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Thermal entanglement in two-atom Tavis–Cummings model with taking into account the dipole-dipole interaction

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Abstract – **Background.** Interest in the study of entangled states of systems of natural and artificial atoms (qubits) interacting with selected modes of microwave resonators is associated with their use as logic elements of quantum computers. At the same time, the most important task of the physics of quantum computing is the choice of the most effective mechanisms for manipulating and controlling the entangled states of qubits in such devices. **Aim.** The dynamics of the entanglement of two dipole-coupled superconducting Josephson qubits induced by a thermal noise of a coplanar resonator is studied for various initial states of the qubits. **Methods.** Based on the exact solution of the quantum Liouville equation for the whole density matrix of the system under consideration, the time behavior of the qubit entanglement parameter (negativity) is found for chaotic thermal, pure separable, and entangled initial states of qubits. **Results.** It is shown that the entanglement of qubits induced by the thermal noise of the resonator is possible for both the chaotic thermal states and separable states of qubits, except the case when both qubits are excited. It has also been found that, for small values of the dipole–dipole interaction parameter, taking this interaction into account leads to an increase in the degree of entanglement. For values of the dipole-dipole interaction parameter greater than some limit value, the opposite effect takes place. It is found that for entangled initial states of qubits, the inclusion of direct interaction has a small effect on the entanglement dynamics. It is shown that the initial coherence of qubit states can lead to a significant increase in the degree of their entanglement in the presence of a dipole–dipole interaction. **Conclusion.** The dipole-dipole interaction can be used as an effective mechanism for qubits entanglement manipulation and controlling.
Keywords – superconducting qubits; coplanar cavity; thermal field; dipole-dipole interaction; entanglement; negativity.

Introduction

The study of entangled states is currently one of the most pressing problems of quantum theory because of the wide possible applications of such states in quantum information science, quantum teleportation, quantum cryptography, and quantum dense coding [1–3]. A method for generating atomic entangled states is the interaction of natural and artificial atoms (impurity spins, superconducting Josephson rings, quantum dots, etc.) with the quantum electromagnetic fields of resonators [4]. At the same time, special attention has recently been paid to studying the possibility of entangling atoms with light by interaction with a thermal electromagnetic field. It is generally believed that the thermal state of this field contains minimal information about the system and can be considered “chaotic.” Moreover, multimode thermal fields have often been used to analyze the decoherence of quantum atomic systems. However, Kim et al. [5] showed that such an uncorrelated thermal field can entangle atoms in a resonator. They studied the evolution of two identical two-level atoms interacting resonantly with a single-mode thermal field in a lossless resonator and calculated the atomic entanglement parameter as a function of time. The degree of entanglement of atoms depends considerably on their initial states. If one atom is initially in the ground state and the other atom is in the excited state, then the thermal field can lead to a noticeable degree of atom–atom entanglement. In contrast, if both atoms are initially in excited states, then in the resonant approximation, their entanglement is impossible. A similar effect occurs for the diatomic model with multiphoton transitions [6; 7]. Zang [8] generalized the work of Kim et al. [5] to the case where the transition frequencies in atoms are slightly detuned from the frequency of the thermal field mode of the resonator and studied how such detuning affects atomic entanglement. This study revealed that with a suitable choice of detuning for the initial state of the atomic system, where one atom is in the excited state and the other atom is in the ground state, the entanglement of atoms can approach the maximum value. In addition, substantial atomic entanglement can be achieved even when both atoms are initially in excited states.

The dipole–dipole interaction of atomic systems is well-known as a natural mechanism for the occurrence of atomic entanglement. The dipole–dipole interaction of atoms, in particular, can considerably increase the degree of entanglement of two atoms

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interacting with the thermal field mode in an ideal resonator through single-photon transitions [9] and two-photon degenerate [13] and nondegenerate transitions [10; 11]. For artificial atoms, the dipole–dipole interaction can be considerably greater than that for ordinary atoms and ions. For example, for superconducting Josephson qubits, the effective dipole–dipole constant (inductive interaction of superconducting qubits) can considerably exceed not only the qubit–photon interaction constant but also the initial transition energy between the levels of the qubit itself [12; 13]. An interesting consideration is the influence of the dipole–dipole interaction of qubits in various initial states on the maximum degree of their resonator thermal field-induced entanglement. When studying the exact dynamics of qubits in a resonator (references in [14–17]), the authors used various methods to solve the quantum time Schrödinger equation for the full wave function or the quantum Liouville' equation for the full density matrix, depending on the initial states of the qubits and the resonator field. In contrast, for the resonant two-qubit model with single-photon transitions and direct dipole–dipole interaction of qubits, an exact solution to the equation for the evolution operator was found [9]. In this case, the authors of this study used the revealed exact solution for the evolution operator to calculate the parameter of qubit entanglement induced by the thermal field of the resonator for the simplest case, one of the qubits is prepared in an excited state and the other is in the ground state.

