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Survival of the strongest in a sequential truel

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**Abstract.** Sequential truel is a duel-like game between three players. Each player is endowed with his own marksmanship. At each turn, a player whose turn is to shoot can target any of the remaining alive opponents or abstain from shooting. The game ends when there is only one player alive or when all alive players abstained from shooting one after another. The single survivor obtains the “survivor” prize 1, while the payoff of all other players is 0. In the case the truel ends due to “inactivity”, all the players receive the payoff of 0. It is a deeply studied game with paradoxical finding that the weakest player has the highest probability of surviving in many settings, especially when the player can abstain from shooting. Here we present an explicit construction that contradicts this finding. There exists a mixed strategy subgame perfect equilibrium in which the strongest player has the highest probability of survival. This equilibrium exists for a specific order of play, in which the two stronger opponents act before the weakest one. When it exists, there are multiple subgame-perfect equilibria, including the existing stationary construction, in which two stronger opponents target each other.

**Keywords:** sequential truel, subgame perfect equilibrium, survival of the weakest.

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## 1. INTRODUCTION

A truel is a duel-like competition between three players, in which the players fight for survival or for a prize by targeting (shooting at) opponents. Only one opponent can be targeted at a time. Originally, the sequential truel appeared as a brain teaser and has been part of many collections of mathematical puzzles (Phillips, 1937; Kinnaird, 1946; Gardner, 1966; Mosteller, 1987). The problem statement can be written as follows. Each of the three players is endowed with marksmanship — the probability of eliminating the opponent with a shot. Three players are different, with marksmanships of 0.3, 0.5, and 1 (or 0.5, 0.8, and 1). Whom should they target? Who wins? These problems have a paradoxical answer — the weakest player has the highest chance of surviving while the strongest player (a sure shot) has the lowest survival odds. There is also an extra layer to the problem — the weakest player might do even better by intentionally abstaining, e.g., by shooting in the ground or in the air.

M. Shubik (Shubik, 1964, 43–46) coined the term *truel* and described it as an example of a game in which the pursuit of individual goals leads to a paradoxical result. D. Kilgour (Kilgour, 1975) initiated a game-theoretic analysis of sequential truels as infinitely repeated games with and without abstentions. He considered undominated Nash equilibria in stationary strategies and showed that if the players marksmanships are different from each other, then the equilibrium is essentially unique, and for many values of the parameters the weakest player has the strongest survival odds. This happens because for two stronger opponents it is optimal to target each other, while the weakest player can simply stay aside and wait until one

of the stronger opponents is eliminated. Many variations of truels have been considered: sequential truels with arbitrary duel values (Kilgour, 1978), simultaneous truels (Kilgour, 1972), truels with random order of shooting, finitely many rounds or bullets, possibilities for cooperation, target-undetectable or silent truels (Kilgour, Brams, 1997; Bossert, Brams, Kilgour, 2002). P. Amengual and R. Toral (Amengual, Toral, 2006) analyse different versions of the sequential game, and analyse them as Markov chains with three absorbing states (i.e. the possible victories of each of the players). A qualitatively robust prediction appears to follow from existing results: whenever there is a fight, the two strongest opponents shoot at each other, and, unless the marksmanships are dramatically different, the weakest player has the highest survival odds.

R. Shubik (Shubik, 1954) used a three-person duel to model ‘what sort of individual is best suited to survive, when every man acts for himself and by himself’. He pointed out immediate applications of truels to the analysis of elections, political competition, and multi-country interaction, arguing that a weaker country can benefit if it gets itself in the middle of a non-cooperative conflict between two stronger countries. Truels and  $N$ -els were used for modeling decisions in intense conflict situations (Cole, Phillips, Hartman, 1977), evolutionary biology to explain persistent extensive variation in competitive skills (Archetti, 2012), dynamic targeted competition, where actions can be directed at a particular opponent (Dubovik, Parakhonyak, 2014), negative advertising, especially in political campaigning (Skaperdas, Grofman, 1995; Richman, 2020). (Toral, Amengual, 2005) consider a variant of a truel — a game of opinion making or dynamic persuasion, in which players try to convince the others in their opinion on some matter and so all players are active until the consensus is reached. In all these applications the central feature is the advantage of the weakest player. Recently, in the article M. Wegener and E. Mutlu (Wegener, Mutlu, 2021) offered a model of network formation and evolution, in which players of three possible types (marksmanships) engage in truels with their neighbors. The authors assume that players target the strongest among the remaining opponents but not of the similar type, and show that the evolutionary pressures over network actually lead to strongest types surviving more often.

