_u = \frac{A^3\left(c^2 + \gamma hu^2\right)^{3/2}}{ah\gamma c^2},$$

where $c > 0$ is an arbitrary number; $0 < \cos\varphi \le 1$, i.e. $0^\circ \le \varphi < 90^\circ$.

According to formulas (5):

$$N_u = -\frac{qaA}{L} = -\frac{q\sqrt{c^2 + \gamma hu^2}}{h\gamma c^2}\left[a^2c^2 + \gamma hu^2\left(a^2 + \gamma h\right)\right],$$

$$S = -\frac{luq}{2c^2}(1 - 2v)\frac{3\left[a^2c^2 + \gamma hu^2\left(a^2 + \gamma h\right)\right] + c^2\gamma h}{\sqrt{a^2c^2 + \gamma hu^2\left(a^2 + \gamma h\right)}},$$

$$N_v = q\frac{l^2 v(1-v)\sqrt{c^2+\gamma h u^2}}{2c^2\left[a^2c^2+\gamma h u^2\left(a^2+\gamma h\right)\right]^2}\left[a^2c^4\left(3a^2+\gamma h\right)+9\gamma h a^2 c^2 u^2\left(a^2+\gamma h\right)+\right.$$

$$\left.+6\gamma^2 h^2 u^4 (a^2+\gamma h)^2\right].$$

5. Comparison of Results of Momentless Analysis of Five Cylindrical Thin Shells in the Cross-Sections of the Shells (in the Longitudinally Middle Section of the Shells, i.e. at $v = 0.5$)

As noted earlier, all shells are assumed to have the same overall dimensions, i.e. $h = 4\ m$, $a = 5\ m$, $l = 20\ m$. In this case, the formulas for calculating the internal membrane forces per unit length of the coordinate line take the following form:

- *parabolic cylindrical thin shell*:

$$N_u = -8q\left(0.39+u^2\right),\quad S = \frac{10uq\left(2v-1\right)}{\sqrt{0.39+u^2}},\quad N_v = \frac{9.77v\left(1-v\right)}{\left(0.39+u^2\right)^2}q;$$

- *cylindrical thin shell with semi-ellipse in cross-section*:

$$N_u = -2.25q\left(2.78-u^2\right)\sqrt{1-u^2},$$

$$S = \frac{49.2-22.5u^2}{\sqrt{2.78-u^2}}qu\left(1-2v\right),$$

$$N_v = \frac{qv\left(v-1\right)\sqrt{1-u^2}}{\left(2.78-u^2\right)^2}\left(300u^4-1250u^2+910\right);$$

- *cylindrical thin shell with incomplete ellipse in cross-section* ($a_1 = 5.98\ m$, $h_1 = 8.84\ m$, $\beta = 0.7$):

$$N_u = -q\left(4.04+3.36u^2\right)\sqrt{1-0.7u^2},$$

$$S = \frac{14.1qu\left(1-2v\right)}{\sqrt{1+0.83u^2}}\left(0.37+u^2\right),$$

$$N_v = q\frac{18.35v\left(v-1\right)\sqrt{1-0.7u^2}}{\left(1+0.83u^2\right)^2}\left(1+9.22u^2+5.1u^4\right);$$

- *circular cylindrical thin shell* ($R = 5.125\ m$):

$$N_u = -q\sqrt{R^2-25u^2} = -qR\cos\varphi,$$

$$S = \frac{1000qu\left(1-2v\right)}{R},$$

$$N_v = 400qv\left(v-1\right)\sqrt{R^2-25u^2}\ /\ R^2;$$

- *hyperbolic cylindrical thin shell* ($c = 2$ *m*, $\gamma = 8$ *m*, $b = 0.88$ *m*; $0.68 \le \cos\varphi \le 1$, i.e. $0^o \le \varphi \le 47^o$):

$$N_u = -1.56q\sqrt{1+8u^2}\left[1+18.24u^2\right],$$

$$S = -qu(1-2v)\frac{107\left(1+0.17u^2\right)}{\sqrt{1+18.24u^2}},$$

$$N_v = \frac{107q\sqrt{1+8u^2}}{\left(1+18.24u^2\right)^2}v(1-v)\left(1+38.36u^2+466u^4\right).$$

The values of shear and normal forces per unit length of coordinate lines *u* and *v* in the z = *l*/2 section of cylindrical shells are presented in Tabl.

Momentless analysis results

No.	**Shell type**	**Force/*q***	***u* = 0**	***u* = ± 0.4**	***u* = ±1 (support along the length of the shell)**
	***v* = 0.5 (*z* = *l*/2 = 10 m section)**				
1.	Parabolic cylindrical thin shell	N_u/q S/q N_v/q	–3.12 0 16.05	–4.4 0 8.07	–11.12 0 1.27
2.	Cylindrical thin shell with semi-ellipse in cross-section	N_u/q S/q N_v/q	–6.25 0 –29.4	–5.42 0 –24.05	0 0 0
3.	Cylindrical thin shell with incomplete ellipse in cross-section	N_u/q S/q N_v/q	–4.04 0 –4.59	–4.3 0 –8.8	–4.05 0 –11.6
4.	Circular cylindrical thin shell	N_u/q S/q N_v/q	–5.125 0 –19,51	–4.72 0 –18	–1.13 0 –4.28
5.	Hyperbolic cylindrical thin shell	N_u/q S/q N_v/q	–1.56 0 26.75	–9.23 0 50.14	–90 0 109

S o u r c e: made by S.N. Krivoshapko

Using the momentless theory, normal forces N_u in a hyperbolic cylindrical shell are determined by the first formula (5):

$$N_u = ZR_u = -q\cos\varphi\, R_u,$$

where the radius of curvature $R_u = 1.56$ m at $u = 0$, but $R_u = 131.8$ m on the contour $u = \pm1$, therefore this normal force takes on a larger value $N_u = -90q$. The momenless theory does not allow to specify a boundary condition on these edges. Taking into account that at the apex $u = 0$, $\varphi = 0^o$, $\cos\varphi = 1$, this results in $N_u/q = R_u$, i.e. the values in the $u = 0$ column of Table in the first row for N_u show the radius of curvature of a coordinate line R_u in its apex.

All the above formulas in Sections 4 and 5 should be used with caution. Biderman V.L. [14] found that for open circular cylindrical shells, the momentless theory can be used in exceptional cases if the membrane boundary conditions on longitudinal edges are automatically satisfied. The scope of application of the momentless theory is limited to short shells with respect to shell radius *R* (see Section 4.2.3). Apparently, for non-circular cylindrical shells, the applicability of the momentless theory, by analogy with circular shells, can be limited by the relation:

$$\frac{l}{a} \ll \sqrt{\frac{a}{h}}.$$

Although in some cases the formulas for the momentless theory provide acceptable results, and the possibilty of their application needs to be confirmed by more accurate calculation methods. In many cases, it is possible to incorporate the solutions of the momentless theory equations into the approximate solution of the general equations by splitting the latter into the momentless stress state equations and the edge effect.

In monograph [6], a similar analysis is performed on four cylindrical roof shells under dead load according to the momentless theory and it is concluded that the most optimal shell is the cylindrical shell with an incomplete ellipse in the cross-section, since this shell transfers a relatively small normal load to the supporting beams.

