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DIFFERENTIAL EQUATIONS

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PERSONALITIES OF THE SCIENCE

TO THE EIGHTY-FIFTH ANNIVERSARY OF NIKOLAI ALEKSEEVICH IZOBOV

January 23, 2025 is the 85th anniversary of Nikolai A. Izobov, an outstanding scientist, world-known expert in the field of ordinary differential equations, academician of the National Academy of Sciences of the Republic of Belarus, professor, doctor of physical and mathematical sciences, member of the editorial board of the journal "Differential Equations", a major organizer of science and education.

Nikolai Alekseevich was born in the village of Krasyni, Liozna district, Vitebsk region. He graduated from high school with honors in 1958, and in 1965, completed his studies at the Mathematics Department of the Belarusian State University, specializing in differential equations – a field to which he dedicated his entire subsequent scientific career.. In 1966, N. A. Izobov entered the postgraduate program; and in 1967 he brilliantly defended his Ph.D. thesis under the supervision of Prof. Y. S. Bogdanov. In 1979, he defended his doctoral dissertation at Leningrad University, the abstract of that (as one of the best dissertations) was published in the journal "Mathematical Notes". In 1980, Nikolai Alekseevich was elected a corresponding member of the Academy of Sciences of the BSSR, and in 1994 – a full member of the National Academy of Sciences of Belarus for 10 years.

Since November 1980, N.A. Izobov has been working at the Institute of Mathematics of the National Academy of Sciences of Belarus at the following positions: senior researcher (1980–1986), head of the stability theory laboratory (1986–1993), head of the differential equations department (1993–2010), and chief researcher (since 2010 up to the present time). In addition, during 1996–1999, he was Head of the Department of Higher Mathematics, Faculty of Applied Mathematics, Belarusian State University. Since

1994, he headed the Expert Council on Mathematics of the Higher Attestation Commission of the Higher Attestation Commission of the Republic of Belarus.

At present, Nikolai is a member of the editorial boards of the scientific journals “Differential Equations” (in 1969–1990 he was deputy editor-in-chief of this journal), “Memoirs on Differential Equations and Mathematical Physics”, “Vesci Natsiyanalnaia Akademi nauk Belarusi. Series of Physics and Mathematics”, “Proceedings of the Institute of Mathematics”.

The main topics of Nikolai Alekseevich’s research activities are: the theory of Lyapunov characteristic indices, the theory of stability by linear approximation, linear Koppel–Conti systems, Emden–Fowler equations and linear Pfaff systems. He introduced the notions of exponential exponents and sigma exponents of a linear system, which are nowadays called Izobov exponents.

N. A. Izobov published about 250 scientific papers, including 3 monographs, one of which was published in Cambridge. More than 20 candidate and doctoral theses were prepared and defended under his supervision.

Nikolai Alekseevich was awarded the Order of Francysk Skaryna (2000), the Diploma of the Council of Ministers of the Republic of Belarus (2000), the V. M. Ignatovsky Medal of Honor of the National Academy of Sciences of Belarus (2020), the State Prize of the Republic of Belarus for the series of works “Investigation of asymptotic properties of differential and discrete systems” (2000), the Prize of the International Academic Publishing Company “Nauka/Interperiodica” for the best publication in its journals (diploma signed by the President of the Russian Academy of Sciences Y.S. Osipov (2009)), and also the prize of the National Academy of Sciences of Belarus for the series of works “Modern development of the first Lyapunov method: theory and applications” (2013).

We wish dear Nikolai Alekseevich good health, vigor, long active years of life and success in all his endeavors.

Editorial Board

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PARTIAL DIFFERENTIAL EQUATIONS

MODEL PROBLEM IN A STRIP FOR THE HYPERBOLIC DIFFERENTIAL-DIFFERENCE EQUATION

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Abstract. The paper investigates the question of the existence of a classical solution to the initial value problem with incomplete initial data on the boundary of the strip for a hyperbolic differential-difference equation. The equation contains a superposition of a differential operator and a translation operator with respect to a spatial variable that varies along the entire real axis. Using the Gelfand–Shilov operational scheme, a solution to the problem was obtained in explicit form.

Keywords: *hyperbolic equation, differential-difference equation, translation operator, initial problem, operational scheme, Fourier transform*

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1. INTRODUCTION. PROBLEM STATEMENT

The interest in the study of functional-differential and, in particular, differential-difference equations and problems for them is due to two reasons. First, for such generalizations of differential equations some methods "working well" for classical equations are inapplicable, and also there appear qualitatively new effects in the solutions, that have no place in classical cases. Secondly, such equations are encountered in a variety of applications (mechanics of a deformable solid body, processes of vortex formation and formation of complex coherent spots, modeling of crystal lattice vibrations, nonlinear optics, neural networks, etc.), including those that cannot be described by classical models of mathematical physics. Significant results in the study of problems for functional- differential equations of various classes were obtained by A. L. Skubachevskii [1, 2], V. V. Vlasov [3, 4], A. B. Muravnik [5], A. V. Razgulin [6], L. E. Rossovskii [7], V. Zh. Sakbaev [8] and other authors.

We will call according to [1] a *differential-difference* equation containing both differential operators and shift operators.

To date, problems for elliptic (both in bounded and unbounded domains) and parabolic differential-difference equations have been studied in detail. Hyperbolic differential-difference equations have been studied to a much lesser extent. In [9, 10], two-dimensional hyperbolic equations with a shift operator in the senior derivative acting on a spatial variable are considered for the first time. The purpose of this paper is to construct explicitly, using the known operational scheme [11], the solution of the model initial problem in the strip for such an equation.

