Physics of Wave Processes and Radio Systems
2025, vol. 28, no. 2, pp. 87–94

DOI 10.18469/1810-3189.2025.28.2.87-94
UDC 621.391.1:621.395
Original research

Received 12 December 2024
Accepted 13 January 2025
Published 30 June 2025

Spectral solutions for QS with distribution laws in the form of probabilistic mixtures

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Abstract – **Background.** QS are the main mathematical tool for modelling data transmission systems, which are not without reason called queuing networks. The need to regulate such characteristics of mass service systems as waiting time in a queue or queue length is due to the improvement of the quality of operation of data transmission systems. The ability to regulate these characteristics allows minimising the waiting time in the queue in the buffers of transmitting devices, as well as the volumes of buffer memory itself. To demonstrate this possibility, the paper examines queuing systems formed by both conventional distribution laws in the form of probability mixtures and time-shifted distribution laws. **Aim.** In this work, the hyperexponential and hyper-Erlangian distributions of the second order were chosen as components of the QS. Based on these distribution laws, numerical-analytical models were constructed for two queuing systems with normal and shifted distribution laws, with the derivation of a solution for the main characteristic of the queuing system – the average waiting time in the queue. As is known, the remaining characteristics of the QS are derivatives of the average waiting time. **Methods.** The paper uses a shift of the distribution laws to the right from the zero point. To derive a solution for the average waiting time in a queue, the classical method of spectral solution of the Lindley integral equation is used based on the Laplace transform of the distribution laws that form the considered QS. The obtained calculation formulas for the average waiting time in a queue allow us to calculate the characteristics of such systems for a wide range of changes in teletrafic parameters. **Results.** The obtained results can be used in modern teletrafic theory in the design and modelling of various promising data transmission systems, including the volumes of buffer memory of transmitting devices. **Conclusion.** The shift of the distribution laws in time leads to a decrease in their variation coefficients. Due to the quadratic dependence of the average waiting time on the variation coefficients of the arrival and service time intervals, a noticeable decrease in the average waiting time follows in systems with time shifts.

Keywords – ordinary and shifted hyper exponential and hyper-Erlang distribution laws; Lindley integral equation; spectral decomposition method; Laplace transform.

Introduction

The spectral solution method of Lindley's integral equation plays an important role in the study of G/G/1 systems. This method is most accessible in specific examples presented in the classics of queueing theory [1].

This article is devoted to the analysis of QS $H_2/HE_2/1$, formed by two flows described by ordinary and right-shifted from the zero point density functions of hyperexponential and hyper-Erlangian distribution laws of the second order.

In their early works, the authors clearly show that in systems formed by shifted distribution laws, the average waiting time becomes shorter than in conventional systems with the same load factor. This is achieved by the fact that the coefficients of variation of arrival times c_λ and service times c_μ for shifted distribution laws become smaller when the shift parameter $t_0 > 0$ is introduced. Thus, the distribution shift operation transforms conventional Markovian queuing systems into a non-Markovian G/G/1 system.

The results of works [2–4] in the field of QS with shifted distributions, together with [1], made it possible to develop a method of spectral decomposition of the solution of Lindley's integral equation for the considered systems $H_2/HE_2/1$.

In queuing theory, studies of G/G/1 systems are relevant because they are actively used in modern teletrafic theory, and moreover, it is impossible to obtain solutions for such systems in finite form for the general case. The spectral decomposition method for solving Lindley's integral equation plays an important role in the study of G/G/1 systems, and most of the results in queueing theory have been obtained using this method.

One form of Lindley's integral equation looks like this [1]:

$$W(y)=\begin{cases}\int\limits_{-\infty}^{y} W(y-u)\,dC(u), & y\ge 0;\\ 0, & y<0,\end{cases}$$

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where *W*(*y*) *is* the probability distribution function (PDF) of the waiting time for a request in a queue; $C(u) = P(\tilde{u} < u)$ is the PDF of the random variable $\tilde{u} = \tilde{x} - \tilde{t}$, , where, in turn, $\tilde{x}$ is the random service time for a request; $\tilde{t}$ is the random variable representing the time interval between requests.