6. The System of Design Equations of the General Theory of Thin Cylindrical Shells in Terms of Displacements

Five equations of equilibrium (1), six constitutive equations (2) and six geometric equations (3) were derived for the calculation of thin elastic cylindrical shells using the general linear theory.

Substituting the geometric equations (3) into the constitutive equations (2), the following is obtained:

$$N_u = C\left(\frac{1}{A}\frac{\partial u_u}{\partial u} - \frac{u_z}{R_u} + \frac{\nu}{l}\frac{\partial u_v}{\partial v}\right);$$

$$N_v = C\left(\frac{1}{l}\frac{\partial u_v}{\partial v} - \nu\frac{u_z}{R_u} + \frac{\nu}{A}\frac{\partial u_u}{\partial u}\right);$$

$$S = \frac{1-\nu}{2}C\left(\frac{1}{A}\frac{\partial u_v}{\partial u} + \frac{1}{l}\frac{\partial u_u}{\partial v}\right);$$

$$M_u = -D\left[\frac{1}{A}\frac{\partial}{\partial u}\left(\frac{1}{A}\frac{\partial u_z}{\partial u} + \frac{u_u}{R_u}\right) + \frac{\nu}{l^2}\frac{\partial^2 u_z}{\partial v^2}\right];$$

$$M_v = -D\left[\frac{1}{l^2}\frac{\partial^2 u_z}{\partial v^2} + \frac{\nu}{A}\frac{\partial}{\partial u}\left(\frac{1}{A}\frac{\partial u_z}{\partial u} + \frac{u_u}{R_u}\right)\right];$$

$$H = -D(1-\nu)\frac{1}{l}\frac{\partial}{\partial v}\left(\frac{1}{A}\frac{\partial u_z}{\partial u} + \frac{u_u}{2R_u}\right). \quad (6)$$

The values of the shear forces are determined from the last two equilibrium equations (1):

$$Q_u = \frac{1}{l}\left(\frac{\partial H}{\partial v} - \frac{l}{A}\frac{\partial M_u}{\partial u}\right), \quad Q_v = \frac{1}{A}\left(\frac{\partial H}{\partial u} - \frac{A}{l}\frac{\partial M_v}{\partial v}\right). \quad (7)$$

From the last three equations (6) and equations (7), it can be seen that internal moments M_u, M_v, H and shear forces Q_u and Q_v do not depend on displacement u_v.

Substituting the expressions for internal forces and moments into the first three equilibrium equations (1) results in a system of three partial differential equations in terms of displacements of the middle surface of the eighth order, which are usually written in a reduced form:

$$L_{11}^2 u_u + L_{12}^2 u_v + L_{13}^3 U_z + AlX = 0\ ,$$

$$L_{21}^2 u_u + L_{22}^2 u_v + L_{23}^1 U_z + AlY = 0\ ,$$

$$L_{31}^3 u_u + L_{32}^1 u_v + L_{33}^4 U_z - AlZ\ ,$$

where L_{ij}^k ... are differential operators, the upper index of which shows the maximum order of the corresponding partial derivative. In a more extended form these three equations can be expressed as

$$\frac{\partial}{\partial u}\left(\frac{l}{A}\frac{\partial u_u}{\partial u}+\nu\frac{\partial u_v}{\partial v}-l\frac{u_z}{R_u}\right)+\frac{(1-\nu)}{2}\frac{\partial}{\partial v}\left(\frac{\partial u_v}{\partial u}+\frac{A}{l}\frac{\partial u_u}{\partial v}\right)+$$

$$+\frac{h^2}{12R_u}\left\{\frac{(1-2\nu)}{l}\frac{\partial^3 u_z}{\partial u\partial v^2}+\frac{(1-\nu)\mathrm{A}}{2l}\frac{u_u}{R_u}-l\frac{\partial}{\partial u}\left[\frac{1}{A}\frac{\partial}{\partial u}\left(\frac{1}{A}\frac{\partial u_z}{\partial u}+\frac{u_u}{R_u}\right)\right]\right\}+\frac{Al}{C}X=0,$$

$$\frac{12A}{h^2R_u}\left(\frac{l}{A}\frac{\partial u_u}{\partial u}+\nu\frac{\partial u_v}{\partial v}-l\frac{u_z}{R_u}\right)+$$

$$+\frac{\partial}{\partial u}\left\{\frac{l}{A}\frac{\partial}{\partial u}\left[\frac{1}{A}\frac{\partial}{\partial u}\left(\frac{1}{A}\frac{\partial u_z}{\partial u}+\frac{u_u}{R_u}\right)\right]+\frac{\nu}{lA}\frac{\partial^3 u_z}{\partial u\partial v^2}-\frac{(1-\nu)}{l}\frac{\partial^2}{\partial v^2}\left(\frac{2}{A}\frac{\partial u_z}{\partial u}+\frac{u_u}{R_u}\right)\right\}+$$

$$+\frac{1}{l}\frac{\partial^2}{\partial v^2}\left[\frac{A}{l^2}\frac{\partial^2 u_z}{\partial v^2}+\nu\frac{\partial}{\partial u}\left(\frac{1}{A}\frac{\partial u_z}{\partial u}+\frac{u_u}{R_u}\right)\right]-\frac{Al}{D}Z=0,$$

$$\frac{(1-\nu)l}{2}\frac{\partial}{\partial u}\left(\frac{1}{A}\frac{\partial u_v}{\partial u}\right)+\frac{\partial}{\partial v}\left(\frac{A}{l}\frac{\partial u_v}{\partial v}+\frac{(1+\nu)}{2}\frac{\partial u_u}{\partial u}-\nu\mathrm{A}\frac{u_z}{R_u}\right)+\frac{Al}{C}Y=0. \tag{8}$$

Thus, the equilibrium equations (1) of cylindrical shells are expressed in terms of displacements u_u, u_v, u_z of the middle surface. The system of three partial differential equations (8) with variable coefficients is obtained. The system is of the eighth order. The use of geometric equations (3) guarantees the satisfaction of the strain compatibility conditions in the mid-layer of the shell.

The system of three differential equations in displacements for developable shells, i.e., for shells of zero Gaussian curvature defined in a non-orthogonal conjugate coordinate system, is given in monograph [15], but for the case of zero Poisson's ratio ($\nu = 0$). Taking $\nu = 0$ and the average coefficient of the first quadratic form $F = 0$, equations (8) can be reduced to the form presented in monograph [15].

In general form, equations (8) will probably never be applied to the calculation of cylindrical shells defined in lines of curvature. These equations are derived here only to illustrate the displacement method for cylindrical shells in the form of analytic surfaces. Their application to circular cylindrical shells, the geometry of which is described in Section 2.2.3, will be shown only in Section 7.

7. Existing Approaches to the Calculation of Thin Elastic Cylindrical Shells Using Analytical Methods During the “Golden Age of Shells”

It is commonly believed that the “Golden age of shells” fell on 1924–1970s. It was at this time that reliable analytical methods for the calculation of thin elastic shells emerged, which are related to the requests of the construction practice for shell design and making it possible to describe the behavior of shells under external static loads more or less accurately.