In this paper we challenge the existing findings regarding the survival of the weakest. We present a novel subgame-perfect equilibrium construction for a sequential truel in which the strongest player has the largest survival odds. We depart from the existing literature in three key elements: 1) we allow players to use non-stationary (and non-Markovian) strategies; 2) we consider mixed strategies; 3) we explicitly define the conditions and outcomes for a ‘peace’ scenario — the game ends if all the three players abstain in a row. This equilibrium exists for a specific order of play, in which the two stronger opponents act before the weakest one. When it exists, there are multiple subgame-perfect equilibria including the existing stationary construction in which two stronger opponents target each other.

In equilibrium, the strongest opponent, player  $\gamma$  abstains if the middle-skilled player  $\beta$  has not targeted  $\gamma$  previously; player  $\beta$  abstains if the weaker player  $\alpha$  targeted  $\gamma$  in the previous round, and randomizes between abstaining and targeting  $\gamma$ , if  $\alpha$  shot at  $\beta$ . If both stronger players abstain, the weaker player  $\alpha$  has to target someone and randomizes between the opponents. The player  $\alpha$ ’s incentives to target  $\beta$  are provided by the expectation that  $\beta$  then targets  $\gamma$  with some positive probability, in which case  $\alpha$  ends up in a duel shooting first. In turn, randomization by  $\alpha$  is needed to provide incentives for  $\gamma$  to abstain. In essence, the two stronger players tacitly collude to force  $\alpha$  to target someone. The strongest player benefits when  $\alpha$  targets  $\beta$ , which can happen in a subgame-perfect non-stationary equilibrium!

## 2. SEQUENTIAL TRUEL

There are three players competing for being the sole survivor (winner) in a sequential duel-like game. At the beginning of the truel all players are alive. At each turn, a player whose turn is to shoot can target any of the remaining alive opponents or abstain from shooting (delope, e.g. shoot at the air or shoot at the ground). A targeted player is eliminated with probability equal to the marksmanship of the shooter. All the players observe who was the intended target and what is the outcome of the shot. The order of shooting is chosen in advance, and once and for all. No player can have two turns while some other (alive) player had none. Eliminated players do not shoot. The truel ends when all but one player was eliminated or when all alive players abstained from shooting one after another. The single survivor obtains the ‘survivor’ prize, the payoff to which is normalized to 1, while the payoff of all the other players is 0. In the case the truel ends due to ‘inactivity’, all the players receive the payoff of 0. The solution concept is subgame perfect equilibrium (SPE).

In our game-theoretic analysis we consider not only pure strategies for players that specify a particular target (or abstaining) after any history that can arise but also mixed strategies, where each player can randomize

over its pure strategies and, in particular, randomly choose its target (or abstain) with some probabilities when called to act. The payoff from a mixed strategy is an expectation of the profits from pure strategies<sup>1</sup>.

We refer to the players by their marksmanship,  $\alpha$ ,  $\beta$ ,  $\gamma$ , and suppose that  $\alpha < \beta < \gamma$ ,  $\gamma = 1$ . Let  $\alpha^* = 1 - \alpha$  and  $\beta^* = 1 - \beta$  be the probabilities of surviving a shot from a respective player. Denote by  $seq$  the order (sequence) of shooting. For instance, if  $seq = \beta\gamma\alpha$ , then player  $\beta$  shoots first, then player  $\gamma$  shoots (if alive), then player  $\alpha$ , then again player  $\beta$  and so on. We write  $seq(k)$  with  $k = 0, 1$  or  $2$  to highlight the number of successive abstentions prior to the first player in  $seq$  shooting. Finally, let  $y_i^{seq}$  and  $z_i^{seq}$  be, respectively, the lowest and the highest SPE payoffs to player  $i$  if the sequence of shooting is  $seq$ . For instance,  $y_\alpha^{\alpha\beta\gamma(1)}$  is the minimal SPE payoff of player  $\alpha$  for the sequence  $\alpha\beta\gamma$  for the subgame following one abstention (by player  $\gamma$ ).

Some of our parametric assumptions, such as  $\gamma = 1$ , unequal marksmanships, and 0 payoffs to the ‘peace’ outcome, are made to simplify the exposition. For example, all results (with slight modification in statements) are true for  $\gamma$  in small neighbourhood of 1, but will require more technical details.

Some other parametric assumptions, such as three abstentions mean ‘peace’ are needed to complete specification of the game, which in turn allows for analysis of mixed-strategy equilibria. It would be interesting to know, how equilibria are affected when the number of abstentions that terminate the game or the ‘peace’ payoff vary, but this is left for future analysis.