In this brief description of the method for solving Lindley's equation, we will adhere to the approach and notation used by the author [1]. To do this, we will denote the Laplace transforms of the density functions of the intervals between arrivals and service times by $A^*(s)$ and $B^*(s)$, respectively. The essence of solving Lindley's integral equation using the spectral decomposition method is to find an expression for $A^*(-s)\cdot B^*(s) - 1$ in the form of a product of two factors that would give a rational function of *s*. Therefore, to find the distribution law of the waiting time, the following spectral decomposition is required:

$$A^*(-s)\cdot B^*(s) - 1 = \psi_+(s)/\psi_-(s),$$

where $\psi_+(s)$ and $\psi_-(s)$ are some rational functions of *s* that can be decomposed into factors. The functions $\psi_+(s)$ and $\psi_-(s)$ must satisfy special conditions according to [1].

1. Problem statement

The paper sets the task of finding a solution for the average waiting time of requests in a queue in QS $H_2/HE_2/1$ and $H_2^-/HE_2^-/1$ with shifted hyper-exponential (H_2^-) and hyper-Erlang (HE_2^-) input distributions using the classical spectral decomposition method. For other systems, the application of this method is considered in [2-4]. Issues of approximation of distribution laws are discussed in detail in [5; 6; 8-10].

2. Solution to the problem

Consider the system $H_2/HE_2/1$, formed by the hyper-exponential and hyper-Erlang distribution laws with density functions

$$a(t) = p\lambda_1 e^{-\lambda_1 t} + (1-p)\lambda_2 e^{-\lambda_2 t} \quad (1)$$

$$b(t) = 4q\mu_1^2 t e^{-2\mu_1 t} + 4(1-q)\mu_2^2 t e^{-2\mu_2 t}. \quad (2)$$

Distribution laws (1) and (2) are the most general distributions of non-negative continuous random variables, since they provide a wide range of variation coefficients.

The Laplace transform of functions (1) and (2) is

$$A^*(s) = p\frac{\lambda_1}{s+\lambda_1} + (1-p)\frac{\lambda_2}{s+\lambda_2},$$

$$B^*(s) = q\left(\frac{2\mu_1}{s+2\mu_1}\right)^2 + (1-q)\left(\frac{2\mu_2}{s+2\mu_2}\right)^2.$$

Then the spectral decomposition for the system $H_2/HE_2/1$ is transformed as

$$\frac{\psi_+(s)}{\psi_-(s)} = \left[p\frac{\lambda_1}{\lambda_1 - s} + (1-p)\frac{\lambda_2}{\lambda_2 - s}\right]\times$$

$$\times\left[q\left(\frac{2\mu_1}{2\mu_1+s}\right)^2 + (1-q)\left(\frac{2\mu_2}{2\mu_2+s}\right)^2\right] - 1.$$

The first factor on the right-hand side in square brackets is equal to

$$\left[p\frac{\lambda_1}{\lambda_1 - s} + (1-p)\frac{\lambda_2}{\lambda_2 - s}\right] =$$

$$= \frac{\lambda_1\lambda_2 - [p\lambda_1 + (1-p)\lambda_2]s}{(\lambda_1 - s)(\lambda_2 - s)} = \frac{a_0 - a_1 s}{(\lambda_1 - s)(\lambda_2 - s)},$$

where the intermediate parameters are $a_0 = \lambda_1\lambda_2$ *and* $a_1 = p\lambda_1 + (1-p)\lambda_2$. Similarly, we represent the second factor

$$\left[q\left(\frac{2\mu_1}{2\mu_1+s}\right)^2 + (1-q)\left(\frac{2\mu_2}{2\mu_2+s}\right)^2\right] =$$

$$= \frac{q\left(16\mu_1^2\mu_2^2 + 16\mu_1^2\mu_2 s + 4\mu_1^2 s^2\right)}{(2\mu_1+s)^2(2\mu_2+s)^2} +$$

$$+ \frac{(1-q)\left(16\mu_1^2\mu_2^2 + 16\mu_1\mu_2^2 s + 4\mu_2^2 s^2\right)}{(2\mu_1+s)^2(2\mu_2+s)^2} =$$