In this work, we investigated the influence of the dipole–dipole interaction of qubits on the dynamics of their entanglement in a two-qubit resonance single-photon model induced by the resonator thermal field for various initial states of qubits, namely, Bell-type, thermal, and coherent entangled states.

1. Model and its exact solution

Let us consider two identical superconducting qubits A and B that resonantly interact with the common quantum single-mode electromagnetic field of an ideal microwave coplanar resonator. We will consider the direct dipole–dipole interaction of qubits because, for superconducting qubits, the constant of this interaction can substantially exceed the qubit–field interaction constant. The Hamiltonian of the interaction of the system in the rotating wave approximation can then be represented as

$$H = \hbar g \sum_{i=A,B} (\sigma_i^+ a + a^+ \sigma_i^-) + \hbar J(\sigma_A^+ \sigma_B^- + \sigma_A^- \sigma_B^+). \quad (1)$$

Here, σ_1^z and σ_2^z are the inversion operators for qubits A and B, respectively; $\sigma_i^+ = |+\rangle_{ii}\langle -|$ and $\sigma_i^- = |-\rangle_{ii}\langle +|$ are the operators of transitions between the excited $|+\rangle_i$ and ground $|-\rangle_i$ states in the i-th qubit ($i = A, B$), while a^+ and a are the creation and destruction operators of photons of the resonator mode of the field; g is the interaction constant between the qubit and the field; and J is the constant of direct dipole–dipole interaction of qubits.

Let us assume that the resonator field is initially in a single-mode state with a density matrix

$$\rho_F(0) = \sum_n p_n \, |n\rangle\langle n|, \quad (2)$$

where the weight coefficients are $p_n = \bar{n}^n / (1+\bar{n})^{n+1}$. Here, $\bar{n}$ is the average number of thermal photons in the resonator $\bar{n} = (\exp[\hbar\omega_i / k_B T] - 1]^{-1}$, where k_B is Boltzmann's constant, and T is the equilibrium temperature of the resonator.

As the initial state of the qubits, we choose

a) pure separable coherent states of the form

$$|\Psi(0)\rangle_{AB} = |\Psi(0)\rangle_A \otimes |\Psi(0)\rangle_B, \quad (3)$$
$$|\Psi(0)\rangle_A = \cos\theta_A \, |+\rangle_A + \sin\theta_A \, |-\rangle_A,$$
$$|\Psi(0)\rangle_B = \cos\theta_B \, |+\rangle_B + sin\theta_B \, |-\rangle_B,$$

where θ_i are the parameters that determine the initial coherence of the i-qubit state;

b) mixed states with a density matrix of the form

$$\rho(0)_{AB} = \prod_{i=A,B} \lambda \, |+\rangle_{ii}\langle +| + (1-\lambda) \, |-\rangle_{ii}\langle -|, \quad (4)$$

where $\frac{\lambda}{1-\lambda} = \exp[\hbar\omega_0 / k_B T]$ and ω_0 is the resonant frequencies of the transition between the first excited state and the ground state of the qubit; and

c) Bell-type entangled states

$$|\Psi(0)\rangle_{AB} = \cos\theta \, |+,-\rangle + \sin\theta \, |-,+\rangle, \quad (5)$$
$$|\Psi(0)\rangle_{AB} = \cos\theta \, |+,+\rangle + \sin\theta \, |-,-\rangle, \quad (6)$$

where θ is a parameter that determines the initial degree of qubit entanglement.

The time-dependent density matrix of the system under study can be determined by solving the following quantum Liouville equation:

$$i\hbar \frac{\partial \rho}{\partial t} = [H, \rho] \quad (7)$$

with the initial condition

$\rho(0) = \rho_{AB}(0) \otimes \rho_F(0)$.

In the case of the pure initial states of qubits,

$\rho_{AB}(0) = |\Psi(0)\rangle_{AB \, AB}\langle \Psi(0)|$.