There are six possible sequences of players, which can be divided in two groups: either  $\beta$  acts immediately after  $\alpha$  or vice versa. Since our main goal is to present an explicit construction of equilibria in which the strongest player obtains high payoffs we will concentrate on the first case. Once we present the new equilibrium for sequence  $\beta\gamma\alpha$  in Section 6, as a corollary, we obtain a similar construction for sequence  $\gamma\beta\alpha$ , in which  $\alpha$  acts immediately after  $\beta$ . Since we need to stimulate player  $\alpha$  to shoot in our equilibrium, it is crucial that  $\alpha$  must be the last player to shoot.

**Lemma 1.** *The minimal and maximal SPE payoffs of players  $\alpha$  and  $\beta$  satisfy the following inequalities*

$$y_\alpha^{\alpha\cdots} \geq \alpha y_\alpha^{\beta\alpha} = \alpha \frac{\beta^*\alpha}{1-\alpha^*\beta^*}, y_\beta^{\beta\cdots} \geq \beta y_\beta^{\alpha\beta} = \beta \frac{\alpha^*\beta}{1-\alpha^*\beta^*}, z_\alpha^{\cdots} \leq y_\alpha^{\alpha\beta} = \frac{\alpha}{1-\alpha^*\beta^*}, z_\beta^{\cdots} \leq y_\beta^{\beta\alpha} = \frac{\beta}{1-\alpha^*\beta^*}.$$

(Dots here denote any of the appropriate order of shooting.)

**P r o o f.** The minimal SPE payoff of player  $\alpha$  (or  $\beta$ ) cannot be lower than what the player  $\alpha$  (or  $\beta$ ) can obtain by targeting  $\gamma$  and, if successful, ending up in a duel shooting second. Clearly, there would be no abstentions in any duel. Similarly, the maximal possible payoff a player can get cannot be higher than that from a duel against the weakest of the opponents shooting first. ■

### 3. PLAYER $\gamma$ STRATEGIES

**Lemma 2.** *In any SPE player  $\gamma$  does not target  $\alpha$  if  $\beta$  is alive. Player  $\gamma$ ’s minimal SPE payoff for any sequence of moves in which  $\gamma$  shoots first is bounded from below:  $y_\gamma^{\gamma\cdots} \geq \alpha^*$ .*

**P r o o f.** If player  $\gamma$  shoots at another player, the target is eliminated and  $\gamma$  ends up in a duel with the remaining player where that player shoots first. A duel with the weakest opponent is preferred. Thus, player  $\gamma$  can guarantee payoff of  $\alpha^*$  by targeting  $\beta$ . ■

Can  $\gamma$  obtain a higher payoff than  $\alpha^*$  in equilibrium when  $\gamma$  is the first to act? For this to happen,  $\gamma$  has to shoot in the air and one of the opponents has to target the other one with positive probability in the future, so that  $\gamma$  reaches a duel shooting first. Otherwise, if players  $\alpha$  and  $\beta$  target  $\gamma$  or abstain each time they act, player  $\gamma$  has to target someone to end up in a duel (shooting second) to obtain positive payoff, which is at most  $\alpha^*$ .

### 4. PLAYER $\beta$ STRATEGIES

So, from here on we consider sequences of players in which  $\beta$  shoots immediately after  $\alpha$ .

<sup>1</sup> For example, consider two possible strategies of the player  $\alpha$ : always target  $\beta$  (if not eliminated, otherwise target  $\gamma$ ) and always target  $\gamma$  (if not eliminated, otherwise target  $\beta$ ). Fix some strategies of the players  $\beta$  and  $\gamma$ . Payoffs in the game (expected probability of being a sole survivor) are well defined once all the strategies are specified, and let  $\pi_{\alpha,\beta}$  and  $\pi_{\alpha,\gamma}$  be the payoffs to the player  $\alpha$  from the two considered pure strategies. Suppose that the player  $\alpha$  considers a mixed strategy: flip a coin and if heads target  $\beta$  always and if tails target  $\gamma$  always. Then the player  $\alpha$ ’s payoff from this strategy is simply  $0.5 \pi_{\alpha,\beta} + 0.5 \pi_{\alpha,\gamma}$ . Of course, more complex mixed strategies can be considered, e.g., those that implement randomization at any time a player acts. In the example, the player  $\alpha$  can flip a coin between two players (if available) any time it has a shot. In equilibrium, a player’s mixed strategy has to give the maximal and identical payoff to the pure strategies it randomizes over.