In addition to specific theories of cylindrical shells (V.Z. Vlasov, L. Donnel, A.A. Umansky, H.M. Mushtari, S.M. Feinberg), V.V. Novozhilov [16], at that time, proposed a complex resulting equation that includes all specific theories of cylindrical shells, including those strenghthened by stiffeners.

7.1. Circular Cylindrical Thin Shell

The largest number of analytical methods for the calculation of thin cylindrical shells based on the general linear theory of elastic shells has been proposed for circular cylindrical shells. The equations and formulas of the theory of circular cylindrical shells are mainly written in terms of z and s or $\xi = z/R$ and $\theta = s/R$, where s is the distance along the middle surface from the initial position to the corresponding point. As a result, $A = B = 1$ in equations (1) – (3), and $R_u = R = \text{const}$.

V.Z. Vlasov [17] introduced function Φ(ξ,θ), through which displacements u_u, u_v, u_z are expressed. This allowed to turn the first two equations of equilibrium (1) into identities, and the third equation of equilibrium took the form of a partial differential equation with respect to the introduced function Φ(ξ,θ). This function was expressed as a double trigonometric sine series, which corresponds to a shell with hinged supports along the contour. The extension of the solution to shells with other boundary conditions is possible, but requires cumbersome calculations [18].

In the calculation of open circular shells of length *l*, the load and displacements were decomposed into Fourier series with period *l*:

$$u_v = \sum_{n=0}^{\infty} u_{vn}(s)\cos\frac{n\pi z}{l};\;\; u_u = \sum_{n=1}^{\infty} u_{un}(s)\sin\frac{n\pi z}{l};$$

$$u_z = \sum_{n=1}^{\infty} u_{zn}(s)\,sin\frac{n\pi z}{l};$$

$$Y = \sum_{m=0}^{\infty} Y_m(\mathrm{s})\cos\frac{m\pi z}{l};\quad X_m = \sum_{m=1}^{\infty} X_m(s)\sin\frac{n\pi z}{l};$$

$$Z = \sum_{m=1}^{\infty} Z_m(s)\sin\frac{n\pi z}{l},$$

where the known decomposition coefficients X_m, Y_m, Z_m can be determined using the formulas given by V.Z. Vlasov [17]. Substituting these values into equations (8) for open circular cylindrical shells, three simple differential equations with respect to u_{un}, u_{vn}, u_{zn} are obtained. In $z = 0$ и $z = l$ planes, only the free support conditions will be satisfied, i.e., $u_u = uz = 0$ and $N_v = M_v = 0$.

For a closed circular cylindrical shell loaded with axisymmetric load, the solution can be represented as a single cosine series, which leads to an eighth order ordinary differential equation with respect to variable $\xi = z/R$. Its solution is sought in the form of the sum of hyper-geometric functions of ξ.

Lurye A.I. [19] proposed the calculation of the optimal strenghthening of a circular hole in a circular cylindrical shell.

In [20], solutions in the framework of technical and semi-integral shell theories using the method of separation of variables are demonstrated. The solution methods and simplifying assumptions for closed and open circular shells are different. In this work, it is noted that in many cases calculations for building shells give excessive accuracy from the practical standpoint, therefore it was possible to introduce a number of simplified versions of the shell theory, which allowed to solve a certain class of problems at that time. For example, V.Z. Vlasov [17] and his students introduced additional static and geometric hypotheses, and A.L. Goldenweiser [21] and V.V. Novozhilov [22] and their followers revealed conditions under which some terms of the design equations turn out to be insignificant.

In the semi-momenltess theory of cylindrical shells, it is assumed that $M_v = H = Q_v = 0$ and $\varepsilon_u = \varepsilon_{uv} = 0$, due to which the design equations of shells are simplified. The semi-momentless theory is discussed in more detail in [20; 23]. In [20] it is argued that one can introduce additional assumptions that do not have a noticeable effect on the behavior of closed cylindrical shells, in particular, one can assume that the Poisson's ratio is zero ($\nu = 0$) or one can discard relatively small values compared to a unit. W. Flügge [23] gave recommendations for the calculation of multi-wave reinforced concrete circular cylindrical shells with flexible or high sides.

Different assumptions for the calculation of shallow and non-shallow short open shells and medium-length and long shells were proposed.

During this period, many researchers studied buckling of circular cylindrical shells with closed contour under compressive and shear loads applied at the ends [23] and under pure shear [23].

The theory of multilayer shells also appeared at this time. The main directions in the development of the theory of multilayer shells were presented in 1972 by E.I. Grigolyuk and F.A. Kogan, as described in article [24]. In this article, the first works on the calculation of cylindrical shells are mentioned.

7.2. Elliptic Cylindrical Thin Shell

In [20], it is stated that cylindrical thin shells of non-circular cross-section were generally supported by flexible diaphragms at the transverse ends and the longitudinal edges were strenghthened with edge elements. The performance of such a system depends largely on the $l/(2a)$ ratio. In long shells ($l/(2a) > 4$), the external load is taken up almost exclusively by the N_u, N_v, S forces. In medium-length shells ($4 > l/(2a) > 1$), in addition to these membrane forces, bending moment M_u is significant, while bending moment M_v and torque H play a minor role and may not even be taken into account. In short shells ($l/(2a) < 1$) the effect of bending moment M_u is relatively small, but the role of bending moment M_v increases. The mentioned features of shell behavior allow simplifying their calculation by replacing the actual structure with a simplified model. For example, short shells can be calculated as beams of length l. In contrast to circular cylindrical shells, the exact calculation of non-circular shells was extremely difficult in the mentioned period of time [20].

Most of the approximate methods were based on the idea of replacing the shell by prismatic folds. Basically, the methods differ from each other by the hypotheses about the degree of influence of certain internal forces, moments, or strains on the behavior of the considered shell. Studies on the use of computers for the calculation of the approximating folds began to appear [25].

7.3. Parabolic Cylindrical Thin Shell

The contents of the first two paragraphs of the previous Section 7.2, which deals with elliptic cylindrical shells, are fully consistent with parabolic cylindrical shells as well.

In the considered time interval, studies of multilayer shallow parabolic cylindrical shells, taking into account plastic deformations, have begun, e.g. [26].

7.4. Hyperbolic Cylindrical Thin Shell

Hyperbolic cylindrical surface has been known for a long time, but only geometers have used it in their research. Not a single structure has been built. Not a single scientific article devoted to the strength analysis of a shell in the form of this surface has been found.

7.5. Conclusion of Section 7

E. Freyssinet, a French bridge engineer, designed and built the first parabolic cylindrical shell, which is technically a system of parabolic arches, to cover a 30-meter span of a factory in Monluson (France) in 1905. In Russia, long and short cast-in-situ cylindrical shells have been used in industrial buildings since 1928. For example, the first reinforced concrete cylindrical shell was erected under the direction of Prof. M.A. Novogorodsky over a water tank in Baku in 1925. After, cylindrical shells were used for the building of the Kharkov post office (1928), Moscow automobile depot (1929) and Rostov agricultural machinery plant (1931) [27]. A large number of circular and elliptic cylindrical shells made of reinforced concrete and brick constructed for the roofs of industrial buildings in 1930–1970 in Italy are presented in review article [28]. The advent of reinforced concrete and public demand for large-span structures for warehouses, hangars and public buildings led to the need of static calculation methods for these rigid shells. It was during the "Golden age of shells" that the fundamental discipline — shell theory — emerged, initial hypotheses and assumptions were accepted, resolving equations adequately describing the behavior of shells were formed, simplifications of the equations of the shell theory were proposed, and qualitative studies of a number of specific problems appeared.