Let us represent the formal solution of Eq. (7) as

$\rho(t) = U^+(t)\rho(0)U(t)$,

where the evolution operator of a system with the Hamiltonian of Eq. (1) in the basis

$|-,-\rangle,\ \ |+,-\rangle,\ \ |-,+\rangle,\ \ |+,+\rangle$

has the form [12]

$$U(t)=\begin{pmatrix} U_{11} & U_{12} & U_{13} & U_{14}\\ U_{21} & U_{22} & U_{23} & U_{24}\\ U_{31} & U_{32} & U_{33} & U_{34}\\ U_{41} & U_{42} & U_{43} & U_{44}\end{pmatrix}, \quad (8)$$

where

$$U_{11}=1+2a\frac{A}{\lambda}a^{+},\quad U_{14}=2a\frac{A}{\lambda}a,$$

$$U_{41}=2a^{+}\frac{A}{\lambda}a^{+},\quad U_{44}=1+2a^{+}\frac{A}{\lambda}a\ ,$$

$$U_{12}=U_{13}=a\ \frac{B}{\theta},\quad U_{21}=U_{31}=\frac{B}{\theta}a^{+},$$

$$U_{24}=U_{34}=\frac{B}{\theta}a\ ,\quad U_{42}=U_{43}=a^{+}\frac{B}{\theta},$$

$$U_{22}=U_{33}=\frac{\exp\left[-i\frac{g}{2}(\alpha+\theta)t\right]}{4\theta}\times$$
$$\times\left\{[1-\exp(ig\theta t)]\alpha+2\theta\exp(i\frac{g}{2}(3\alpha+\theta)t]+\right.$$
$$\left.+\theta[1+\exp(ig\theta t)]\right\},$$

$$U_{23}=U_{32}=\frac{\exp\left[-i\frac{g}{2}(\alpha+\theta)t\right]}{4\theta}\times$$
$$\times\left\{[1-\exp(ig\theta t)]\alpha-2\theta\exp(i\frac{g}{2}(3\alpha+\theta)t]+\right.$$
$$\left.+\theta[1+\exp(ig\theta t)]\right\},$$

$$A=\exp\left[-i\frac{g\alpha}{2}t\right]\left\{\cos\left(\frac{g\theta}{2}t\right)+i\frac{\alpha}{\theta}\sin\left(\frac{g\theta}{2}t\right)\right\}-1,$$

$$B=\exp\left[-i\frac{g}{2}(\alpha+\theta)t\right]\left[1-\exp(ig\theta t)\right],$$

$$\alpha=\frac{J}{g},\quad \lambda=2(aa^{+}+a^{+}a\ \ ),\quad \theta=\sqrt{8(a\ \ a^{+}+a^{+}a\ \ )+\alpha^{2}}.$$

Having the evolution operator of Eq. (8), we can determine an explicit form of the temporal density matrix of the system under consideration for any initial states of the qubits. In this study, we use the exact solution of the quantum Liouville equation to study the time dynamics of qubit entanglement. To date, strict entanglement criteria have been obtained in quantum information science for two-qubit systems. One of these criteria is the Peres–Horodecki criterion or negativity [18; 19]. In this study, to quantify the degree of qubit entanglement, we use negativity in the form

$$\varepsilon=-2\sum\mu_i^{-}, \quad (9)$$

where μ_i^{-} are the negative eigenvalues of the reduced two-qubit density matrix partially transposed over the variables of one qubit $\rho_{AB}^{T_1}(t)$. The two-qubit reduced density matrix can be determined by averaging the full density matrix of the system over the field variables:

$$\rho_{AB}(t)=Tr_F\rho(t).$$

Using the explicit form of the evolution operator after rather cumbersome calculations for the reduced qubit density matrix, we obtain in a two-qubit basis $|-,-\rangle,\ |+,-\rangle,\ |-,+\rangle,\ |+,+\rangle$

$$\rho_A(t)=\begin{pmatrix} \rho_{11}(t) & \rho_{12}(t) & \rho_{13}(t) & \rho_{14}(t)\\ \rho_{12}^{*}(t) & \rho_{22}(t) & \rho_{23}(t) & \rho_{24}(t)\\ \rho_{13}^{*}(t) & \rho_{23}^{*}(t) & \rho_{33}(t) & \rho_{34}(t)\\ \rho_{14}^{*}(t) & \rho_{24}^{*}(t) & \rho_{34}^{*}(t) & \rho_{44}(t)\end{pmatrix}, \quad (10)$$

$$\rho_{11}(t)=\sum\nolimits_{n=1}^{\infty}p_n\left[\rho_{11}\left(1+2(n+1)Q_{n+1}(t)\right)+\right.$$
$$\left.+(\rho_{11}+\rho_{23}+\rho_{32}+\rho_{33})\,|\,S_n\,|^{2}\ n+\rho_{44}4(n-1)Q_{n-1}(t)\right],$$

$$\rho_{12}(t)=\sum\nolimits_{n=0}^{\infty}p_n\left[(\rho_{24}+\rho_{24})S_n(t)S_{n-1}^{*}(t)n+\right.$$
$$\left.+(\rho_{12}U_{22,n}^{*}+\rho_{13}U_{23,n}^{*})\left(1+2(n+1)Q_{n+1}(t)\right)\right],$$