$$= \frac{b_0 + b_1 s + b_2 s^2}{(2\mu_1+s)^2(2\mu_2+s)^2},$$

where the intermediate parameters are $b_0 = 16\mu_1^2\mu_2^2$, $b_1 = 16\mu_1\mu_2[q\mu_1 + (1-q)\mu_2]$, $b_2 = 4[q\mu_1^2 + (1-q)\mu_2^2]$. Then the desired expression for the spectral decomposition will be written as

$$\frac{\psi_+(s)}{\psi_-(s)} = \frac{(a_0 - a_1 s)(b_0 + b_1 s + b_2 s^2)}{(\lambda_1 - s)(\lambda_2 - s)(2\mu_1+s)^2(2\mu_2+s)^2} - \quad (3)$$

$$- \frac{(\lambda_1 - s)(\lambda_2 - s)(2\mu_1+s)^2(2\mu_2+s)^2}{(\lambda_1 - s)(\lambda_2 - s)(2\mu_1+s)^2(2\mu_2+s)^2}.$$

The polynomial in the numerator on the right-hand side of the decomposition (3) always has one zero $s = 0$ [1]. In this case, the free term of the decomposition is also equal to 0: $a_0 b_0 - 16\lambda_1\lambda_2\mu_1^2\mu_2^2 \equiv 0$.

In the numerator of the fraction on the right-hand side of the expansion, we obtain a sixth-degree polynomial $-s(s^5 - c_4 s^4 - c_3 s^3 - c_2 s^2 - c_1 s - c_0)$, whose coefficients are equal to:

$$c_0 = a_0 b_1 - a_1 b_0 + b_0(\lambda_1 + \lambda_2) - 16 a_0 \mu_1 \mu_2 (\mu_1 + \mu_2), \tag{4}$$

$$c_1 = a_0 b_2 - a_1 b_1 - b_0 - 4 a_0 (\mu_1^2 + \mu_2^2) + 16(\lambda_1 + \lambda_2)(\mu_1 + \mu_2)\mu_1 \mu_2 - 16 a_0 \mu_1 \mu_2,$$

$$c_2 = 4(\lambda_1 + \lambda_2)[(\mu_1 + \mu_2)^2 + 2\mu_1 \mu_2] - 4(\mu_1 + \mu_2)(a_0 + 4\mu_1 \mu_2) - a_1 b_2,$$

$$c_3 = 4(\lambda_1 + \lambda_2)(\mu_1 + \mu_2) - 4[(\mu_1 + \mu_2)^2 + 2\mu_1 \mu_2] - a_0,$$

$$c_4 = \lambda_1 + \lambda_2 - 4(\mu_1 + \mu_2).$$

The coefficients (4) were obtained using Mathcad symbolic operations, since the numerator of the decomposition (3) contains 42 terms even after introducing intermediate parameters. Apparently, the lack of results for the system under consideration is explained by the large amount of work involved in the calculations.

Let us isolate the polynomial in the numerator of the decomposition (3):

$$s^5 - c_4 s^4 - c_3 s^3 - c_2 s^2 - c_1 s - c_0, \tag{5}$$

since determining its roots is the main part of the spectral decomposition method.

An investigation of the polynomial (5) with coefficients (4) using Viète's formulas confirms the existence of four negative real roots and one positive root, or, instead of the former, two negative real roots and two complex conjugate roots with negative real parts. An examination of the sign of the lowest coefficient of polynomial (5) shows that $c_0 > 0$ always in the case of a stable system when $0 < \rho < 1$. Taking into account the minus sign in the polynomial before the coefficient c_0, Viète's formulas do not contradict the fact that there are four negative roots of polynomial (5).

By denoting the negative roots of the polynomial (5) or their negative real parts for convenience as $-s_1$, $-s_2$, $-s_3$, $-s_4$, and the positive root as s_5, the ratio $\psi_+(s) / \psi_-(s)$ can finally be decomposed into the following factors:

$$\frac{\psi_+(s)}{\psi_-(s)} = \frac{-s(s+s_1)(s+s_2)(s+s_3)(s+s_4)(s-s_5)}{(\lambda_1 - s)(\lambda_2 - s)(2\mu_1 + s)^2 (2\mu_2 + s)^2}. \tag{6}$$

Taking into account the special conditions [1], we accept the function $\psi_+(s)$ as

$$\psi_+(s) = \frac{s(s+s_1)(s+s_2)(s+s_3)(s+s_4)}{(2\mu_1 + s)^2 (2\mu_2 + s)^2},$$

since the zeros of the polynomial (5) are: $s = 0$, $-s1$, $-s_2$, $-s_3$, $-s_4$, and the double poles $s = -2\mu_1$, $s = -2\mu_2$ lie in the region $\mathrm{Re}(s) \le 0$, and for the function $\psi_-(s)$ – $\psi_-(s) = -\frac{(\lambda_1 - s)(\lambda_2 - s)}{(s - s_5)}$, since its zeros and pole lie in the region $Re(s) < D$.