Let us denote by $D = \{(x,t) : x \in \mathbb{R}, 0 < t < T\}$ the area of the coordinate plane Oxt, where $T > 0$ is a given real number, let $\overline{D} = \{(x,t) : x \in \mathbb{R}, 0 \le t \le T\}$.

We need to find the function $u(x,t) \in C^1(\overline{D}) \cap C^2(D)$, satisfying the equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = a^2 \frac{\partial^2 u(x-h,t)}{\partial x^2}, \quad (x,t) \in D, \tag{1}$$

where $a > 0, h \neq 0$ are given real numbers, and the initial condition

$$u(x,0) = 0, \quad x \in \mathbb{R}. \tag{2}$$

Definition. We will call the *classical solution of the problem* (1), (2) a function $u(x,t)$, continuous and continuously differentiable on the variables x and t in the set $\overline{D}$; twice continuously differentiable on x and t in D; satisfying at each point of the region D the relation (1); such that for each point $x_0 \in \mathbb{R}$ the limit of the function $u(x_0,t)$ at $t \to +0$ exists and is equal to zero.

2. CONSTRUCTING A SOLUTION TO THE PROBLEM

To find the solution of the problem (1), (2) according to the operational scheme [11] we apply, to equation (1) and initial condition (2) (formally), the Fourier transform on the variable x, acting according to the rule

$$\hat{u}(\xi,t) := F_x[u(x,t)] = \int_{-\infty}^{+\infty} u(x,t)e^{i\xi x}\,dx.$$

As a result, we obtain the problem in Fourier images

$$\frac{d^2\hat{u}(\xi,t)}{dt^2} + a^2\xi^2 e^{ih\xi}\hat{u}(\xi,t) = 0, \tag{3}$$

$$\hat{u}(\xi,0) = 0, \quad \xi \in \mathbb{R}. \tag{4}$$

The characteristic roots of the equation corresponding to equation (3) are determined by the formula

$$k_{1,2} = \pm ia\xi e^{(ih\xi/2)},$$

then the general solution of equation (3) has the form

$$\hat{u}(\xi,t) = C_1(\xi)\cos(a\xi e^{(ih\xi/2)}t) + C_2(\xi)\sin(a\xi e^{(ih\xi/2)}t),$$

where $C_1(\xi)$ and $C_2(\xi)$ are arbitrary constants depending on the parameter $\xi \in \mathbb{R}$. Substituting this function into the initial condition (4), we obtain $C_1(\xi) = 0$. Since problem (3), (4) is a problem with incomplete initial data, let us assume that

$$C_2(\xi) = (a\xi e^{(ih\xi/2)})^{-1}$$

and write down the final form of its solution:

$$\hat{u}(\xi,t) = \frac{\sin(a\xi e^{(ih\xi/2)}t)}{a\xi e^{(ih\xi/2)}}, \quad \xi \in \mathbb{R}.$$

Applying now the inverse Fourier transform to the found function (formally), we obtain by analogy with [12] the following relations:

$$\begin{aligned}
F_\xi^{-1}[\hat{u}(\xi,t)] &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{u}(\xi,t)e^{-ix\xi}\,d\xi = \\
&= \frac{1}{2\pi a}\int_{-\infty}^{+\infty} \frac{\sin(a\xi t e^{(ih\xi/2)})}{\xi e^{(ih\xi/2)}} e^{-ix\xi}\,d\xi = \\
&= \frac{1}{2\pi a}\left[\int_0^{+\infty} \frac{\sin(a\xi e^{(-ih\xi/2)}t)}{\xi} e^{i(x+h/2)\xi}\,d\xi \; + \right. \\
&\quad \left. + \int_0^{+\infty} \frac{\sin(a\xi e^{(ih\xi/2)}t)}{\xi} e^{-i(x+h/2)\xi}\,d\xi\right] = \\
&= \frac{1}{2\pi a}\int_0^{+\infty} \left[\frac{\sin((at\cos(h\xi/2) + x + h/2)\xi)}{\xi e^{(-at\xi\sin(h\xi/2))}} \; + \right. \\
&\quad \left. + \frac{\sin((at\cos(h\xi/2) - x - h/2)\xi)}{\xi e^{(at\xi\sin(h\xi/2))}}\right] d\xi.
\end{aligned} \tag{5}$$

Remark 1. If we put $h = 0$ in (5), then we obtain $\theta(at - |x|)/(2a)$ — the *fundamental solution* of the wave operator $\partial^2/\partial t^2 - a^2\partial^2/\partial x^2$, where θ is the Heaviside function.

Since the obtained improper integral in (5) diverges, we introduce, according to [11], the *regularizer* $f(\xi)$ for expression (5) — a function satisfying the conditions:

1) $f(\xi)$ is positively defined and continuous on the set $[0,+\infty)$;

2) for any number $\epsilon > 0$ there are the following equations

$$\lim_{\xi\to+\infty} f(\xi)e^{at\xi\sin(h\xi/2)}\xi^{\epsilon}=0,\quad \lim_{\xi\to+\infty} f(\xi)e^{-at\xi\sin(h\xi/2)}\xi^{\epsilon}=0; \tag{6}$$

3) the integrals converge at any value of $t\in[0,T]$

$$\int_0^{+\infty} f(\xi)e^{at\xi\sin(h\xi/2)}d\xi,\quad \int_0^{+\infty} f(\xi)e^{-at\xi\sin(h\xi/2)}d\xi; \tag{7}$$

4) the integrals converge at any value of $t\in(0,T]$

$$\int_0^{+\infty} f(\xi)\xi e^{at\xi\sin(h\xi/2)}d\xi,\quad \int_0^{+\infty} f(\xi)\xi e^{-at\xi\sinh\xi/2}d\xi. \tag{8}$$

An example of such a function satisfying conditions 1)–4) is, for example, the function $f(\xi)=\xi^{\beta}e^{(-CT\xi)}$, where $\beta\geq 0$ and $C>a>0$ are any real constants.