8. Information on Publications After 2000 about Calculation of Cylindrical Shells with Second-Order Algebraic Middle Surfaces and Their Application in Architecture and Construction

8.1. Application of Cylindrical Shells with Second-Order Algebraic Middle Surfaces

Many review [3] and scientific [11] papers, monographs [29] and reference books [30] are devoted to the application of cylindrical shells, structures and cylindrical surfaces in architecture, construction and mechanical engineering. In all these works, there are sections on structures and products in the form of second-order cylindrical algebraic surfaces.

8.2. Information on Publications After 2000 About Calculation of Cylindrical Shells With Second-Order Algebraic Middle Surfaces

As already mentioned in the Introduction, in the last quarter of a century, almost no studies have been conducted to determine the optimal cylindrical shell among the five cylindrical shells with the same overall dimensions and boundary conditions and subjected to the same external load. There is only paper [32], where the stability of the computational algorithm is studied when one type of cylindrical surface is replaced with another, which can simplify the computation. It is argued that the replacement of hyperbolic and elliptic cylindrical shells with a parabolic shell simplifies the calculations.

All other scientific articles are devoted to studies on the calculation of specific cylindrical shells of the same shape for strength, stability, and dynamics, i.e., each article is devoted to only one type of closed or open, thin or thick cylindrical shells. The most complete review of works on the calculation of three degenerate cylindrical shells in the form of second-order algebraic surfaces is given in article [11], which contains 14 references on the subject.

Almost all new studies are devoted to circular cylindrical shells. It is noted in [33] that, despite the extensive literature on the development of shell theory, the comparative analysis of analytical and numerical solutions is not sufficiently covered. One of the first works devoted to this direction in the study of natural frequencies and vibration modes of closed circular cylindrical shells is [33].

Very interesting results have been obtained in the experimental study of the behavior of masonry cylindrical roofs under the settlement of one of the longitudinal rectilinear supports, due to which longitudinal cracks appear [34].

Many papers claim that the momentless theory provides acceptable results for thin shells with a ratio of thickness to internal diameter equal to 0.1 and having the shape of shells of rotation [13]. Examples of using the momentless theory for cylindrical shells subjected to axisymmetric loading are given mainly in textbooks for students of various fields [13; 35; 36].

In monograph [6], the maximum stresses and displacements in an elliptical tube caused by uniform normal pressure are determined. It is found that the maximum normal stresses in the direction of the ellipse will occur at the ends of the minor semi-axis, the shear stresses will reach the highest value at the shell supports. The normal axial stresses reach their maximum in the middle of the span. The authors of monograph [6] point out that the results obtained by the momentless shell theory are not applicable to long pipes. Similar conclusions were made in Section 5 and in the “Results” section of the presented paper.

With a major excess of studies on open or closed circular cylindrical shells, there are works whose authors believe that non-circular shells are an interesting object for research, for example, elliptic, in which the values of semi-axes R_x and R_y act as variable parameters. At their specific values, it is possible to achieve a decrease or increase in the minimum natural frequency relative to the cylindrical configuration without changing the mode of vibration. Paper [37] shows the variation of natural frequencies of vibration depending on the type of boundary conditions set at the edges of an elliptic shell and the frequencies of vibration of a cantilever steel non-circular shell at different lengths of one semi-axis.

The aeroelastic stability of closed cylindrical shells with an elliptical cross-section streamlined from the outside by a supersonic gas flow is investigated in article [38] in a three-dimensional setting using the mathematical model and its numerical realization based on the finite element method. Using the developed program, the influence of kinematic boundary conditions, dimensions, and the ellipticity parameter on the critical characteristics of buckling in the form of flutter was analyzed. It is demonstrated that in the cantilever case, there exist such ellipse semi-axis ratios that provide higher aeroelastic stability boundaries as compared to a similar circular configuration.

Finite element modeling and buckling analysis of a composite closed elliptic cylindrical lattice shell under axial compression and lateral bending were studied for spacecraft in [31]. Here, the natural vibration modes of the considered product are obtained and it is found that the structure and geometry of the object affect the magnitude of the critical load and the natural vibration modes.

There are studies on the determination of strength of parabolic cylindrical shells. For example, FEM was used to numerically investigate the behavior of a straight parabolic cylindrical shell with supports at four corners, subjected to dynamic loading, when increasing the shell height or its thickness [39]. The tabular and graphical results of the study were obtained using the standard SAP 2000 software.

A numerical study, performed together with the experimental one, proves the effect of cracks in parabolic cylindrical reinforced concrete shells on the static and dynamic behavior of the studied shells [40]. The obtained results will help to determine the choice of shell repair times.

In [41], it is argued that parabolic cylindrical panels made of memory polymers are very useful in spacecraft that are subjected to dynamic (vibration) loads.

One of the first attempts to select an optimal shell among shells with different cross-sections was made in [42], where single-layer shells with a triangular lattice with circular, catenary and parabolic cross-sections under a distributed load such as self-weight were considered. It is established that the shells of greatest mass will have a cross-section in the form of a catenary curve. The same article presents graphs of the relationship between the mass of the three considered lattice shells and the height. The parabolic shell will be the most economical in terms of material. If the maximum deflection of the structure is taken as the main criterion, the shell in the shape of a catenary curve has an advantage over the circular and parabolic shapes.

An interesting approach to the calculation of long open cylindrical building structures of non-circular cross-section is proposed in [43], where the stress-strain state of a structure obtained using a beam model with supports at two ends and using the equations of linear shell theory is studied.

8.3. Conclusion of Section 8

The presented brief review of the studies of cylindrical roofs and closed circular and elliptic shells carried out over the last 25 years shows a decline in interest in their investigation. The only exceptions are closed circular cylindrical shells, which have found applications in underground structures [44] and in mechanical engineering [45]. However, there are still unsolved problems. For example, as mentioned above, there is still no answer to the question of the optimal shell with respect to the five cylindrical shells shown in Figures 1–4 and Figure 5, which are subjected to some static or dynamic load. There are practically no studies on the determination of radial displacements of a hollow cylinder under the combined action of internal pressure, centrifugal forces, and temperature effects, which is very important in the manufacture of products in the field of mechanical engineering [46].

To determine the strength and dynamic characteristics of cylindrical objects, numerical methods and standard computer systems are almost always used. Analytical methods are developed only for individual cases.

The form of cylindrical shells, shell structures and cylindrical external outlines of buildings, defined by second-order algebraic surfaces, is used very widely by architects and civil and mechanical engineers in the XXI century.

A brief review of studies on the strength, stability, and dynamics of cylindrical shells conducted over the last 25 years confirms the conclusions of V.V. Novozhilov and other scientists in the field of mechanics

[6] that the "Shell Theory" fundamental discipline becomes one of the sections of applied mathematics. In this case, the theory is used only to write out the initial system of equations.