Next, using the spectral decomposition method, we determine the constant

$$K = \lim_{s \to 0} \frac{\psi_+(s)}{s} = \frac{s_1 s_2 s_3 s_4}{16 \mu_1^2 \mu_2^2}.$$

The constant K determines the probability that the request entering the system will find it free. Using the function $\psi_+(s)$ and the constant K, we will determine the Laplace transform of the waiting time $W(y)$:

$$\Phi_+(s) = \frac{K}{\psi_+(s)} = \frac{s_1 s_2 s_3 s_4 (s + 2\mu_1)^2 (s + 2\mu_2)^2}{16 s \mu_1^2 \mu_2^2 (s + s_1)(s + s_2)(s + s_3)(s + s_4)}.$$

Then, the Laplace transform for the waiting time density function will be the function $s \cdot F_+(s)$, i.e.

$$W^*(s) = \frac{s_1 s_2 s_3 s_4 (s + 2\mu_1)^2 (s + 2\mu_2)^2}{16 \mu_1^2 \mu_2^2 (s + s_1)(s + s_2)(s + s_3)(s + s_4)}. \tag{7}$$

The desired average waiting time in the queue is equal to the value of the derivative of the Laplace transform (9) of the density function with a minus sign at the point $s = 0$:

$$-\left.\frac{dW^*(s)}{ds}\right|_{s=0} = \frac{1}{s_1} + \frac{1}{s_2} + \frac{1}{s_3} + \frac{1}{s_4} - \frac{1}{\mu_1} - \frac{1}{\mu_2}.$$

Finally, the average waiting time in the queue for QS $H_2/HE_2/1$

$$\bar{W} = \frac{1}{s_1} + \frac{1}{s_2} + \frac{1}{s_3} + \frac{1}{s_4} - \frac{1}{\mu_1} - \frac{1}{\mu_2}. \tag{8}$$

From expression (7), it is also possible to determine the higher-order moments for the waiting time, if necessary. The second derivative of the transform (7) at the point $s = 0$ gives the second initial moment of the waiting time, which allows us to determine the variance of the waiting time. Considering the definition of jitter in telecommunications as the spread of the waiting time around its mean value [7], we can thus determine jitter through variance. This is an important result for the analysis of delay-sensitive traffic.

Now let us move on to the study of the $H_2/HE_2/1$ system with shifted input distributions, i.e., a system with time delay. Unlike a conventional system, we will denote such a system as $H_2^- / HE_2^- / 1$. To do this, we will consider the density functions of the input flow intervals and service times:

$$a(t) = p\lambda_1 e^{-\lambda_1(t-t_0)} + (1-p)\lambda_2 e^{-\lambda_2(t-t_0)}, \quad (9)$$

$$b(t) = 4q\mu_1^2(t-t_0)e^{-2\mu_1(t-t_0)} + \quad (10)$$
$$+ 4(1-q)\mu_2^2(t-t_0)e^{-2\mu_2(t-t_0)}.$$

The density functions (9) and (10) are shifted to the right of the zero point by an amount $t_0 > 0$ by hyperexponential and hyper-Erlang distributions of the second order. To find the average waiting time in the queue for this system, we will prove the following statement.

Statement. The spectral decompositions $A^*(-s) \cdot B^*(s) - 1 = \psi_+(s) / \psi_-(s)$ for the systems $H_2^- / HE_2^- / 1$ and $H_2/HE_2/1$ coincide completely and have the form (6).