Remark 2. The fulfillment of the equations (6) entails [13, p. 102] the convergence of the integral integrals

$$\int_0^{+\infty} \frac{f(\xi)}{\xi}e^{at\xi\sin(h\xi/2)}d\xi,\quad \int_0^{+\infty} \frac{f(\xi)}{\xi}e^{-at\xi\sin(h\xi/2)}d\xi. \tag{9}$$

3. KEY FINDINGS

Lemma. *If conditions* 1)–4) *are satisfied, the function*

$$G(x,t):=\int_0^{+\infty}\left[\frac{f(\xi)\sin((at\cos(h\xi/2)+x+h/2)\xi)}{\xi e^{-at\xi\sin(h\xi/2)}}+\frac{f(\xi)\sin((at\cos(h\xi/2)-x-h/2)\xi)}{\xi e^{at\xi\sin(h\xi/2)}}\right]d\xi \tag{10}$$

satisfies equation (1) *in the classical sense*.

Proof. The integrand in (10) is continuous on the set $[0,+\infty)$ as a composition of continuous functions (there is no singularity at the point $\xi=0$ due to the limit relation $\sin\alpha/\alpha\to 0$ at $\alpha\to 0$).

Let's investigate the convergence of the integral

$$\int_0^{+\infty} F(x,t;\xi)d\xi:=\int_0^{+\infty}\frac{f(\xi)\sin((at\cos(h\xi/2)+x+h/2)\xi)}{\xi e^{-at\xi\sin(h\xi/2)}}d\xi. \tag{11}$$

In view of condition 1)

$$\left|\int_0^{+\infty} F(x,t;\xi)d\xi\right|\leqslant\int_0^{+\infty}\frac{f(\xi)}{\xi}e^{at\xi\sin(h\xi/2)}d\xi,$$

then by virtue of the fulfillment of condition 2) and, as a consequence, of Remark 2, the integral (11) converges.

Let us now check that function (11) satisfies equation (1). For this purpose, we differentiate (11) formally under the sign of the integral over the variables t and x up to the second order:

$$\int_0^{+\infty} F_x(x,t;\xi)d\xi=\int_0^{+\infty} f(\xi)\cos((at\cos(h\xi/2)+x+h/2)\xi)e^{at\xi\sin(h\xi/2)}d\xi; \tag{12}$$

$$\int_0^{+\infty} F_{xx}(x,t;\xi)d\xi=-\int_0^{+\infty} f(\xi)\xi\sin((at\cos(h\xi/2)+x+h/2)\xi)e^{at\xi\sin(h\xi/2)}d\xi,$$

then

$$\int_0^{+\infty} F_{xx}(x-h,t;\xi)d\xi=-\int_0^{+\infty} f(\xi)\xi\sin((at\cos(h\xi/2)+x-h/2)\xi)e^{at\xi\sin(h\xi/2)}d\xi. \tag{13}$$

Next,

$$\int_0^{+\infty} F_t(x,t;\xi)d\xi = a\int_0^{+\infty} f(\xi)\big[\cos(h\xi/2)\cos((at\cos(h\xi/2)+x+h/2)\xi)+$$
$$+\sin(h\xi/2)\sin((at\cos(h\xi/2)+x+h/2)\xi)\big]e^{at\xi\sin(h\xi/2)}d\xi=$$
$$= a\int_0^{+\infty} f(\xi)\cos((at\cos(h\xi/2)+x)\xi)e^{at\xi\sin(h\xi/2)}d\xi; \tag{14}$$
$$\int_0^{+\infty} F_{tt}(x,t;\xi)d\xi = -a^2\int_0^{+\infty} f(\xi)\xi\big[\cos(h\xi/2)\sin((at\cos(h\xi/2)+x)\xi)-$$
$$-\sin(h\xi/2)\cos((at\cos(h\xi/2)+x)\xi)\big]e^{at\xi\sin(h\xi/2)}d\xi=$$
$$= -a^2\int_0^{+\infty} f(\xi)\xi\sin((at\cos(h\xi/2)+x-h/2)\xi)e^{at\xi\sin(h\xi/2)}d\xi. \tag{15}$$

Substituting the found derivatives (13) and (15) into the relation (1), we are convinced of its validity.

Let us examine the integral (12) for uniform convergence. We have

$$\int_0^{+\infty} |F_x(x,t;\xi)|d\xi \le \int_0^{+\infty} f(\xi)e^{(at\xi\sin(h\xi/2))}d\xi.$$

Since the integral in the right-hand side of the inequality converges due to condition 3), and the integrand in it does not depend on the variable x, then by virtue of the Weierstrass sign the integral (12) converges uniformly on the variable x at any finite interval $[x_1,x_2]\subset\mathbb{R}$.

Similarly, from the estimation

$$\int_0^{+\infty} |F_{xx}(x-h,t;\xi)|d\xi \le \int_0^{+\infty} f(\xi)\xi e^{at\xi\sin(h\xi/2)}d\xi,$$

condition 4) and the independence of the integrand from x in the right-hand side of the last inequality results in the uniform convergence of the integral (13) on the variable x on any interval $[x_1,x_2]\subset\mathbb{R}$. This means that the differentiation under the sign of the integral in (11) on the variable x up to and including the second order was legitimate.