$$\rho_{12}(t)=\sum\nolimits_{n=0}^{\infty}p_n\left[(\rho_{24}+\rho_{24})S_n(t)S_{n-1}^{*}(t)n+\right.$$
$$\left.+(\rho_{12}U_{23,n}^{*}+\rho_{13}U_{22,n}^{*})\left(1+2(n+1)Q_{n+1}(t)\right)\right],$$

$$\rho_{12}(t)=\sum\nolimits_{n=1}^{\infty}p_n\rho_{14}\left(1+2(n+1)Q_{n+1}(t)\right)\left(1+2nQ_{n-1}(t)\right),$$

$$\rho_{22}(t)=\sum\nolimits_{n=0}^{\infty}p_n\left[\rho_{11}(n+1)\left|S_{n+1}(t)\right|^{2}+\right.$$
$$\left.+\rho_{44}n\left|S_{n-1}(t)\right|^{2}+\rho_{22}\,|\,U_{22,n}\,|^{2}\right]+$$
$$+\sum\nolimits_{n=0}^{\infty}p_n[\rho_{23}U_{22,n}U_{23,n}^{*}+$$
$$+\rho_{23}^{*}U_{23,n}U_{22,n}^{*}+\rho_{33}\,|\,U_{23,n}\,|^{2}],$$

$$\rho_{33}(t)=\sum\nolimits_{n=1}^{\infty}p_n\left[\rho_{11}(n+1)\left|S_{n+1}(t)\right|^{2}+\rho_{44}n\left|S_{n-1}(t)\right|^{2}\right]+$$
$$+\sum\nolimits_{n=1}^{\infty}p_n[\rho_{22}\,|\,U_{23,n}\,|^{2}+\rho_{23}U_{23,n}U_{22,n}^{*}+$$
$$+\rho_{23}^{*}U_{22,n}U_{23,n}^{*}+\rho_{33}\,|\,U_{22,n}\,|^{2}],$$

$$\rho_{33}(t)=\sum\nolimits_{n=1}^{\infty}p_n\left[\rho_{11}(n+1)\left|S_{n+1}(t)\right|^{2}+\rho_{44}n\left|S_{n-1}(t)\right|^{2}\right]+$$
$$+\sum\nolimits_{n=1}^{\infty}p_n[\rho_{22}\,|\,U_{23,n}\,|^{2}+\rho_{23}U_{23,n}U_{22,n}^{*}+$$
$$+\rho_{23}^{*}U_{22,n}U_{23,n}^{*}+\rho_{33}\,|\,U_{22,n}\,|^{2}],$$

$$\rho_{24}(t)=\sum_{n=1}^{\infty}p_n\left[(\rho_{24}U_{22,n}+\rho_{34}U_{23,n})\left(2nQ_{n-1}(t)+1\right)+\right.$$
$$\left.+(\rho_{12}+\rho_{13})(n+1)S_{n+1}(t)S^{*}_{n-1}(t)\right],$$
$$\rho_{24}(t)=\sum_{n=1}^{\infty}p_n\left[(\rho_{34}U_{22,n}+\rho_{24}U_{23,n})\left(2nS_{n-1}(t)+1\right)^{*}+\right.$$
$$\left.+(\rho_{12}+\rho_{13})(n+1)S_{n+1}(t)S^{*}_{n-1}(t)\right],$$
$$\rho_{44}(t)=1-\rho_{11}(t)-\rho_{22}(t)-\rho_{33}(t).$$

Here,

$$U_{23,n}(t)=\frac{\mathrm{Exp}\left[-i\frac{(\alpha+\theta_n)}{2}t\right]}{4\theta_n}\left((1-\mathrm{Exp}[i\theta_n t])\alpha-\right.$$
$$\left.-2\theta_n\mathrm{Exp}\left[i\frac{(3\alpha+\theta_n)t}{2}\right]+\theta_n(1+\mathrm{Exp}[i\theta_n t])\right),$$
$$U_{22,n}=\frac{\mathrm{Exp}\left[-i\frac{(\alpha+\theta_n)}{2}t\right]}{4\theta_n}\left((1-\mathrm{Exp}[i\theta_n t])\alpha+\right.$$
$$\left.+2\theta_n\mathrm{Exp}\left[i\frac{(3\alpha+\theta_n)t}{2}\right]+\theta_n(1+\mathrm{Exp}[i\theta_n t])\right),$$
$$Q_n=\frac{A_n}{\lambda_n},\quad S_n=\frac{B_n}{\theta_n},$$
$$A_n=\exp\left[-i\frac{g\alpha}{2}t\right]\left\{\cos\left(\frac{g\theta_n}{2}t\right)+i\frac{\alpha}{\theta_n}\sin\left(\frac{g\theta_n}{2}t\right)\right\}-1,$$
$$B_n=\exp\left[-i\frac{g}{2}(\alpha+\theta_n)t\right]\left[1-\exp(ig\theta_n t)\right],$$