An investigation of recent publications shows that the calculation of cylindrical shells is currently associated with the consideration of multilayered walls, geometrical and physical nonlinearities, new structural materials [41] and new types of external static, dynamic [47] and thermal [46] loads, sometimes after the adoption of refining hypotheses [48].

After circular cylindrical shells, the most popular are parabolic cylindrical shells, especially as roofs for warehouses, industrial and exhibition halls, and elliptic cylindrical shells for mechanical engineering projects. Open cylindrical shells play a much more important role in construction.

Section 8 summarizes only the main directions of research on cylindrical shells in the form of second-order algebraic surfaces and which have been presented in publications over the last 25 years. Additional information on the topic can be found in review and scientific articles [11; 49; 50; 51].

9. Results

A comparative analysis of five thin shells under a static load of self-weight type according to the momentless shell theory has been performed.

1. Three formulas (4) in explicit form are obtained for the determination of internal membrane forces in the five shells defined by second-order algebraic surfaces with boundary conditions in terms of forces, acceptable for the momentless theory of thin shells.

2. Three formulas (5) in explicit form are obtained for the determination of internal membrane forces in the five shells defined by second-order algebraic surfaces with two specified boundary conditions in terms of forces at two opposite ends for normal forces in the form $N_v = 0$ involving the momentless theory of shells.

3. The main geometric characteristics of the five second-order middle algebraic surfaces are presented.

4. It is shown that in the framework of the approximate momentless theory of shells, the parabolic cylindrical thin shell (Figure 1) and the cylindrical thin shell with an incomplete ellipse in the cross-section (Figure 4) perform best under external load similar to self-weight, and the greatest internal forces arise in the cylindrical thin shell with a semi-ellipse in the cross-section (Figure 2).

5. It has been established that in cylindrical shells, the axial normal forces N_v determined using the momentless theory depend significantly on the square of the shell length l, which is clearly seen from the third formulas of systems (4) and (5). This is contrary to common sense, so it is necessary to agree with the conclusions of many scientists in the field of mechanics and the author, that the application area of the momentless theory is limited to short cylindrical shells, or to take into account that long shells behave according to the beam model with hinged supports at the ends [23]. However, cylindrical shells cannot be calculated as beams with curvilinear contour, because they generally do not follow the law of plane sections and do not obey the hypothesis of invariability of the cross-sectional shape. Deplanation of cross-sections occurs.

6. The known statements of the momentless shell theory that the forces determined in the shell regions farthest from the supports, for which boundary conditions cannot be set, are the closest to the true internal forces in shells, are confirmed. In the considered cases, these regions are located around the coordinate line $u = 0$.

The following has been done with respect to the general linear theory of shells:

7. A system of three partial differential equations (8) of the eighth order with respect to the displacements of the middle surface of a cylindrical shell defined in the curvilinear coordinate system u $(-1 \le u \le 1)$, v $(0 \le v \le 1)$ is derived for the first time for possible future application.

8. It is shown that the internal bending moments M_u, M_v and torque H as well as shear forces Q_u, Q_v do not depend on displacements u_v along the straight cylinder generators.

9. It was found that no studies on the calculation and application of hyperbolic cylindrical roofs have been published after 2000.

10. Conclusion

With the modern development of numerical calculation methods and computer technology, most approximate methods of calculation of curvilinear civil and mechanical engineering structures of shell kind, including the momentless theory of shells, are practically not used. However, virtually all textbooks on structural mechanics of shells have sections devoted to the momentless theory of shells. The materials contained in this article can be used in the teaching process to illustrate the applicability of approximate methods of shell analysis.

In the practice of thin shell design, the above materials are unlikely to be used, if only for preliminary assignment of thicknesses of cylindrical shells. All well-known scientists in the field of mechanics related to the design of thin shells state that when selecting the shell shape, one should strive for the shell behavior to be close to momentless.

Having the formulas for calculating the values of internal forces, it is possible to determine the boundary conditions at the edges of the shell in terms of displacements and forces, by satisfying which it is possible to obtain a momentless stress-strain state in the considered shell. However, the required boundary conditions can be created only theoretically, but it is necessary to strive for it.

It may seem that the momentless theory of cylindrical shells is practically useless due to a large number of constraints. However, in a number of cases it allows to obtain simple and sufficiently accurate solutions. In particular, it is applicable to the calculation of shells supported by spandrels, since the length of the compartments between them is set relatively small. The momentless stress state is taken as the basic stress state, and the stability of cylindrical shells in the basic stress state is investigated.

References

1. Lee Y.S. Review on the cylindrical shell research. *Transactions of the Korean Society of Mechanical Engineers, A.* 2009;33(1):1–26. http://doi.org/10.3795/KSME-A.2009.33.1.1

2. Krivoshapko S.N. History of research and application of conical surfaces and shells. *News of Higher Educational Institutions. Construction*. 2024;12(792):123–143. (In Russ.) http://doi.org/10.32683/0536-1052-2024-792-12-123-143

3. Mamieva I.A. Analytical surfaces for parametrical architecture in contemporary buildings and structures. *Academia. Architecture and Construction*. 2020;(1):150–165. (In Russ.) EDN: KNYKTY

4. Gbaguidi Aisse G.L. Influence of the geometrical research of surfaces of revolution and translation surfaces on design of unique structures. *Structural Mechanics of Engineering Constructions and Buildings* 2019;15(4):308–314. http://doi.org/10.22363/1815-5235-2019-15-4-308-314

5. Krivoshapko S.N. Optimal shells of revolution and main optimizations. *Structural Mechanics of Engineering Constructions and Buildings.* 2019;15(3):201–209. http://doi.org/10.22363/1815-5235-2019-15-3-201-209

6. Novozhilov V.V., Chernikh K.F., Mikhailovskiy E.I. *Linear Theory of Thin Shells*. Leningrad: Politechnika Publ.; 1991. (In Russ.) ISBN 5-7325-0127-4

7. Krivoshapko S.N., Ivanov V.N. Simplified selection of optimal shell of revolution. *Structural Mechanics of Engineering Constructions and Buildings*. 2019;15(6):438–448. http://doi.org/10.22363/1815-5235-2019-15-6-438-448

8. Sahu R.R., Gupta P.K. Blast diffusion by different shapes of domes. *Defense Science Journal*. 2015;65(1):77–82. http://doi.org/10.14429/dsj.65.6908

9. Tupikova E.M. Shells in the form of algebraic ruled surfaces on a rhombic base. *Structural Mechanics of Engineering Constructions and Buildings*. 2023;19(5):510–519. (In Russ.) http://doi.org/10.22363/1815-5235-2023-19-5-510-519

10. Hayakawa K., Ohsaki M. Form generation of discrete piecewise developable surface and its interior boundaries using local gauss map. *Summaries of Technical Papers of Annual Meeting.* Architectural Institute of Japan, 2023;1:997–998. (In Japan.)