Let us now evaluate the integral (14):

$$\int\limits_0^{+\infty} |F_t(x,t;\xi)|\,d\xi \le a\int\limits_0^{+\infty} f(\xi)e^{at\xi\sin(h\xi/2)}\,d\xi \le \begin{cases} a\int\limits_0^{+\infty} f(\xi)e^{at_2\xi\sin(h\xi/2)}\,d\xi, & \sin(h\xi/2)\ge 0,\\ a\int\limits_0^{+\infty} f(\xi)e^{at_1\xi\sin(h\xi/2)}\,d\xi, & \sin(h\xi/2)<0.\end{cases}$$

The integrals in the right-hand side of the relations converge according to condition 3), and the integrand expressions in them do not depend on t, hence, the integral (14) converges uniformly on any interval $[t_1,t_2]\subset[0,T]$.

From the assessment

$$\int\limits_0^{+\infty} |F_{tt}(x,t;\xi)|\,d\xi \le \begin{cases} a^2\int\limits_0^{+\infty} f(\xi)\xi e^{at_2\xi\sin(h\xi/2)}\,d\xi, & \sin(h\xi/2)\ge 0,\\ a^2\int\limits_0^{+\infty} f(\xi)\xi e^{at_1\xi\sin(h\xi/2)}\,d\xi, & \sin(h\xi/2)<0\end{cases}$$

and condition 4) it follows that the integral (15) converges uniformly on any segment $[t_1,t_2]\subset(0,T]$. Thus, the differentiation (15) under the sign of the integral over the variable t up to and including the second order is valid.

Similarly it can be shown, in view of conditions 1) and 2), that the non-singular integral converges

$$\int_0^{+\infty} H(x,t;\xi)d\xi := \int_0^{+\infty} \frac{f(\xi)\sin((at\cos(h\xi/2)-x-h/2)\xi)}{\xi e^{at\xi\sin(h\xi/2)}}d\xi \tag{16}$$

and that function (16) satisfies equation (1), differentiating directly (16) under the sign of integral on variables x and t up to the second order inclusive and substituting the found derivatives $H_{tt}(x,t;\xi)$ and $H_{xx}(x-h,t;\xi)$ into (1). In this case, by virtue of conditions 3) and 4), the integrals $H_x(x,t;\xi)$ and $H_{xx}(x,t;\xi)$ converge uniformly on the variable x at any segment $[x_1,x_2]\subset\mathbb{R}$ and the integrals $H_t(x,t;\xi)$ and $H_{tt}(x,t;\xi)$ converge uniformly at any segment $[t_1,t_2]$ of the sets $[0,T]$ and $(0,T]$, respectively.

Thus, it is shown that function (10) exists at every point of the domain D and satisfies equation (1) in the classical sense. The lemma is proved.

On the basis of the lemma the following is true.

Theorem. *If conditions* 1)–4) *are satisfied, the function*

$$u(x,t)=\frac{1}{2\pi a}\int_{-\infty}^{+\infty}G(x-\tau,t)u_0(\tau)d\tau, \tag{17}$$

where $G(x,t)$ *is defined by equality* (10), $u_0(x)$ *is any integrable function on the whole number line, satisfies equation* (1) *in the classical sense and the limit relation*

$$\lim_{t\to+0}u(x_0,t)=0$$

for any value of $x_0\in\mathbb{R}$.

Proof. Function (17) has the form

$$u(x,t)=\frac{1}{2\pi a}\int_{-\infty}^{+\infty}u_0(\tau)\int_0^{+\infty}\left[\frac{\sin((at\cos(h\xi/2)+x-\tau+h/2)\xi)}{\xi e^{-at\xi\sin(h\xi/2)}}+\right.$$
$$\left.+\frac{\sin((at\cos(h\xi/2)-x+\tau-h/2)\xi)}{\xi e^{at\xi\sin(h\xi/2)}}\right]d\xi d\tau.$$

Since $u_0(x)\in L_1(\mathbb{R})$, it is sufficient to show that $|G(x-,t)|\leqslant const$, that is true, due to condition 2) and Remark 2, for the existence of the function (17) in the domain D. In view of the proved lemma, function (17) is a classical solution of equation (1). Note also that, by virtue of the same lemma, the function (17) belongs to the class $C^1(\overline{D})\cap C^2(D)$ (the integrand in (17) is continuous), the integrals $u_x(x,t)$ and $u_{xx}(x,t)$ converge uniformly on the variable x at any finite segment $[x_1,x_2]\subset\mathbb{R}$, the integrals $u_t(x,t)$ and $u_{tt}(x,t)$ converge uniformly on t at any finite segment $[t_1,t_2]$ of the sets $[0,T]$ and $(0,T]$, respectively (the integral $u_t(x,t)$ converges on the boundary $t=0$).

Let $x_0\in\mathbb{R}$. In (17) we substitute the variable by the formula $(x_0-\tau)/t=\eta$ and get

$$u(x_0,t)=\frac{t}{2\pi a}\int_{-\infty}^{+\infty}G(t\eta,t)u_0(x_0-t\eta)d\eta,$$

whence at $t\to+0$ follows the evaluation of $|u(x_0,t)|<\varepsilon$ for any arbitrarily small number $\varepsilon>0$. The theorem is proved.

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CONFLICT OF INTERESTS

The author of this paper declares that he has no conflict of interests.