and

$$\lambda_n=2(2n+1),\quad \theta_n=\sqrt{8(2n+1)+\alpha^2}.$$

The initial values of the elements of the two-qubit density matrix for the coherent initial state of qubits of Eq. (3), the thermal state of Eq. (4), and the entangled states of Eqs. (5) and (6) are given as follows:

$$\rho_{11}=\sin\theta_A\sin\theta_B,\quad \rho_{12}=\rho_{13}=\rho_{14}=\rho_{24}=\rho_{34}=0,$$
$$\rho_{44}(0)=\cos\theta_A\cos\theta_B,\quad \rho_{23}(0)=\cos\theta_A\sin\theta_B,$$
$$\rho_{32}(0)=\cos\theta\sin\theta,\quad \rho_{33}(0)=\sin^2\theta;$$
$$\rho_{12}=\rho_{13}=\rho_{14}=\rho_{23}=\rho_{24}=\rho_{34}=0,$$
$$\rho_{11}=\lambda^2,\quad \rho_{22}=\rho_{33}=\lambda(1-\lambda),\quad \rho_{44}=(1-\lambda)^2;$$
$$\rho_{11}=\rho_{12}=\rho_{13}=\rho_{14}=\rho_{24}=\rho_{34}=0,$$
$$\rho_{22}(0)=\cos^2\theta,\quad \rho_{23}(0)=\cos\theta\sin\theta,$$
$$\rho_{32}(0)=\cos\theta\sin\theta,\quad \rho_{33}(0)=\sin^2\theta$$

and

$$\rho_{11}(0)=\sin^2\theta,\quad \rho_{44}(0)=\cos^2\theta,\quad \rho_{23}(0)=\cos\theta\sin\theta,$$
$$\rho_{12}=\rho_{13}=\rho_{14}=\rho_{22}=\rho_{24}=\rho_{33}=\rho_{34}=0.$$

The two-qubit matrix for Eq. (10), partially transposed over the variables of one qubit, has the form

$$\rho_A^T(t)=\begin{pmatrix}\rho_{11} & \rho_{12} & \rho_{13} & \rho^{*}_{23}\\ \rho^{*}_{12} & \rho_{22} & \rho^{*}_{14} & \rho_{24}\\ \rho^{*}_{13} & \rho_{14} & \rho_{33} & \rho_{34}\\ \rho_{23} & \rho^{*}_{24} & \rho^{*}_{34} & \rho_{34}\end{pmatrix}. \tag{11}$$

We have revealed explicit equations for the eigenvalues of the matrix of Eq. (11) partially transposed over the variables of one qubit. In this work, these equations are not given, because of their extreme cumbersomeness. In this case, all four eigenvalues can take negative values and, accordingly, must be considered in the sum of Eq. (9). The results of the computer modeling of the negativity parameter are presented in Figs. 1–6.

2. Results and discussion

Figure 1 presents the negativity as a function of dimensionless time gt for the coherent initial state of Eq. (3) with $\theta_1=\pi/4$, $\theta_2=-\pi/4$, and the incoherent initial state of qubits of the form $|+,-\rangle$. The parameter of the dipole–dipole interaction of qubits is chosen as $\alpha=0,1$. The average number of photons is $\bar{n}=10$. The figure clearly shows that the initial coherence of qubits substantially increases the degree of entanglement of qubits, induced by the resonator thermal field, compared with the initial states without coherence. Figure 2 presents the negativity as a function of dimensionless time for the coherent initial state of Eq. (3) with $\theta_1=\pi/4$, $\theta_2=-\pi/4$, and the incoherent initial state of qubits of the form $|+,-\rangle$. The parameter of the dipole–dipole interaction of qubits is chosen as $\alpha=0,1$. The average number of photons is $\bar{n}=10$. Figures 1 and 2 demonstrate that with increasing thermal field intensity, the maximum degree of qubit entanglement decreases much more strongly for noncoherent initial states.