11. Krivoshapko S.N. Shells, shell structures, and curvilinear erections in the form of algebraical surfaces of the second order. *Structural Mechanics and Analysis of Constructions.* 2024;6:2–18. (In Russ.) http://doi.org/10.37538/0039-2383.2024.6.2.18

12. Rabinskiy L.N., Shoumova N.P., Zhavoronok S.I. Analytical mechanics of membrane shells: a review. *Applied Mathematical Sciences.* 2016;10(44):2189–2204. http://doi.org/10.12988/ams.2016.64158

13. Olshevskiy A.A. *Structural Mechanics of Shells.* Bryansk, 2019. (In Russ.)

14. Biderman V.L. *Mechanics of Thin-Walled Structures. Statics.* Moscow: Mashinostroenie Publ.; 1977. (In Russ.) Available from: https://dwg.ru/dnl/6392 (accessed: 02.05.2024).

15. Krivoshapko S.N. *Geometry of Ruled Surfaces with Cuspidal Edge and Linear Theory of Analysis of Tangential Developable Shells*: Monograph. Moscow: RUDN Publ.; 2009. (In Russ.) ISBN 978-5-209-03087-4

16. Novozhilov V.V. Analysis of Cylindrical Shells. *Izv. AN SSSR. OTN*. 1946;(6):803–816.

17. Vlasov V.Z. *General Shell Theory and its Application in Technics*. Moscow; Leningrad: Gostehizdat Publ.; 1949.

18. Chernikh K.F. *Linear Theory of Shells*. Part 1. Leningrad; 1962.

19. Lurye A.I. Concentration of stresses around a hole on the surface of the circular cylinder. *PMM.* 1946;10(3):307–406.

20. Prokopovich I.E., Slezinger I.N., Shteinberg M.V. *Analysis of Thin Elastic Cylindrical Shells and Prizm Hipped Plate Structures.* Kiev: Budivel'nik Publ.; 1967.

21. Goldenveizer A.L. *Theory of Elastic Thin Shells*. New York: Published by Pergamon Press; 1961.

22. Novozhilov V.V. *Theory of Thin Shells*. Leningrad: Sudpromgiz Publ.; 1962.

23. Flügge W. *Statik und Dynamik der Schalen*. Springer-Verlag; 1957.

24. Grigolyuk E.I., Kulikov G.M. General direction of development of the theory of multilayered shells. *Mechanics of Composite Materials.* 1988;2:231–241. https://doi.org/10.1007/BF00608158

25. Slezinger I.N. Matrix Form of Analysis of Elastic Cylindrical Shells and Hipped Plate Structures. *The Fifth All-Union Conference on Theory of Plates and Shells*. Moscow: Nauka Publ.; 1965.

26. Yokoo Y., Nakamura T., Matsui T. Limit analysis of shallow parabolic cylindrical shells. *Transactions of the Architectural Institute of Japan.* 1964;106:10–19. https://doi.org/10.3130/aijsaxx.106.0_10

27. Krivoshapko S.N., Christian A. Bock Hyeng, Mamieva I.A. Chronology of erection of the earliest reinforced concrete shells. *International Journal of Research and Reviews in Applied Sciences*. 2014;18(2):95–108. EDN: UVEJCD

28. Currà E., Giannetti I., Russo M. Self-supporting thin shells in Italy. Space and structure of the industrial buildings (1930-1970). *Informes de la Construcción.* 2024;76(575):6893. https://doi.org/10.3989/ic.6893

29. Krasic S. *Geometrijske Površi u Arhitekturi*. Gradevinsko-arhitektonski fakultet Univerzitet u Nišu; 2012. ISBN 978-86-88601-02-3

30. Polański S., Pianowski L. *Surface development in technology. Computer-Aided Design.* Warsaw: Wydawnictwo Naukowe PWN; 2001. (In Polish) ISBN 83-01-13362-7

31. Lopatin A.V., Morozov E.V., Shatov A.V. Buckling and vibration of composite lattice elliptical cylindrical shells. *Proceedings of the Institution of Mechanical Engineers. Part L: Journal of Materials Design and Applications.* 2019;233(7): 1255-1266. http://doi.org/10.1177/1464420717736549

32. Bakusov P.A., Semenov A.A. Analysis of the stability of the computational algorithm to a change in the geometric parameters of cylindrical shell structures. *PNRPU Mechanics Bulletin.* 2021;1:12–21. (In Russ.) http://doi.org/10.15593/perm.mech/2021.1.02

33. Dyachenko I.A., Mironov A.A., Sverdlik Yu.M. Comparative analysis of theories and finite-element models of calculations of natural vibrations of cylindrical shells. *Transportnie Sistemy.* 2019;3(10):55–63. (In Russ.) http://doi.org/10.46960/62045_2019_3_55

34. Orlov A.S., Rubtsova E.G., Shcheglov A.S. Analysis of deformations of cylindrical vaulted roofing of religious buildings. *Russian Journal of Building Construction and Architecture*. 2022;3(55):109–120. http://doi.org/10.36622/VSTU.2022.55.3.011

35. Monahov V.A. *Theory of Plates and Shells*. Penza: PGUAS; 2016. (In Russ.)

36. Blaauwendraad J., Hoefakker J.H. Application of membrane theory to circular cylindrical shells. *Structural Shell Analysis. Solid Mechanics and Its Applications*. 2014;200:29–50. https://doi.org/10.1007/978-94-007-6701-0_4

37. Lekomtsev S.V. Finite-element algorithms for calculation of natural vibrations of three-dimensional shells. *Computational continuum mechanics.* 2012;5(2):233–243. (In Russ.) http://doi.org/10.7242/1999-6691/2012.5.2.28

38. Bochkarev S.A., Lekomtsev S.V., Matveenko V.P. Aeroelastic stability of cylindrical shells with elliptical cross-section. *Mechanics of Solids.* 2020;55(5):728–736. (In Russ.) http://doi.org/10.31857/S0572329920050049

39. Sahu R., Kumar B., Shrotriya D. Dynamic study of parabolic cylindrical shell: A parametric study. *International Journal of Trend in Scientific Research and Development.*2021;5(4):1545–1548. Available from: www.ijtsrd.com/papers/ijtsrd43638.pdf (accessed: 02.05.2024).

40. Robles S.I., Ortega N.F. Damage evaluation in shells from changes in its static parameters. *The Open Construction and Building Technology Journal.* 2011;5:182–189. http://doi.org/10.2174/1874836801105010182

41. Li H., Guo D., Tzou H. Light-induced vibration control of parabolic cylindrical shell panels. *Journal of Intelligent Material Systems and Structures. Engineering, Materials Science, Physics.* 2017;28(20):2947–2958. http://doi.org/10.1177/1045389X17704912

42. Ivanova S.S., Umnova O.V. A study of the influence of the steel grid shell's height to its weight and deflection. *Diary of Science.* 2017;12(12):16. (In Russ.) EDN: YMELPK

43. Sehlström A., Olsson K., Williams C. Should Torroja's prestressed concrete Alloz aqueduct be thought of as a beam or a shell? *Engineering Structures*. 2022;264:114425. http://doi.org/10.1016/j.engstruct.2022.114425

44. Kosytsyn S.B., Akulich V.Yu. Numerical analysis of cylindrical shell stability inter-acting with inhomogeneous soil. *Structural Mechanics of Engineering Constructions and Buildings.* 2021;17(6):608–616. (In Russ.) http://doi.org/10.22363/1815-5235-2021-17-6-608-616

45. Rustamova M.A. Determination of natural vibration frequencies of reinforced cylindrical shell. *Structural Mechanics of Engineering Constructions and Buildings.* 2021;17(6):628–638. (In Russ.) http://doi.org/10.22363/1815-5235-2021-17-6-628-638

46. Hudoley A.L., Baran O.A., Goman A.M. Determination of radial displacements of cylindrical shell under thermo-forced loading. *Vestnik of Brest state technical university. Series: Mechanical engineering.* 2006;(4):9–13. (In Russ.) Available from: https://rep.bstu.by/handle/data/12497 (accessed: 02.05.2024).