**Lemma 3.** *For all sequences of players in which player  $\beta$  shoots immediately after  $\alpha$ , player  $\beta$  never targets  $\alpha$ .*

**P r o o f.** Suppose not, and that there exists an SPE in which  $\beta$  targets  $\alpha$ . Consider a subgame where this happens. If  $\beta$  eliminates  $\alpha$ , then  $\gamma$  eliminates  $\beta$  in the next round with probability 1. Therefore,  $\beta$  has to expect a substantial continuation payoff  $u_\beta$  following a miss:  $\beta^* u_\beta \geq y_\beta^{\beta\gamma\alpha}$ , or, from Lemma 1

$$u_\beta \geq \frac{\beta}{\beta^*} \times \frac{\alpha^* \beta}{1 - \alpha^* \beta^*}. \quad (1)$$

Let us focus on  $seq = \gamma\alpha\beta$  and the equilibrium in which  $\beta$  obtains  $u_\beta$ . As follows from Lemma 2, player  $\gamma$  has to abstain with positive probability for  $\beta$  to get a positive payoff as otherwise  $\gamma$  eliminates  $\beta$ . Player  $\gamma$  then receives at least  $\alpha^*$ , while  $\alpha$  who moves next obtains at least  $y_\alpha^{\alpha\beta\gamma}$ . Combined, the sum of payoffs of all three players has to be at least  $y_\alpha^{\alpha\beta\gamma} + u_\beta + \alpha^*$ . From Lemma 1 and inequality (1), this sum exceeds

$$\alpha \frac{\beta^* - \alpha^* \beta^*}{1 - \alpha^* \beta^*} + \frac{\beta}{\beta^*} \frac{\alpha^* \beta}{1 - \alpha^* \beta^*} + \alpha^* = \frac{\alpha^* \beta + \alpha \beta^* \beta^*}{\beta^* (1 - \alpha^* \beta^*)} = \frac{\beta - \alpha \beta + \alpha - 2\alpha \beta + \alpha \beta^2}{\alpha + \beta - 2\alpha \beta - \beta^2 + \alpha \beta^2} > 1,$$

since  $\beta^2 > \alpha \beta$  — that is a contradiction.

Thus, we have shown that for considered sequences of players in any SPE neither  $\beta$  nor  $\gamma$  target  $\alpha$ . This is consistent with previous findings (Shubik, 1954; Kilgour, 1975; Toral, Amengual, 2005), yet here we have derived it without assumptions of stationarity, pure strategies, or necessity of targeting someone.

## 5. PLAYER $\alpha$ STRATEGIES

We explore first, what player  $\alpha$  can get by abstaining.

**Lemma 4.** *For any SPE, for any subgame in which player  $\alpha$  has a chance to abstain without ending the game, we have (here  $k = 0$  or  $1$ )  $y_\alpha^{\alpha\beta\gamma(k)} \geq \alpha$ .*

*In addition, for sequences  $\gamma\alpha\beta(0)$  and  $\gamma\alpha\beta(2)$  player  $\gamma$  targets  $\beta$ , while for sequence  $\beta\gamma\alpha(1)$  player  $\beta$  targets  $\gamma$ .*

**P r o o f.** Suppose player  $\alpha$  abstains. If  $\beta$  also abstains, then  $\gamma$  targets  $\beta$  (who obtains payoff 0), while  $\alpha$  ends up in the duel with  $\gamma$  shooting first, getting payoff  $\alpha$ . Thus,  $\beta$  will not abstain, and target  $\gamma$  instead.

If  $\beta$  eliminates  $\gamma$ , then  $\alpha$  obtains  $y_\alpha^{\alpha\beta} = \alpha / (1 - \alpha^* \beta^*) > \alpha$ . If  $\beta$  misses and  $\gamma$  targets  $\beta$ , then  $\alpha$  obtains payoff  $\alpha$ . Finally, if  $\beta$  misses and  $\gamma$  abstains, we come to the subgame in which  $\alpha$  can again abstain, and, as above, either guarantee herself at least  $\alpha$  or reach a subgame in which she can abstain, and so on. In expectation over all possible outcomes, player  $\alpha$  obtains at least  $\alpha$  in any SPE whenever she (he) can abstain.

Player  $\gamma$  does not abstain if  $\alpha$  can abstain next round, and thus  $\gamma$  targets  $\beta$  for sequence  $\gamma\alpha\beta(0)$ . Indeed, since  $\alpha$  can abstain and guarantee herself payoff  $\alpha$ , by targeting  $\beta$ , player  $\gamma$  obtains  $\alpha^*$ . If  $\gamma$  abstains, then  $\beta$  obtains a positive payoff, in this case, what remains to  $\gamma$  is strictly smaller than  $\alpha^*$ .