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PARTIAL DIFFERENTIAL EQUATIONS

INSTABILITY AND STABILIZATION OF SOLUTIONS OF A STOCHASTIC MODEL OF VISCOELASTIC FLUID DYNAMICS

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Abstract. The instability and stability of solutions of the stochastic system describing the flow of a viscoelastic liquid are investigated. It is shown that for certain values of the parameters included in the equations of the system, the existence of unstable and stable invariant spaces. For unstable case, the stabilization problem is solved based on the feedback principle.

Keywords: *Sobolev type stochastic equation, invariant space, stabilization*

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1. INTRODUCTION. PROBLEM STATEMENT

Let $D \subset \mathbb{R}^n$ be a bounded region with boundary ∂D of class C^∞. Let's consider the following model of viscoelastic incompressible fliud flow in $D \times \mathbb{R}$:

$$(\lambda - \nabla^2)u_t = \nu\nabla^2 u - \nabla p, \quad \nabla u = 0; \tag{1}$$

$$u(x,t) = 0, \quad (x,t) \in \partial D \times \mathbb{R}; \quad u(x,0) = u_0, \quad x \in D,$$

where $u(x,t) = (u_1(x,t), u_2(x,t), \ldots, u_n(x,t))$ and p are the velocity and pressure vectors, respectively. System (1) is a linearization of the system

$$(\lambda - \nabla^2)u_t = \nu\nabla^2 u - (u\nabla)u - \nabla p, \quad \nabla u = 0,$$

obtained by A.P. Oskolkov [1] to describe the flow of viscous liquids possessing elasticity property. Redefining ∇p by p, we write the system (1) in the following form

$$(\lambda - \nabla^2)u_t = \nu\nabla^2 u - p, \quad \nabla(\nabla)u = 0. \tag{2}$$

Here, the parameter λ characterizes elastic properties, and ν characterizes viscous properties. In [2], it was shown that the parameter λ can take negative values. In [3], a physical model of fluid flow with negative viscosity was constructed, so we will assume further that $\nu \in \mathbb{R}$.

It has been experimentally shown that the flow of polymer solutions and melts has the property of instability (see the review [4] and the bibliography therein). This instability can have a significant impact on the material processing technologies and the quality of final products. One of the causes of this instability is inlet pulsations ("inlet instability"). Note that polymer solution and melts are viscoelastic fluids. We will investigate the instability and stability of the flow of an incompressible viscoelastic fluid described by system (2) with stochastic initial data. As an initial condition, we choose a random variable

$$\eta(0) = \eta_0, \tag{3}$$

and we will consider the system (2) as a stochastic equation of the Sobolev type

$$L\overset{\circ}{\eta} = M\eta. \tag{4}$$

The solution of the stochastic equation is a stochastic process that is not differentiable at any point. Therefore, as the derivative of the stochastic process η we will consider the Nelson–Glicklich derivative $\overset{\circ}{\eta}$ [5]. At present, a large number of works devoted to the study of stochastic equations of Sobolev type are known. Let us note some of them. The solvability of the Cauchy problem for equation (4) is studied in [6] (in the case of a relatively bounded operator), [7] (in the case of a relatively sectorial operator) and [8] (in the case of a relatively radial operator). In [9], stochastic linear equations of Sobolev type of high order are considered; in [10, 11], the "initial-finite" problem for equation (4) is investigated; in [12], the stability of equation (4) is studied. In [13–15], numerical experiments on finding stable and unstable solutions of stochastic nonclassical equations that can be represented in the form (4) were carried out.

The deterministic system (2) has been studied in various aspects. The study of its solvability was started in [1] under the condition that the parameters $\lambda, \nu \in \mathbb{R}_+$. In [16], the question of existence of solutions was solved using the phase space method at $\lambda \in \mathbb{R}\backslash\{0\}$ and $\nu \in \mathbb{R}_+$; the existence of an exponential dichotomy of solutions was shown. In [17], the initial-final problem for a linear system of Oskolkov equations was studied.

The purpose of this paper is to study the instability and stability of solutions of the stochastic system (2) in the case when the parameters $\lambda, \nu \in \mathbb{R}\backslash\{0\}$, and to solve the problem of stabilization of unstable solutions. In Section 2, we give abstract results on the existence of solutions of equation (4) and their stability. In Section 3, the system (2) in spaces of random K-values is considered, and the solvability of the stochastic system (2) is shown. In Section 4, the existence of stable and unstable invariant spaces is proved, the problem of stabilization of unstable solutions by the feedback principle is solved.

2. INVARIANT SPACES OF THE STOCHASTIC EQUATION OF SOBOLEV TYPE

By $\mathbf{L}_2$ we denote the space of random variables ξ with zero mathematical expectation and finite variance, and by $\mathbf{CL}_2$ we denote the space of continuous stochastic processes η. We fix $\eta \in \mathbf{CL}_2$ and $t \in \mathfrak{I}$, where $\mathfrak{I}$ is some interval, and through $\mathcal{N}_t^\eta$ we denote the σ-algebra generated by η and $\mathbf{E}_t^\eta = \mathbf{E}(\cdot|\mathcal{N}_t^\eta)$. Let us define the *Nelson–Glicklich derivative* of the stochastic process η at the point $t \in \mathfrak{I}$ as the limit

$$\overset{\circ}{\eta}(\,\cdot\,,\omega) = \frac{1}{2}\left[\lim_{\Delta t\to+0}\mathbf{E}_t^\eta\left(\frac{\eta(t+\Delta t,\cdot)-\eta(t,\cdot)}{\Delta t}\right) + \lim_{\Delta t\to+0}\mathbf{E}_t^\eta\left(\frac{\eta(t,\cdot)-\eta(t-\Delta t,\cdot)}{\Delta t}\right)\right],$$

if it converges in the uniform metric on $\mathbb{R}$. By $\mathbf{C}^l\mathbf{L}_2$ we denote the space of stochastic processes whose Nelson–Glicklich derivatives are a.s. (almost surely) continuous on $\mathfrak{I}$ up to order l inclusive.