Figure 3 presents negativity as a function of dimensionless time for the coherent initial state of Eq. (3) with $\theta_1=\pi/4$, $\theta_2=-\pi/4$ in the absence of dipole–dipole interaction of qubits (dashed line) and in the presence of such interaction in the case of $\alpha=0,1$ (solid line). The average number of photons is $\bar{n}=10$. The figure reveals that in the case of an initial coherent state, the degree of entanglement of qubits sharply increases only in the presence of dipole–dipole interaction of qubits. Thus, the initial coherence of qubit states and dipole–dipole interaction can be simultaneously used to effectively control the de-

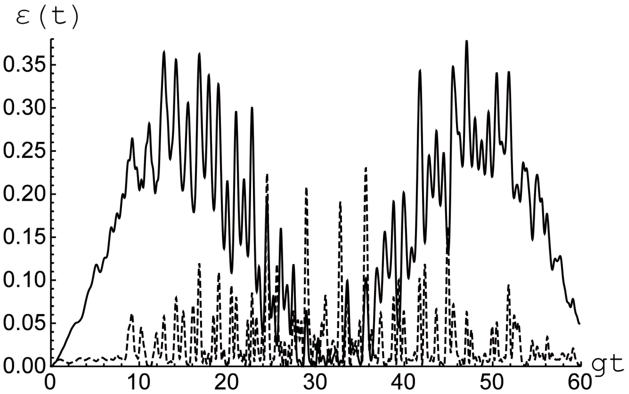

Fig. 1. Negativity as a function of dimensionless time gt for the coherent state (3) with $\theta_1 = \pi/4$, $\theta_2 = -\pi/4$ (solid line) and the incoherent state $|+,-\rangle$ (dashed line). Dipole-dipole communication parameter $\alpha = 0{,}1$. Average number of thermal photons $\bar{n} = 10$

Рис. 1. Отрицательность как функция безразмерного времени gt для когерентного состояния (3) с $\theta_1 = \pi/4$, $\theta_2 = -\pi/4$ (сплошная линия) и некогерентного состояния $|+,-\rangle$ (штриховая линия). Параметр диполь-дипольного взаимодействия $\alpha = 0{,}1$. Среднее число тепловых фотонов $\bar{n} = 10$

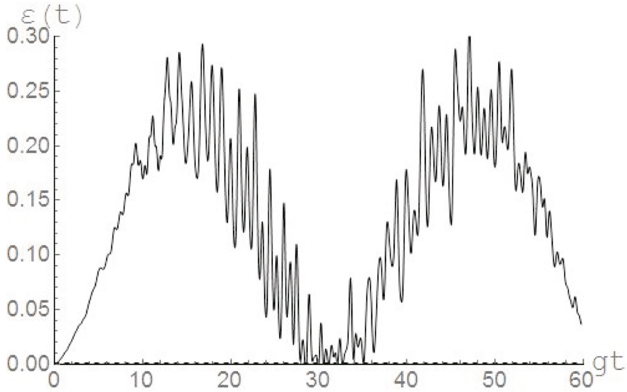

Fig. 2. Negativity as a function of dimensionless time gt for the coherent state (3) with $\theta_1 = \pi/4$, $\theta_2 = -\pi/4$ (solid line) and incoherent state $|+,-\rangle$ (dashed line). Dipole-dipole communication parameter $\alpha = 0{,}1$. Average number of thermal photons $\bar{n} = 20$

Рис. 2. Отрицательность как функция безразмерного времени gt для когерентного состояния (3) с $\theta_1 = \pi/4$, $\theta_2 = -\pi/4$ (сплошная линия) и некогерентного состояния $|+,-\rangle$ (штриховая линия). Параметр диполь-дипольного взаимодействия $\alpha = 0{,}1$. Среднее число тепловых фотонов $\bar{n} = 20$

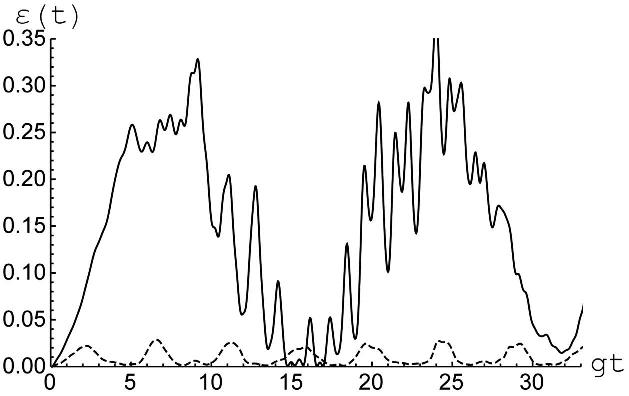

Fig. 3. Negativity as a function of dimensionless time gt for the coherent state (3) with $\theta_1 = \pi/4$, $\theta_2 = -\pi/4$. The dipole-dipole interaction parameter $\alpha = 0$ (dashed line) and $\alpha = 0{,}2$ (solid line). Average number of thermal photons $\bar{n} = 10$