Proof. For the system $H_2^- / HE_2^- / 1$, the spectral decomposition will have the form

$$\frac{\psi_+(s)}{\psi_-(s)} = [p\frac{\lambda_1}{\lambda_1 - s} + (1-p)\frac{\lambda_2}{\lambda_2 - s}]e^{t_0 s} \times$$
$$\times [q\left(\frac{2\mu_1}{2\mu_1 + s}\right)^2 + (1-q)\left(\frac{2\mu_2}{2\mu_2 + s}\right)^2]e^{-t_0 s} - 1 =$$
$$= [p\frac{\lambda_1}{\lambda_1 - s} + (1-p)\frac{\lambda_2}{\lambda_2 - s}] \times$$
$$\times [q\left(\frac{2\mu_1}{2\mu_1 + s}\right)^2 + (1-q)\left(\frac{2\mu_2}{2\mu_2 + s}\right)^2] - 1.$$

The exponents with opposite signs are zeroed, and the shift operation is thus levelled out.

Thus, the spectral decompositions of the solution of Lindley's integral equation for the two systems under consideration coincide. The statement is proven.

Corollary. The calculated expression for the average waiting time for a system with shifted distributions will have exactly the same form as for a system with normal distributions, but with changed parameters due to the time shift operation [2–4].

Now let us determine the numerical characteristics and, through them, the unknown parameters of distributions (9) and (10) using the method of moments. To do this, we write down their Laplace transforms:

$$A^*(s) = [p\frac{\lambda_1}{\lambda_1 + s} + (1-p)\frac{\lambda_2}{\lambda_2 + s}]e^{-t_0 s},$$

$$B^*(s) = [q\left(\frac{2\mu_1}{2\mu_1 + s}\right)^2 + (1-q)\left(\frac{2\mu_2}{2\mu_2 + s}\right)^2]e^{-t_0 s}.$$

The first derivative of function $A^*(s)$ with a minus sign at point $s = 0$ gives the values of the average interval of claim arrival

$$\overline{\tau}_\lambda = p\lambda_1^{-1} + (1-p)\lambda_2^{-1} + t_0, \quad (11)$$

and the second derivative gives the second initial moment of this interval

$$\overline{\tau_\lambda^2} = t_0^2 + 2t_0[\frac{p}{\lambda_1} + \frac{(1-p)}{\lambda_2}] + 2[\frac{p}{\lambda_1^2} + \frac{(1-p)}{\lambda_2^2}]. \quad (12)$$

Then the square of the coefficient of variation of the arrival interval will be equal to

$$c_\lambda^2 = \frac{[(1-p^2)\lambda_1^2 - 2\lambda_1\lambda_2 p(1-q) + p(2-p)\lambda_2^2]}{[t_0\lambda_1\lambda_2 + (1-p)\lambda_1 + p\lambda_2]^2}. \quad (13)$$

Proceeding similarly with distribution (10), we determine the corresponding characteristics for the service time.

$$\overline{\tau}_\mu = q\mu_1^{-1} + (1-q)\mu_2^{-1} + t_0, \quad (14)$$

$$\overline{\tau_\mu^2} = t_0^2 + 2t_0[\frac{q}{\mu_1} + \frac{(1-q)}{\mu_2}] + \frac{3}{2}[\frac{q}{\mu_1^2} + \frac{(1-q)}{\mu_2^2}], \quad (15)$$

$$c_\mu^2 = \frac{\mu_1^2 - 2q\mu_2(\mu_1 - \mu_2) + q(1-2q)(\mu_1 - \mu_2)^2}{2[t_0\mu_1\mu_2 + (1-q)\mu_1 + q\mu_2]^2}. \quad (16)$$

The mechanism for determining the parameters of distributions (1), (2), (9) and (10) using both the first two initial moments and the first three initial moments is described in detail in [3] and [4], respectively. Here we give the ready-made expressions for these parameters. For distribution (9), we find the unknown parameters using the expressions:

$$p = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{(\overline{\tau}_\lambda - t_0)^2}{2[(\overline{\tau}_\lambda - t_0)^2 + c_\lambda^2 \overline{\tau}_\lambda^2]}},$$

$$\lambda_1 = 2p / (\overline{\tau}_\lambda - t_0), \quad \lambda_2 = 2(1-p) / (\overline{\tau}_\lambda - t_0),$$

and for distribution (10) –

$$q = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{3(\overline{\tau}_\mu - t_0)^2}{8[(\overline{\tau}_\mu - t_0)^2 + c_\mu^2 \overline{\tau}_\mu^2]}},$$

$$\mu_1 = 2q / (\overline{\tau}_\mu - t_0), \quad \mu_2 = 2(1-q) / (\overline{\tau}_\mu - t_0).$$

These expressions imply that the shift parameter is limited by the condition $t_0 < \overline{\tau}_\mu < \overline{\tau}_\lambda$. Furthermore, the applicability of the system $H_2^- / HE_2^- / 1$ is determined not by the negativity of the two sub-expressions for p and q.