In turn, if player  $\beta$  knows that  $\alpha$  is able to abstain, she will expect that  $\alpha$  abstains and then  $\gamma$  targets  $\beta$ . So, player  $\beta$  should target  $\gamma$  and not abstain in this case. ■

Thus, having an option to abstain allows player  $\alpha$  to guarantee herself a decent payoff as the other players target each other. Will  $\alpha$  actually abstain in equilibrium? Suppose that  $\beta$  and  $\gamma$  target each other also when  $\alpha$  targets one of them (as in (Kilgour, 1975)). Then, for  $\alpha$  it is better to target  $\gamma$ , since the duel with the weaker of the two opponents is preferred. By targeting  $\gamma$ , player  $\alpha$  obtains payoff  $\alpha y_\alpha^{\beta\alpha} + \alpha^* [\beta y_\alpha^{\alpha\beta} + \beta^* \alpha]$ , and by abstaining she gets  $\beta y_\alpha^{\alpha\beta} + \beta^* \alpha$ .

Therefore, targeting  $\gamma$  is better for  $\alpha$ , if

$$y_\alpha^{\beta\alpha} > \beta y_\alpha^{\alpha\beta} + \beta^* \alpha; \quad \frac{\beta^* \alpha}{1 - \alpha^* \beta^*} > \beta \frac{\alpha}{1 - \alpha^* \beta^*} + \beta^* \alpha;$$

$$\beta^* \alpha > \alpha(\beta + \beta^* - \alpha^* \beta^* \beta^*); \quad \beta^* > 1 - \alpha^* \beta^* \beta^*, \quad \beta < \alpha^* \beta^* \beta^*.$$

We have shown the following lemma.

**Lemma 5.** *A subgame perfect equilibrium in which neither  $\beta$  nor  $\gamma$  abstain exists. In this equilibrium, player  $\beta$  targets  $\gamma$ , player  $\gamma$  targets  $\beta$ , and player  $\alpha$  targets  $\gamma$  if  $\beta < \alpha^* \beta^* \beta^*$  and abstains if  $\beta > \alpha^* \beta^* \beta^*$ .*

It is trivial to check that the described strategies indeed form an SPE. Note that if  $\beta > \beta^* \beta^*$  or  $\beta > 0.5(3 - \sqrt{5}) \approx 0.3820$ , player  $\alpha$  abstains no matter what  $\alpha$  is.

## 6. NOVEL SPE CONSTRUCTION: SURVIVAL OF THE STRONGEST

Can it happen in equilibrium that players  $\beta$  or  $\gamma$  or both abstain? Consider sequence  $\beta\gamma\alpha$ . To abstain, player  $\beta$  should expect that  $\gamma$  will also abstain with high probability in the next round and that  $\alpha$  later (not necessarily immediately in round 3) will target  $\gamma$ , hoping for a duel with  $\alpha$  in which  $\beta$  shoots first. In turn, for player  $\gamma$  to abstain, player  $\gamma$  should expect that  $\alpha$  will be shooting at  $\beta$  with positive probability, as otherwise  $\gamma$ 's expected payoff is strictly below  $\alpha^*$ . For both  $\beta$  and  $\gamma$  to abstain, player  $\alpha$  has to target someone!

**Theorem (main result).** *For sequence  $\beta\gamma\alpha$  and  $\alpha > 0.5$ , there exists a mixed SPE in which players  $\beta$  and  $\gamma$  abstain with positive probabilities, while player  $\alpha$  mixes between the opponents. The equilibrium utilities  $U_i$  of the players ( $i \in \{\alpha, \beta, \gamma\}$ ) are given by the following formulas:*

$$U_\beta = \beta \frac{\alpha^* \beta}{1 - \alpha^* \beta^*}, \quad U_\gamma = \frac{q^* \alpha + q^* p \alpha^* \alpha^* \beta^*}{\alpha + q^* p \alpha^*}, \quad U_\alpha = 1 - U_\beta - U_\gamma,$$

where  $p = \alpha \beta^* / [\alpha^* (\beta - \alpha^* \beta^* \beta^*)]$ ,  $q = \alpha^* \beta$ .

**P r o o f.** Consider the following profile of strategies — a candidate for the equilibrium. Player  $\gamma$  abstains for  $seq = \gamma\alpha\beta(1)$  (following an abstention by  $\beta$  if it is the only one in a row) and targets  $\beta$  in any other subgame. Player  $\beta$  abstains in round 1 and for any subgame in which  $\alpha$  targeted and missed  $\gamma$  in the previous round. If  $\alpha$  targeted and missed  $\beta$ , then  $\beta$  mixes between abstaining and targeting  $\gamma$ . (This will provide incentives for  $\alpha$  to target  $\beta$ .) For all other subgames, player  $\beta$  targets  $\gamma$ . Finally, player  $\alpha$  mixes shooting at  $\beta$  and  $\gamma$  for  $seq = \alpha\beta\gamma(2)$ , and abstains in all other subgames.