Let $\mathfrak{U}$ and $\mathfrak{F}$ be real separable Hilbert spaces, and let $\{\varphi_k\}$ and $\{\psi_k\}$ denote bases in $\mathfrak{U}$ and $\mathfrak{F}$, respectively. Choose a sequence of random variables $\{\xi_k\} \subset \mathbf{L}_2$ ($\{\zeta_k\} \subset \mathbf{L}_2$), such that $\|\xi_k\|_{\mathbf{L}_2} \le \text{const}$ ($\|\zeta_k\|_{\mathbf{L}_2} \le \text{const}$). The elements of the space $\mathbf{U_K L}_2$ ($\mathbf{F_K L}_2$) of ($\mathfrak{U}$-valued ($\mathfrak{F}$-valued)) random $\mathbf{K}$-variables are vectors $\xi = \sum_{k=1}^\infty \lambda_k\xi_k\varphi_k$ ($\zeta = \sum_{k=1}^\infty \lambda_k\zeta_k\psi_k$), where the sequence $\mathbf{K} = \{\lambda_k\} \subset \mathbb{R}_+$ satisfies $\sum_{k=1}^\infty \lambda_k^2 < +\infty$. The following holds:

Lemma 1 [18]. *The operator* $A \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$ *(linear and continuous) if and only if the operator* $A \in \mathcal{L}(\mathbf{U_K K}_2;\mathbf{F_K L}_2)$.

Let the operators $L \in \mathcal{L}(\mathbf{U_K L}_2;\mathbf{F_K L}_2)$, $M \in Cl(\mathbf{U_K L}_2;\mathbf{F_K L}_2)$. Denote by

$$\rho^L(M) = \{\mu \in \mathbb{C}\colon (\mu L - M)^{-1} \in \mathcal{L}(\mathfrak{F};\mathfrak{U})\}$$

the L-*resolvent set*, and by $\sigma^L(M) = \mathbb{C}\setminus\rho^L(M)$ the L-*spectrum of the operator* M. If the operator M is (L,σ)-bounded, i.e., its L-spectrum is bounded, then there exist projectors

$$P = \frac{1}{2\pi i}\int_\gamma (\mu L - M)^{-1}L\,d\mu \in \mathcal{L}(\mathbf{U_K L}_2), \quad Q = \frac{1}{2\pi i}\int_\gamma L(\mu L - M)^{-1}\,d\mu \in \mathcal{L}(\mathbf{F_K L}_2). \tag{1}$$

Here, the contour $\gamma \subset \mathbb{C}$ bounds a region containing $\sigma^L(M)$.

The projectors (5) split the spaces $\mathbf{U_K}\,\mathbf{L}_2 = \mathbf{U_K^0}\,\mathbf{L}_2 \oplus \mathbf{U_K^1}\,\mathbf{L}_2$ and $\mathbf{F_K L}_2 = \mathbf{F_K^0 L}_2 \oplus \mathbf{F_K^1 L}_2$, where $\mathbf{U_K^0 L}_2$ ($\mathbf{U_K^1 L}_2$) $= \ker P$ ($\operatorname{im} P$), $\mathbf{F_K^0 L}_2$ ($\mathbf{F_K^1 L}_2$) $= \ker Q$ ($\operatorname{im} Q$). Let L_k (M_k) denote the restriction of the operator L (M) to $\mathbf{U_K^k L}_2$, $k = 0,1$. The operators $L_k(M_k) \in \mathcal{L}(\mathbf{U_K^k L}_2, \mathbf{F_K^k L}_2)$, $k = 0,1$; there exist operators $M_0^{-1} \in \mathcal{L}(\mathbf{F_K^0 L}_2, \mathbf{U_K^0 L}_2)$, $L_1^{-1} \in \mathcal{L}(\mathbf{F_K^1 L}_2, \mathbf{U_K^1 L}_2)$. Consider the operators $H = L_0^{-1}M_0$ and $S = L_1^{-1}M_1$. If the operator M is (L,p)-bounded and $H \equiv \mathbb{O}$, $p = 0$ or $H^p \ne \mathbb{O}$, $H^{p+1} \equiv \mathbb{O}$, then it is called an (L,p)-*bounded operator*.

We call a stochastic $\mathbf{K}$-process $\eta \in \mathbf{C}^1(\mathcal{J};\mathbf{U_K L}_2)$ is called a *solution of equation* (4) if a.s. all its trajectories satisfy equation (4) at all $t \in \mathcal{J}$. A solution $\eta = \eta(t)$ of equation (4) a *solution of the Cauchy problem* (3), (4) if equality (3) holds for some random **L**-variable $\eta_0 \in \mathbf{U_L L_2}$. The set $\mathbf{P} \subset \mathbf{U_L L_2}$ is called the *stochastic phase space* of equation (4) if a.s. any trajectory of the solution $\eta = \eta(t)$ lies in **P** pointwise, i.e., $\eta(t) \in \mathbf{P}$ for all $t \in \mathcal{J}$, and for a.e. $\eta_0 \in \mathbf{P}$ there exists a solution to the problem (3), (4).