The algorithm for calculating the average waiting time for given input parameters $\overline{\tau}_\lambda$, $\overline{\tau}_\mu$, c_λ, c_μ, t_0 boils down to sequentially determining the unknown parameters of distributions (9) and (10). Next, we de-

Table 1. Experimental results for the $H_2/HE_2/1$ system
Таблица 1. Результаты экспериментов для системы $H_2/HE_2/1$

Input parameters		Average waiting time	
ρ	(c_λ, c_μ)	QS $N_2/NE_2/1$	QS $N_2/N_2/1$
0,1	(1; 0,71)	0,086	-
	(1;1)	0,111	0,111
	(2;2)	0,446	0,445
	(4;4)	1,791	1,779
	(8;8)	7,173	7,112
0,5	(1; 0,71)	0,755	-
	(1;1)	1,000	1,000
	(2;2)	4,043	4,044
	(4;4)	16,235	16,129
	(8;8)	64,844	64,178
0,9	(1; 0,71)	6,771	-
	(1;1)	9,075	9,000
	(2;2)	36,169	36,200
	(4;4)	144,773	144,833
	(8;8)	577,875	577,861

termine the coefficients of the polynomial (5) using the above expressions (4) and find the required roots with negative real parts $-s_1$, $-s_2$, $-s_3$, $-s_4$. Substituting the absolute values of these roots into expression (8), we determine the average waiting time. The presence of such roots is due to the existence and uniqueness of the spectral decomposition. Numerous experiments have only confirmed this fact.

3. Results of computational experiments

Tables 1 and 2 show the results of calculations in Mathcad for a conventional $H_2/NE_2/1$ system and a system with a delay of $H_2^- / HE_2^- / 1$ for low, medium and high loads $\rho = 0{,}1$; 0,5; 0,9 for a wide range of variation coefficients c_λ, $c_{(\mu)}$ and shift parameter t_0. The results for the conventional system are compared with the data for the similar system $H_2/H_2/1$. Dashes in Table 1 indicate that the $H_2/NE_2/1$ system is not applicable for these parameter values. The results for the system with delay $H_2^- / HE_2^- / 1$ are compared with the results for the conventional system. The load factor ρ in both tables is determined by the ratio of the average intervals $\rho = \bar{\tau}_\mu / \bar{\tau}_\lambda$. The calculations use the normalised service time $\bar{\tau}_\mu = 1$.

The results for the $H_2/NE_2/1$ and $H_2/H_2/1$ systems coincide to the integer parts, but the range of service parameters for the first system is wider than for the second.

The system $H_2^- / HE_2^- / 1$ is also applicable for small values of the coefficients of variation, in particular, when $\rho = 0{,}9$, $c_\lambda = c_\mu = 0{,}2$ $t_0 = 0{,}99$, the average waiting time is only $\bar{W} = 0{,}187$ units of time.

Thus, the range of parameter variation for the system $H_2^- / HE_2^- / 1$ is much wider than that for the conventional system $H_2/HE_2/1$.

Conclusion

The following conclusions can be drawn from the results of the work.

As expected, reducing the coefficients of variation c_λ and c_μ by introducing the shift parameter $t_0 > 0$ into the distribution laws of the input flow and service time results in a noticeable decrease in the average waiting time in systems with delays. Thus, we extend the scope of application of the $H_2/HE_2/1$ system in traffic theory.