For the transition matrix between subgames for the proposed profile of strategies see Figure. Each decision node is marked by the acting player with the number of previous abstentions in the brackets. Only the transitions following misses are shown. Horizontal arrows point to the intended target,  $\uparrow$  stands for abstention. In figure 1  $p$  and  $q$  are the probabilities with which players  $\beta$  and  $\alpha$  target  $\gamma$  in their mixed strategies.

For the suggested profile to be an SPE, the following set of conditions has to be satisfied. Here,  $U_i$  is the equilibrium utility of player  $i$ ,  $p^* = 1 - p$ , and  $q^* = 1 - q$ . By  $U_i(j)$  we denote the utility of player  $i$  using pure strategy shooting  $j$ .

$$U_\beta = \alpha^* U_\beta + \alpha \left[ q \frac{\beta}{1 - \alpha^* \beta^*} + q^* \times 0 \right] = \beta \frac{\alpha^* \beta}{1 - \alpha^* \beta^*}, \quad U_\gamma = q \left[ \alpha \times 0 + \alpha^* U_\gamma \right] + q^* \left[ \alpha \times 1 + \alpha^* p^* U_\gamma + \alpha^* p \beta^* \alpha^* \right] \geq \alpha^*,$$

$$U_\alpha(\gamma) = \alpha \times \frac{\beta^* \alpha}{1 - \alpha^* \beta^*} + \alpha^* U_\alpha = \alpha \times 0 + \alpha^* \left[ p \beta \frac{\alpha}{1 - \alpha^* \beta^*} + p \beta^* \alpha + p^* U_\alpha \right] = U_\alpha(\beta).$$

Probability  $q$  can be found from the condition for player  $\beta$  to be indifferent between targeting  $\gamma$  and abstaining:

$$\alpha \beta \frac{\alpha^* \beta}{1 - \alpha^* \beta^*} = \alpha q \frac{\beta}{1 - \alpha^* \beta^*}, \quad q = \alpha^* \beta.$$

Similarly, from the indifference condition for player  $\alpha$  we can find  $p$ :

$$U_\alpha(\gamma) = \frac{\alpha \beta^*}{1 - \alpha^* \beta^*} = \alpha^* \left[ p \beta \frac{\alpha}{1 - \alpha^* \beta^*} + p \beta^* \alpha + p^* \frac{\alpha \beta^*}{1 - \alpha^* \beta^*} \right];$$

$$\beta^* = \alpha^* (p \beta + p^* \beta^* + p \beta^* (1 - \alpha^* \beta^*)); \quad \beta^* = \alpha^* (p \beta + \beta^* - p \beta^* \alpha^* \beta^*); \quad p = \frac{\alpha \beta^*}{\alpha^* (\beta - \alpha^* \beta^* \beta^*)}.$$

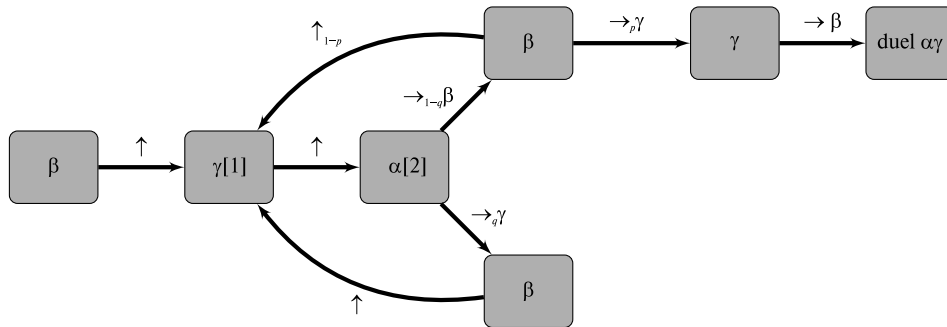


Figure. Transitions in equilibrium

Finally, we need to check that  $\gamma$  prefers abstaining from targeting  $\beta$ . From the expression for  $\gamma$  above we can compute

$$U_{\gamma} = \frac{q^* \alpha + q^* \alpha^* p \beta^* \alpha^*}{1 - \alpha^* q - \alpha^* q^* p^*} = \frac{q^* \alpha + q^* p \alpha^* \alpha^* \beta^*}{\alpha + q^* p \alpha^*}.$$