47. Montes R.O.P., Silva F.M.A., Pedroso L.J. Free vibration analysis of a claim cylindrical shell with internal and external fluid interaction. *Journal of Fluids and Structures*. 2024;125:104079. https://doi.org/10.1016/j.jfluidstructs.2024.104079

48. Firsanov V.V., Vo A.H. Investigation of strengthened cylindrical shells under action of local load on refined theory. *Trudy MAI.* 2018;(102). (In Russ.) EDN: YQONHV

49. Choudhary D.K., Jha A.K., Parihar R.S., Misra R., Udeniyan A. A parametric study of skewed parabolic cylindrical shell at 0º and 30º by staad pro software. *International Journal of Research Publication and Reviews.* 2024;5(7):2847–2855. ISSN 2582-7421

50. Rai R., Pendharkar U. Computer aided analysis of multiple cylindrical shell structure using different parameters. *International Journal of Engineering Research & Technology.* 2012;1(3):1–7. ISSN: 2278-0181

51. Nemeth M.P., Oterkus E., Madenci E. Stress analysis of composite cylindrical shells with an elliptical cutout. *Journal of Mechanics of Materials and Structures*. 2007;2(4):695–727. http://doi.org/10.2140/jomms.2007.2.695

Список литературы

1. *Lee Y.S.* Review on the cylindrical shell research // Transactions of the Korean Society of Mechanical Engineers, A. 2009. Vol. 33. № 1. P. 1–26.

2. *Кривошапко С.Н.* История изучения и применения конических поверхностей и оболочек // Известия вузов. Строительство. 2024. № 12 (792). С. 123–143. http://doi.org/10.32683/0536-1052-2024-792-12-123-143

3. *Мамиева И.А.* Аналитические поверхности для параметрической архитектуры в современных зданиях и сооружениях // Academia. Архитектура и строительство. 2020. № 1. С. 150–165. EDN: KNYKTY

4. *Gbaguidi Aisse G.L.* Influence of the geometrical research of surfaces of revolution and translation surfaces on design of unique structures // Строительная механика инженерных конструкций и сооружений. 2019. Т. 15. № 4. С. 308–314. http://doi.org/10.22363/ 1815-5235-2019-15-4-308-314

5. *Krivoshapko S.N.* Optimal shells of revolution and main optimizations // Строительная механика инженерных конструкций и сооружений. 2019. Т. 15. № 3. С. 201–209. http://doi.org/10.22363/1815-5235-2019-15-3-201-209

6. *Новожилов В.В., Черных К.Ф., Михайловский Е.И.* Линейная теория тонких оболочек. Ленинград : Политехника, 1991. 656 с. ISBN 5-7325-0127-4

7. *Krivoshapko S.N., Ivanov V.N.* Simplified selection of optimal shell of revolution // Строительная механика инженерных конструкций и сооружений. 2019. Т. 15. № 6. С. 438–448 http://doi.org/10.22363/1815-5235-2019-15-6-438-448

8. *Sahu R.R., Gupta P.K.* Blast diffusion by different shapes of domes // Defense Science Journal. 2015. Vol. 65. No. 1. P. 77–82. http://doi.org/10.14429/dsj.65.6908

9. *Тупикова Е.М.* Оболочки в форме алгебраических линейчатых поверхностей на ромбическом плане // Строительная механика инженерных конструкций и сооружений. 2023. Т. 19. № 5. С. 510–519. http://doi.org/10.22363/1815-5235-2023-19-5-510-519

10. *Hayakawa K., Ohsaki M.* Form generation of discrete piecewise developable surface and its interior boundaries using local gauss map // Summaries of Technical Papers of Annual Meeting, Architectural Institute of Japan, 2023, Structures-1, 997–998. (In Jap.)

11. *Кривошапко С.Н.* Оболочки, оболочечные структуры и криволинейные сооружения в форме алгебраических поверхностей 2-го порядка // Строительная механика и расчет сооружений. 2024. № 6. С. 2–18. http://doi.org/10.37538/0039-2383.2024.6.2.18

12. *Rabinskiy L.N., Shoumova N.P., Zhavoronok S.I.* Analytical mechanics of membrane shells: a review // Applied Mathematical Sciences. 2016. Vol. 10. No. 44. P. 2189–2204. http://doi.org/10.12988/ams.2016.64158

13. *Ольшевский А.А.* Строительная механика оболочек. Брянск, 2019. 51 с.

14. *Бидерман В.Л.* Механика тонкостенных конструкций. Статика. Москва : Машиностроение, 1977. 488 с. URL: https://dwg.ru/dnl/6392 (дата обращения: 02.05.2024).

15. *Кривошапко С.Н.* Геометрия линейчатых поверхностей с ребром возврата и линейная теория расчета торсовых оболочек: монография. Москва : Изд-во РУДН, 2009. 357 с. ISBN 978-5-209-03087-4

16. *Новожилов В.В.* Расчет цилиндрических оболочек // Изв. АН СССР. ОТН. 1946. № 6. С. 803–816

17. *Власов В.З.* Общая теория оболочек и ее приложения в технике. Москва ; Ленинград : Гостехиздат, 1949. 784 с.

18. *Черных К.Ф.* Линейная теория оболочек. Ч. 1. Изд-во Ленинградского университета, 1962. 274 с.

19. *Лурье А.И.* Концентрация напряжений в области отверстия на поверхности кругового цилиндра // ПММ. 1946. Т. 10. № 3. С. 307–406.

20. *Прокопович И.Е., Слезингер И.Н., Штейнберг М.В.* Расчет тонких упругих цилиндрических оболочек и призматических складок. Киев : Изд-во Будiвельник, 1967. 240 с.

21. *Гольденвейзер А.Л.* Теория упругих тонких оболочек. Москва : ГТИ,1953. 544 с.

22. *Новожилов В.В.* Теория тонких оболочек. Ленинград : Судпромгиз, 1962. 431 с.

23. *Флюгге В.* Статика и динамика оболочек. Москва : Гос. изд-во литературы по строительству, архитектуре и строительным материалам, 1961. 306 с.

24. *Grigolyuk E.I., Kulikov G.M.* General direction of development of the theory of multilayered shells // Mechanics of Composite Materials. 1988. Vol. 2. P. 231–241. https://doi.org/10.1007/BF00608158

25. *Слезингер И.Н.* Матричная форма расчета гибких цилиндрических оболочек и складок // V Всесоюзная конференция по теории пластин и оболочек. Москва : Наука, 1965.