The scientific novelty of the results obtained lies in the fact that a spectral decomposition of the solu-

Table 2. Experimental results for the $H_2^- / HE_2^- / 1$ system

Таблица 2. Результаты экспериментов для системы $H_2^- / HE_2^- / 1$

Input parameters		Average waiting time			
(c_λ, c_μ)	$(c_\lambda; c_\mu)$	QS $H_2^- / HE_2^- / 1$			QS $H_2/HE_2/1$
		t_0 = 0,99	t_0 = 0,5	t_0 = 0,01	
0,1	(1;0,71)	0,03	0,04	0,09	0,09
	(1;1)	0,06	0,07	0,11	0,11
	(2;2)	0,23	0,36	0,44	0,45
	(4;4)	0,93	1,56	1,79	1,79
	(8;8)	3,74	6,38	7,16	7,17
0,5	(1;0,71)	0,26	0,48	0,75	0,76
	(1;1)	0,51	0,75	0,99	1,00
	(2;2)	2,04	3,15	4,03	4,04
	(4;4)	8,15	12,73	16,17	16,24
	(8;8)	32,62	51,07	64,58	64,84
0,9	(1;0,71)	2,49	6,00	6,77	6,77
	(1;1)	4,73	8,29	9,06	9,08
	(2;2)	18,92	33,20	36,14	36,17
	(4;4)	75,69	123,39	144,63	144,77
	(8;8)	302,78	528,43	577,29	577,88

tion of the Lindley integral equation for the systems under consideration has been obtained and, with its help, a calculation formula for the average waiting time in the queue for this system has been derived in closed form. The data from numerical experiments confirm the complete adequacy of the theoretical results obtained.

The practical significance of the work lies in the fact that the results obtained can be successfully applied in modern teletrafic theory, where delays in incoming traffic packets play a paramount role. To do this, it is necessary to know the numerical characteristics of the incoming traffic intervals and the service time at the level of the first two moments, which is not difficult when using modern traffic analysers.

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Физика волновых процессов и радиотехнические системы
2025. Т. 28, № 2. С. 87–94

DOI 10.18469/1810-3189.2025.28.2.87-94
УДК 621.391.1:621.395
Оригинальное исследование

Дата поступления 12 декабря 2024
Дата принятия 13 января 2025
Дата публикации 30 июня 2025

Спектральные решения для СМО с законами распределений в виде вероятностных смесей

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Аннотация – **Обоснование.** СМО являются основным математическим инструментарием моделирования систем передачи данных, которые недаром называют сетями массового обслуживания. Необходимость регулирования таких характеристик систем массового обслуживания, как время ожидания в очереди или длины очереди, обусловлена повышением качества функционирования систем передачи данных. Возможность регулирования этих характеристик позволяет минимизировать время ожидания в очереди в буферах передающих устройств, а также сами объемы буферной памяти. Для демонстрации такой возможности в работе рассмотрены системы массового обслуживания, сформированные как обычными законами распределений в виде вероятностных смесей, так и сдвинутыми во времени законами распределений. **Цель.** В качестве составляющих СМО в работе выбраны гиперэкспоненциальное и гиперэрлангговское распределения второго порядка. На основе этих законов распределений построены численно-аналитические модели для двух систем массового обслуживания с обычными и сдвинутыми законами распределений с выводом решения для основной характеристики СМО – среднего времени ожидания в очереди. Как известно, остальные характеристики СМО являются производными от среднего времени ожидания. **Методы.** В работе использован сдвиг законов распределений вправо от нулевой точки. Для вывода решения для среднего времени ожидания в очереди использован классический метод спектрального решения интегрального уравнения Линдли на основе преобразования Лапласа законов распределений, формирующих рассмотренные СМО. Полученные расчетные формулы для среднего времени ожидания в очереди позволяют рассчитать характеристики таких систем для широкого диапазона изменения параметров телетрафика. **Результаты.** Полученные результаты могут быть использованы в современной теории телетрафика при проектировании и моделировании различных перспективных систем передачи данных, включая объемы буферной памяти передающих устройств. **Заключение.** Сдвиг законов распределений во времени приводит к уменьшению их коэффициентов вариаций. Из-за квадратичной зависимости среднего времени ожидания от коэффициентов вариаций временных интервалов поступления и обслуживания следует заметное уменьшение среднего времени ожидания в системах с временными сдвигами.

Ключевые слова – обычные и сдвинутые гиперэкспоненциальный и гиперэрланговский законы распределения; интегральное уравнение Линдли; метод спектрального разложения; преобразование Лапласа.

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