Theorem 1 [7]. Let the operator M be (L,p)-bounded, $p \in \{0\} \cup \mathbb{N}$. Then the group

$$U^t = \frac{1}{2\pi i} \int\limits_{\gamma} R^L_\mu(M) e^{\mu t}\, d\mu$$

is the holomorphic resolving group of equation (4)*; the subspace* $\mathbf{U^1_K L_2}$ *is the phase space of equation*(4).

Definition. An invariant subspace $1\mathbf{I}^{s(u)} \subset \mathbf{P}$ is called the *stable (unstable)* invariant space of equation (4) if the condition

$$\|\eta^{s(u)}(t)\|_{\mathbf{U_K L_2}} \le N e^{-\nu(s-t)} \|\eta^{s(u)}(s)\|_{\mathbf{U_K L_2}}$$

holds for $s \ge t$ $(t \ge s)$, $\eta^{s(u)} = \eta^{s(u)}(t) \in \mathbf{I}^1$, and some $N, \alpha \in \mathbb{R}_+$. If the phase space splits into a direct sum $\mathbf{P} = \mathbf{I}^1 \oplus \mathbf{I}^2$, then the solutions $\eta = \eta(t)$ of equation (4) have an *exponential dichotomy*.

Let the operator M be (L,p)-bounded, $p \in \{0\} \cup \mathbb{N}$ and the relative spectrum has the form

$$\sigma^L(M) = \sigma^L_s(M) \oplus \sigma^L_u(M), \tag{6}$$

where

$$\sigma^L_s(M) = \{\mu \in \sigma^L(M) : \mathrm{Re}\,\mu < 0\} \ne \varnothing, \quad \sigma^L_u(M) = \{\mu \in \sigma^L(M) : \mathrm{Re}\,\mu > 0\} \ne \varnothing.$$

Then there are projectors

$$P_{l(r)} = \frac{1}{2\pi i} \int\limits_{\gamma_{l(r)}} R^L_\mu(M)\, d\mu \in \mathcal{L}(\mathbf{U_K L_2}),$$

where the contour $\gamma_{l(r)}$ lies in the left (right) half-plane of the complex plane and bounds a part of the L-spectrum of the operator M $\sigma^L_{s(u)}(M)$. Let us denote by $\mathbf{I}^{(s(u))} = \mathrm{im}\, P_{l(r)}$.

Let the operator M be (L,p)-bounded and condition (6) be satisfied, then $\mathbf{U^1_K L_2} = \mathbf{I}^s \oplus \mathbf{I}^u$. Equation (4) will be considered as a system

$$H\overset{\circ}{\eta}{}^0 = \eta^0, \tag{7}$$

$$L_s \overset{\circ}{\eta}{}^s = M_s \eta^s, \tag{8}$$

$$L_u \overset{\circ}{\eta}{}^u = M_u \eta^u. \tag{9}$$

Remark 1. The operator M is (L,p)-bounded, so the operator H is nilpotent of degree p. Then the solution of equation (7) $\eta^0 = 0$ and the stochastic process $\eta = \eta^s + \eta^u$ is a solution of equation (4), where η^s and η^u are solutions of equations (8) and (9), respectively. Thus, the question of stability and instability of solutions of equation (4) is reduced to the study of stability and instability of solutions of η^s and η^u.

Theorem 2. *Let the operator* M *be* (L,p)*-bounded,* $p \in \{0\} \cup \mathbb{N}$ *and condition* (6) *be satisfied, then the solutions* $\eta = \eta(t)$ *of equation* (4) *have an exponential dichotomy.*

Proof. The solving groups of equations (8) and (9) have the form

$$U^t_l = \frac{1}{2\pi i} \int\limits_{\gamma_l} (\mu L_s - M_s)^{-1} L_s e^{\mu t}\, d\mu, \quad U^t_r = \frac{1}{2\pi i} \int\limits_{\gamma_r} (\mu L_u - M_u)^{-1} L_u e^{\mu t}\, d\mu.$$

Let's denote $\alpha = -\max_{\mu \in \sigma^L_l(M)} \mathrm{Re}\,\mu$ and $\beta = \min_{\mu \in \sigma^L_r(M)} \mathrm{Re}\,\mu$. Then

$$\|U^t_l\|_{\mathcal{L}(\mathbf{U_K L_2})} \le e^{-\alpha t} \int\limits_{\gamma_l} \|(\mu L_s - M_s)^{-1} L_s\|_{\mathcal{L}(\mathbf{U_K L_2})} |d\mu| \le N_l e^{-\alpha t}, \tag{10}$$

$$\|U^t_r\|_{\mathcal{L}(\mathbf{U_K L_2})} \le e^{\beta t} \int\limits_{\gamma_r} \|(\mu L_r - M_r)^{-1} L_r\|_{\mathcal{L}(\mathbf{U_K L_2})} |d\mu| \le N_r e^{\beta t}. \tag{11}$$

Let $s \geq t$. Then the solution η^s of equation (8) can be written as $\eta^s(t) = U_l^{s-t}\eta^s(s)$. By virtue of (10) we have the relations

$$\|\eta^s(t)\|_{U_K L_2} = \|U_l^{s-t}\eta^s(s)\|_{U_K L_2} \leq N_l e^{-\alpha(s-t)}\|\eta^s(s)\|_{U_K L_2}.$$

Further, let $t \geq s$. Then the solution η^u of equation (9) is: $\eta^u(t) = U_r^{t-s}\eta^u(s)$. By virtue of (11) we have

$$\|\eta^u(t)\|_{U_K L_2} = \|U_r^{t-s}\eta^u(s)\|_{U_K L_2} \leq N_r e^{\beta(t-s)}\|\eta^u(s)\|_{U_K L_2} = N_r e^{-\beta(s-t)}\|\eta^s(s)\|_{U_K L_2}.$$

The theorem is proved.