26. *Yokoo Y., Nakamura T., Matsui T.* Limit analysis of shallow parabolic cylindrical shells // Transactions of the Architectural Institute of Japan. 1964. Vol. 106. P. 10–19. https://doi.org/10.3130/aijsaxx.106.0_10

27. *Krivoshapko S.N., Bock Hyeng C.A., Mamieva I.A.* Chronology of erection of the earliest reinforced concrete shells // International Journal of Research and Reviews in Applied Sciences. 2014. Vol. 18. Issue. 2. P. 95–108. ISSN: 2076-734X, EISSN: 2076-7366

28. *Currà E., Giannetti I., Russo M.* Self-supporting thin shells in Italy. Space and structure of the industrial buildings (1930–1970) // Informes de la Construcción. 2024. Vol. 76. No. 575. Article no. 6893. https://doi.org/10.3989/ic.6893

29. *Krasic Sonja.* Geometrijske Površi u Arhitekturi. Gradevinsko-arhitektonski fakultet Univerzitet u Nišu, 2012. 238 p. ISBN 978-86-88601-02-3

30. *Polański S., Pianowski L.* Rozwiniecia powierzchni w technice. Konstrukcje wspomagane komputerowo. Warszawa: Wydawnictwo Naukowe PWN, 2001. 412 p. (на польск.) ISBN 83-01-13362-7

31. *Lopatin A.V., Morozov E.V., Shatov A.V.* Buckling and vibration of composite lattice elliptical cylindrical shells // Proceedings of the Institution of Mechanical Engineers. Part L: Journal of Materials Design and Applications. 2017. Vol. 233. Issue 7. 1255–1266. http://doi.org/10.1177/1464420717736549

32. *Бакусов П.А., Семенов А.А.* Анализ устойчивости вычислительного алгоритма к изменению геометрических параметров цилиндрических оболочечных конструкций // Вестник Пермского национального исследовательского политехнического университета. Механика. 2021. № 1. С. 12–21. http://doi.org/10.15593/perm.mech/2021.1.02

33. *Дяченко И.А., Миронов А.А., Свердлик Ю.М.* Сравнительный анализ теорий и конечно-элементных моделей расчета свободных колебаний цилиндрических оболочек // Транспортные системы. 2019. № 3 (10). С. 55–63. http://doi.org/10.46960/62045_2019_3_55

34. *Orlov A.S., Rubtsova E.G., Shcheglov A.S.* Analysis of deformations of cylindrical vaulted roofing of religious buildings // Russian Journal of Building Construction and Architecture. 2022. Issue 3 (55). P. 109–120. http://doi.org/10.36622/VSTU.2022.55.3.011

35. *Монахов В.А.* Теория пластин и оболочек : учеб. пособие по направлению подготовки 08.05.01 «Строительство уникальных зданий и сооружений». Пенза : ПГУАС, 2016. 252 с.

36. *Blaauwendraad J., Hoefakker J.H.* Application of membrane theory to circular cylindrical shells // Structural Shell Analysis. Solid Mechanics and Its Applications. 2014. Vol. 200. P. 29–50. https://doi.org/10.1007/978-94-007-6701-0_4

37. *Лекомцев С.В.* Конечно-элементные алгоритмы расчета собственных колебаний трёхмерных оболочек // Вычислительная механика сплошных сред. 2012. Т. 5. № 2. С. 233–243 http://doi.org/10.7242/1999-6691/2012.5.2.28

38. *Бочкарёв С.А., Лекомцев С.В., Матвеенко В.П.* Аэроупругая устойчивость цилиндрических оболочек с эллиптическим поперечным сечением // Известия Российской академии наук. Механика твердого тела. 2020. № 5. С. 151–160. http://doi.org/10.31857/S0572329920050049

39. *Sahu R., Kumar B., Shrotriya D.* Dynamic study of parabolic cylindrical shell: A parametric study // International Journal of Trend in Scientific Research and Development. 2021. Vol. 5. Issue 4. P. 1545–1548. URL: www.ijtsrd.com/papers/ijtsrd43638.pdf (дата обращения: 02.05.2024).

40. *Robles S.I., Ortega N.F.* Damage evaluation in shells from changes in its static parameters // The Open Construction and Building Technology Journal. 2011. Vol. 5. P. 182–189. http://doi.org/10.2174/1874836801105010182

41. *Li H., Guo D., Tzou H.* Light-induced vibration control of parabolic cylindrical shell panels // Journal of Intelligent Material Systems and Structures. Engineering, Materials Science, Physics. 2017. Vol. 28. Issue 20. P. 2947–2958. http://doi.org/10.1177/1045389X17704912

42. *Иванова С.С., Умнова О.В.* Исследование влияния высоты стрелы подъема сетчатой оболочки на ее массу и прогиб // Дневник науки. 2017. № 12 (12). С. 16. EDN: YMELPK

43. *Sehlström A., Olsson K., Williams C.* Should Torroja's prestressed concrete Alloz aqueduct be thought of as a beam or a shell? // Engineering Structures. 2022. Vol. 264. No. 13. Article no. 114425. http://doi.org/10.1016/j.engstruct.2022.114425

44. *Косицын С.Б., Акулич В.Ю.* Численный анализ устойчивости цилиндрической оболочки, взаимодействующей с неоднородным окружающим основанием // Строительная механика инженерных конструкций и сооружений. 2021. Т. 17. № 6. С. 608–616. http://doi.org/10.22363/1815-5235-2021-17-6-608-616

45. *Рустамова М.А.* Определение собственных частот колебаний армированной цилиндрической оболочки // Строительная механика инженерных конструкций и сооружений. 2021. Т. 17. № 6. С. 628–638. http://doi.org/10.22363/1815-5235-2021-17-6-628-638

46. *Худолей А.Л., Баран О.А, Гоман А.М.* Расчет радиальных перемещений цилиндрической оболочки при термосиловом нагружении // Вестник Брестского государственного технического университета. Серия: Машиностроение. 2006. № 4. С. 9–13. URL: https://rep.bstu.by/handle/data/12497 (дата обращения: 02.05.2024).

47. *Montes R.O.P., Silva F.M.A., Pedroso L.J.* Free vibration analysis of a claim cylindrical shell with internal and external fluid interaction // Journal of Fluids and Structures. 2024. Vol. 125. https://doi.org/10.1016/j.fluidstructures.2024.104079

48. *Фирсанов В.В., Во А.Х.* Исследование продольно подкрепленных цилиндрических оболочек под действием локальной нагрузки по уточненной теории // Труды МАИ. 2018. № 102. EDN: YQONHV

49. *Choudhary D.K., Jha A.K., Parihar R.S., Misra R., Udeniyan A.* A parametric study of skewed parabolic cylindrical shell at 0º and 30º by staad pro software // International Journal of Research Publication and Reviews. 2024. Vol. 5. Issue 7. P. 2847–2855. ISSN 2582-7421

50. *Rai R., Pendharkar U.* Computer aided analysis of multiple cylindrical shell structure using different parameters // International Journal of Engineering Research & Technology. 2012. Vol. 1. Issue 3. P. 1–7. ISSN: 2278-0181

51. *Nemeth M.P., Oterkus E., Madenci E.* Stress analysis of composite cylindrical shells with an elliptical cutout // Journal of Mechanics of Materials and Structures. 2007. Vol. 2. No. 4. P. 695–727. http://doi.org/10.2140/jomms.2007.2.695