Рис. 3. Отрицательность как функция безразмерного времени gt для когерентного состояния (3) с $\theta_1 = \pi/4$, $\theta_2 = -\pi/4$. Параметр диполь-дипольного взаимодействия $\alpha = 0$ (штриховая линия) и $\alpha = 0{,}2$ (сплошная линия). Среднее число тепловых фотонов $\bar{n} = 10$

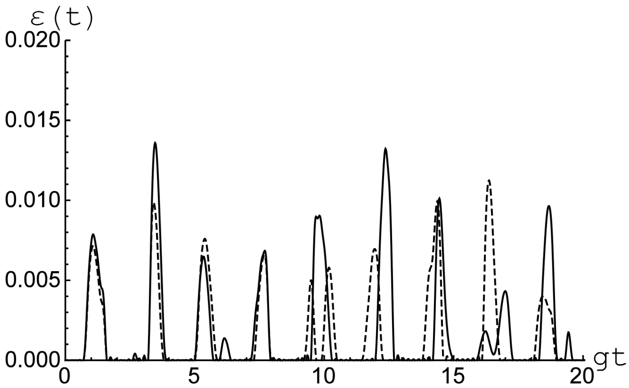

Fig. 4. Negativity as a function of dimensionless time gt for the mixed state of qubits (4) with $\lambda = 0{,}005$. Dipole-dipole interaction parameter $\alpha = 0$ (solid line) and $\alpha = 0{,}5$ (dashed line). Average number of thermal photons $\bar{n} = 0{,}5$

Рис. 4. Отрицательность как функция безразмерного времени gt для смешанного состояния кубитов (4) с $\lambda = 0{,}005$. Параметр диполь-дипольного взаимодействия $\alpha = 0$ (сплошная линия) и $\alpha = 0{,}5$ (штриховая линия). Среднее число тепловых фотонов $\bar{n} = 0{,}5$

gree of their entanglement. This effect can be used to implement effective schemes for using entangled states in the physics of quantum computing. Figure 4 demonstrates negativity as a function of dimensionless time for a chaotic thermal initial distribution of qubits by the energy level of Eq. (4) with $\lambda = 0{,}005$. Even for a chaotic initial state of qubits, the chaotic thermal field of the resonator can induce qubit entanglement. For this state, the maximum degree of entanglement of qubits is registered when $\lambda = 0$ (i.e., for the state $|-,-\rangle$). In this case, as shown in Fig. 4, the inclusion of dipole–dipole interaction increases the maximum degree of entanglement of qubits. For states with $\lambda > 0{,}01$, the entanglement of qubits during their evolution in the thermal resonator does not occur. Figures 5 and 6 show the negativity as a function of dimensionless time for the Bell entangled states of qubits of Eqs. (5) and (6), respectively, selecting in both cases $\theta = \pi/4$. The dashed and solid lines in the figures correspond to the model without dipole–dipole interaction of qubits and the model with dipole-coupled qubits in the case of $\alpha = 1$, respectively. The average number of photons is $\bar{n} = 1$. The figures clearly demonstrate that in the case of initial entangled states of qubits, the inclusion of dipole–dipole interaction does not affect the nature of

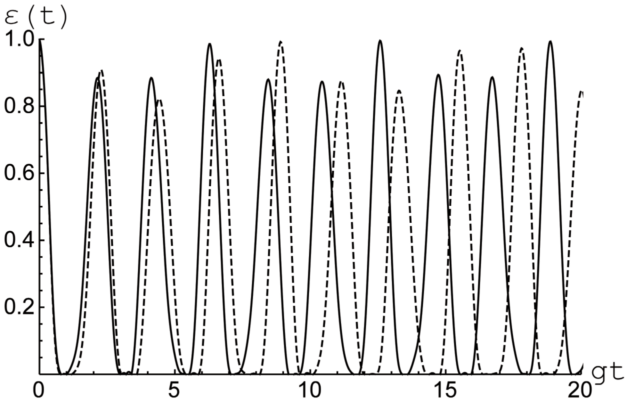

Fig. 5. Negativity as a function of dimensionless time gt for Bell entangled (5) with $\theta = \pi/4$. Dipole-dipole interaction parameter $\alpha = 0$ (dashed line) and $\alpha = 1$ (solid line). Average number of thermal photons $\bar{n} = 1$

Рис. 5. Отрицательность как функция безразмерного времени gt для белловского перепутанного (5) с $\theta = \pi/4$. Параметр диполь-дипольного взаимодействия $\alpha = 0$ (штриховая линия) и $\alpha = 1$ (сплошная линия). Среднее число тепловых фотонов $\bar{n} = 1$