We want to show that  $U_{\gamma} > \alpha^*$ , when  $\alpha > \alpha^*$ . We have

$$\begin{aligned} (U_{\gamma} - \alpha^*)(\alpha + q^* p \alpha^*) &= q^* p \alpha^* \alpha^* (\beta^* - 1) + \alpha (q^* - \alpha^*) = \alpha (\alpha - q) - q^* p \alpha^* \alpha^* \beta = \alpha (\alpha - \alpha^* \beta) - p \alpha^* \alpha^* \beta (1 - \alpha^* \beta) = \\ &= \frac{\alpha}{\beta - \alpha^* \beta^* \beta^*} [(\alpha - \alpha^* \beta)(\beta - \alpha^* \beta^* \beta^*) - (\beta^* \alpha^*)(\beta - \beta \alpha^* \beta)]. \end{aligned}$$

If  $\beta > \alpha > 0.5$ , then  $\alpha > \alpha^*$ ,  $\beta > \beta^*$ , and  $\alpha - \alpha^* \beta > \alpha^* - \alpha^* \beta = \beta^* \alpha^*$ ,  $\beta - \alpha^* \beta^* \beta^* > \beta - \alpha^* \beta \beta$ . Therefore, the difference of products in the above expression is positive. ■

The presented equilibrium construction clearly favors the strongest player as compared to the equilibrium in Lemma 5. In this equilibrium player  $\gamma$  had to rely on both  $\beta$  and  $\alpha$  missing their shots to survive. In the new construction, the same scenario following  $\beta$  targeting  $\gamma$  occurs as well, but with probability less than one, while in a complementary event of  $\alpha$  targeting  $\beta$  and making a shot,  $\gamma$  wins the prize. The probability of this complimentary event is maximized in this equilibrium.

For a specific example, consider  $\alpha = 0.8$ ,  $\beta = 0.9$ ,  $\gamma = 1$ . Then  $q = 0.18$ ,  $p \sim 0.4454$  and players payoffs are  $U_{\alpha} \sim 0.0816$ ,  $U_{\beta} \sim 0.1653$ ,  $U_{\gamma} \sim 0.7531$ . In the equilibrium from lemma 5,  $U_{\alpha}^0 = 1 - U_{\beta}^0 - U_{\gamma}^0 \sim 0.8147$ ,  $U_{\beta}^0 = U_{\beta} \sim 0.1653$ ,  $U_{\gamma}^0 = \beta^* \alpha^* = 0.02$ .

**Corollary 1.** *If  $\alpha > 0.5$  and the sequence of players is  $\beta\gamma\alpha$ , then there exists a continuum of subgame perfect equilibria in which player  $\beta$  earns the same expected payoff as in the equilibrium in Theorem, and payoffs of players  $\alpha$  and  $\gamma$  vary:  $U_{\gamma}$  takes any value from  $\beta^* \alpha^*$  to the value from the Theorem.*

**P r o o f.** Consider the equilibrium construction in the Theorem. Player  $\beta$  in the first round can instead shoot at  $\gamma$  with some probability  $r$ . The higher is  $r$  the higher is  $U_{\alpha}$  (and lower  $U_{\gamma}$ ). For  $r = 1$  we obtain the payoffs as in Lemma 5 with  $U_{\gamma} = \beta^* \alpha^*$ . ■

**Corollary 2.** *For sequence  $\gamma\beta\alpha$  and sufficiently high  $\alpha$ , there exists an SPE with players  $\gamma$  and  $\beta$  abstaining (with probability 1), and  $\gamma$  earning the highest expected payoff.*

**P r o o f.** The equilibrium construction here is similar to the one in Theorem with one correction to account for  $\beta$  shooting immediately after  $\gamma$  and not before. Player  $\gamma$  is going to abstain if  $\alpha$  did not abstain and  $\beta$  did not target  $\gamma$  previously. Player  $\alpha$  is going to mix targets, hoping that  $\beta$  will target  $\gamma$  with positive probability. Differently from above, player  $\beta$  is going to abstain with probability 1 whenever  $\gamma$  abstained in the previous round and  $\alpha$  did not target  $\beta$  in the round before. If  $\alpha$  targets  $\beta$ ,  $\beta$  survives,  $\gamma$  abstains, then  $\beta$  is going to randomize between abstaining and shooting at  $\gamma$ . Similarly to Theorem 1, this will provide incentives for  $\alpha$  to shoot at  $\beta$ . ■

Condition  $\alpha > 0.5$  in Theorem 1 is a simple sufficient condition for the equilibrium we present. The exact condition on  $\alpha$  and  $\beta$  jointly is for the expression at the end of the proof of Theorem to be positive to guarantee that  $U_{\gamma} > \alpha^*$ . If  $\alpha < 0.5$ , when shooting first,  $\gamma$  can get payoff  $\alpha^* > 0.5$  (simply shooting  $\beta$ ) and therefore also get highest payoff among the players.