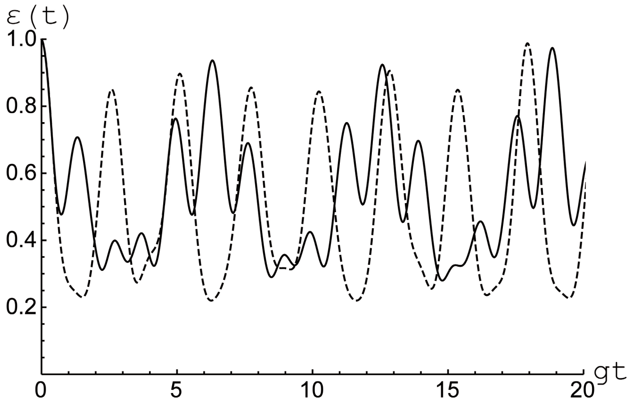

Fig. 6. Negativity as a function of dimensionless time gt for Bell entangled (6) with $\theta = \pi/4$. Dipole-dipole interaction parameter $\alpha = 0$ (dashed line) and $\alpha = 1$ (solid line). Average number of thermal photons $\bar{n} = 1$

Рис. 6. Отрицательность как функция безразмерного времени gt для белловского перепутанного (6) с $\theta = \pi/4$. Параметр диполь-дипольного взаимодействия $\alpha = 0$ (штриховая линия) и $\alpha = 1$ (сплошная линия). Среднее число тепловых фотонов $\bar{n} = 1$

the negativity behavior or the maximum values of the degree of entanglement of qubits.

Conclusion

In this study, we revealed an exact solution to the quantum equation for the evolution of a system of two dipole-coupled superconducting qubits interacting with the quantum thermal electromagnetic field mode of an ideal resonator for coherent, chaotic, and entangled initial states of Bell-type qubits. On the basis of the exact solution of the evolution equation, the qubit entanglement parameter (negativity) could be calculated in analytical form. The results of computer modeling of negativity for the initial coherent state of qubits show that when analyzing the dipole–dipole interaction, considering the initial coherence of the states of qubits leads to a substantial increase in the maximum degree of their entanglement. The thermal field of the resonator was found to even induce the entanglement of qubits in the chaotic initial state. These effects can be used to effectively control the degree of entanglement of qubits, which is necessary for quantum information processing. If the initial states of qubits are entangled, including dipole–dipole interaction does not substantially affect the entanglement of qubits.

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Тепловое перепутывание в двухатомной модели Тависса – Каммингса с учетом диполь-дипольного взаимодействия

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Аннотация – **Обоснование.** Интерес к изучению перепутанных состояний систем естественных и искусственных атомов (кубитов), взаимодействующих с выделенными модами микроволновых резонаторов, связан с их использованием в качестве логических элементов квантовых компьютеров. При этом важнейшей задачей физики квантовых вычислений является выбор наиболее эффективных механизмов контроля и управления перепутанными состояниями кубитов в таких устройствах. **Цель.** В работе исследована динамика перепутывания двух дипольно-связанных сверхпроводящих джозефсоновских кубитов, индуцированного тепловым шумом копланарного резонатора, для различных начальных состояний кубитов. **Методы.** На основе точного решения квантового уравнения Лиувилля для полной матрицы плотности рассматриваемой системы найдено временное поведение параметра перепутывания кубитов (отрицательности) для хаотических тепловых, чистых сепарабельных и перепутанных начальных состояний кубитов. **Результаты.** Показано, что перепутывание кубитов, индуцированное тепловым шумом резонатора, возможно как для хаотического теплового состояния, так и сепарабельных состояний кубитов, за исключением случая, когда оба кубита возбуждены. Установлено также, что для малых значений параметра диполь-дипольного взаимодействия учет такого взаимодействия приводит к увеличению степени перепутывания. Для значений параметра диполь-дипольного взаимодействия, больших некоторого предельного значения, имеет место обратный эффект. Найдено, что для перепутанных начальных состояний кубитов включение прямого взаимодействия слабо влияет на динамику перепутывания. Показано, что начальная когерентность состояний кубитов может приводить к существенному увеличению степени их перепутывания при наличии диполь-дипольного взаимодействия. **Заключение.** Диполь-дипольное взаимодействие может выступать в качестве эффективного механизма контроля и управления перепутаванием кубитов.

Ключевые слова – сверхпроводящие кубиты; копланарный резонатор; тепловое поле; диполь-дипольное взаимодействие; перепутывание; отрицательность.

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