Clearly our equilibrium construction also extends for values of  $\gamma$  less than but close to 1 (while  $\gamma > \beta$ ). When  $\gamma$  is close to 1, the payoffs of all the players are similar to those in Theorem 1, and the equilibrium is supported by probabilities  $q$  and  $p$ , that make players  $\alpha$  and  $\beta$  to randomize their actions. Certainly,  $q$  and  $r$  should be adjusted depending on  $\gamma$ , but as long as they are within  $(0, 1)$  the equilibrium construction works.

With our assumptions of three abstentions leading to “peace” outcome and  $\gamma = 1$  similar mixed strategy equilibria do not exist for other sequences of play, in particular, when  $\alpha$  is the first player to act and there are no previous abstentions. If we consider a different assumption on abstentions, for instance, if the “peace” realizes once some player abstains for the second time following abstentions by every player, then the presented equilibrium construction exists for all sequences of play. Player  $\alpha$  randomizes over  $\beta$  and  $\gamma$ , and if abstains, the other two players abstain as well, forcing player  $\alpha$  to pick a target.

**Corollary 3.** *Suppose that there is no limit on how many abstentions lead to the “peace” outcome. “Peace” is reached only if all three players abstain forever. Then, for any sequences of players and sufficiently high  $\alpha$ , there exists an SPE with players  $\gamma$  and  $\beta$  abstaining (with probability 1), and  $\gamma$  earning the highest expected payoff.*

**P r o o f.** The above constructions can be amended by specifying that both players  $\beta$  and  $\gamma$  necessarily abstain if player  $\alpha$  abstains. Clearly then, abstaining forever is not optimal for player  $\alpha$ . ■

## 7. CONCLUSION

Truels and  $N$ -els are fascinating conflicts with numerous applications that bring in new insights with each new wave of attention to them. The theory of truels started with the survival of the weakest observation which was and continues to be the main driver of interest in truels. It continued with abstentions as a strategic choice which the weakest player is likely to make. An abstention is a practically feasible strategy so it has to be modeled formally. This eventually lead to a cooperative element in the analysis (Bossert, Brams, Kilgour, 2002) — players can potentially cooperate even if they are fighting for survival and only one survives. Our novel equilibrium construction can also be interpreted like that: two strongest players cooperate to force the weakest player to shoot hoping that one of them survives and be the first to shoot in the ensuing duel. In this paper we show: it is not necessary that the stronger opponents target each other in equilibrium and that other types of SPE equilibria exist, with the strongest player having largest odds of surviving and in which multiple people abstain from shooting with positive probability.

Using mixed strategies is another important element of our construction. This allows to break the curse of the strongest player, viz. being everyone's target. In our construction, using a mixed strategy by player  $\alpha$  provides the incentive for the strongest player to abstain from shooting, while using a mixed strategy by player  $\beta$  conditional on some histories of playing provides the incentive for  $\alpha$  to mix. It is easy to overlook such possibilities if the strategies are assumed to be stationary.

Many questions remain for further research. In particular, it would be valuable to characterize the whole set of achievable subgame perfect equilibrium payoffs for different values of parameters for truels. Are there any other kinds of equilibria in  $N$ -els with  $N > 3$  players?

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**Выживание сильнейшего в последовательной труэли**

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**Аннотация.** Последовательная труэль — один из видов дуэли для трех игроков. У каждого игрока — свой уровень меткости. Игроки ходят по очереди, в каждом раунде выбирая в качестве цели одного из оставшихся в живых игроков или — уклоняясь от стрельбы. Игра заканчивается, когда в живых остается только один игрок, или — при трех последовательных уклонениях от стрельбы. Единственный выживший получает выигрыш победителя, равный 1, а остальные получают 0. В случае когда дуэль заканчивается «мирным» способом, все игроки получают выигрыш, равный 0. Эта игра изучена достаточно хорошо, и большинство исследований посвящено парадоксальному выводу о том, что самый слабый игрок имеет наибольшую вероятность выживания во многих условиях, особенно когда у игроков есть возможность уклониться от стрельбы. В данной статье мы представляем конструкцию равновесия, которое опровергает данный вывод. А именно: существует смешанное равновесие, совершенное на подыграх, в котором сильнейший игрок имеет наибольшую вероятность выжить. Данное равновесие существует при определенном порядке игроков, в котором двое сильнейших стреляют перед слабейшим. Если оно существует, то имеется набор равновесий, совершенных на подыграх, включая стационарное, в котором двое сильнейших стреляют друг в друга.

**Ключевые слова:** последовательная труэль, совершенное на подыграх равновесие, выживание слабейшего.

**Классификация JEL:** C7, C73.

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