Corollary 1. *By the conditions of Theorem* 2, *any trajectory of the solution* $\eta^{s(u)} = \eta^{s(u)}(t)$ *of equation* (8) (*equation* (9)) *a.c. lies in the stable (unstable) invariant space* $I^{s(u)}$ *pointwise, i.e.,* $\eta^{s(u)}(t) \in \mathbf{I}^{s(u)}$ *at all* $t \in \mathbb{R}$.

Remark 2. If $\sigma^L_{s(u)}(M) = \varnothing$, то $\mathbf{I}^{s(u)} = \{0\}$.

3. STOCHASTIC SYSTEMTYPE

We will consider the system (2) in the spaces of random K-values. For this purpose we denote by $\mathbb{H}^2 = (W_2^2(D))^n$, $\mathring{\mathbb{H}}^1 = (\mathring{W}_2^2(D))^n$, $\mathbb{L}^2 = (L_2(D))^n$. The closure $\{u \in C^\infty \colon \nabla u = 0\}$ of the lineal $\mathbb{L}^2$ is denoted by $\mathbb{H}_\sigma$, and there exists a splitting $\mathbb{L}^2 = \mathbb{H}_\sigma \oplus \mathbb{H}_\pi$, where $\mathbb{H}_\pi$ is an orthogonal complement to $\mathbb{H}_\sigma$, and $\Pi \colon \mathbb{L}^2 \to \mathbb{H}_\pi$ is an otroprojector corresponding to this complement. The contraction of the projector Π onto $\mathbb{H}^2 \cap \mathring{\mathbb{H}}^1 \subset \mathbb{L}^2$ is a continuous operator $\Pi \colon \mathbb{H}^2 \cap \mathring{\mathbb{H}}^1 \to \mathbb{H}^2 \cap \mathring{\mathbb{H}}^1$. Let us represent the space $\mathbb{H}^2 \cap \mathring{\mathbb{H}}^1 = \mathbb{H}^2_\sigma \oplus \mathbb{H}^2_\pi$, where $\ker \Pi = \mathbb{H}^2_\sigma$, $\operatorname{im} \Pi = \mathbb{H}^2_\pi$. Let us denote $\Sigma = \mathbb{I} - \Pi$. Let us put $\mathfrak{U} = \mathbb{H}^2_\sigma \times \mathbb{H}^2_\pi \times \mathbb{H}_\pi$ and $\mathfrak{F} = \mathbb{H}_\sigma \times \mathbb{H}_\pi \times \mathbb{H}_\pi$. The element $u \in \mathfrak{U}$ has the form $u = (u_\sigma, u_\pi, p)$.

Lemma 2 [2]. *The formula* $A = (-\nabla^2)^n \colon \mathbb{H}^2 \cap \mathring{\mathbb{H}}^1 \to \mathbb{L}^2$ *defines a linear continuous operator with positive discrete spectrum* $\sigma(A)$, *condensing to the point* $+\infty$, *and the mapping* $A \colon \mathbb{H}^2_{\sigma(\pi)} \to \mathbb{H}^2_{\sigma(\pi)}$ *is bijective.*

The formula $B \colon u \to -\nabla(\nabla u)$ *defines a linear continuous surjective operator* $B \colon \mathbb{H}^2 \cap \mathring{\mathbb{H}}^1 \to \mathbb{H}^2_\pi$, *with* $\ker B = \mathbb{H}^2_\sigma$.

The spaces $W_2^2(D)$, $L_2(D)$ are separable Hilbert spaces, so the spaces $\mathfrak{U}$, $\mathfrak{F}$ are separable Hilbert spaces as their finite products. Let us construct the spaces $\mathbf{U_K L}_2$ and $\mathbf{F_K L}_2$. The operators $L, M \in \mathcal{L}(\mathbf{U_K K}_2; \mathbf{F_K L}_2)$ are defined as

$$L = \begin{pmatrix} \Sigma(\lambda\mathbb{I} + A) & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \Pi(\lambda\mathbb{I} + A) & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} \end{pmatrix}, \quad M = \begin{pmatrix} -\nu\Sigma A & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & -\nu\Pi A & -\Pi \\ \mathbb{O} & \Pi B & \mathbb{O} \end{pmatrix}.$$

Then the stochastic system of equations (2) can be viewed as a stochastic linear equation (4). The following is true

Lemma 3. *Operators* $L, M \in \mathcal{L}(\mathbf{U_K K}_2; \mathbf{F_K L}_2)$.

Proof. Clearly, the operators $L, M \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$, with $\operatorname{im} L = \mathbb{H}^2_\sigma \times \mathbb{H}^2_\pi \times \{0\}$, $\ker L = \{0\} \times \{0\} \times \mathbb{H}^2_\pi$, so by virtue of lemma 1, $L, M \in \mathcal{L}(\mathbf{U_K K}_2; \mathbf{F_K L}_2)$.

Lemma 4. *For any* $\lambda \in \mathbb{R} \setminus \sigma(A)$, $\nu \in \mathbb{R}$ *the operator* M is $(L, 1)$-*limited.*

Proof. In [2] it is shown that the operator M is $(L, 1)$-bounded if the operators $L, M \colon \mathfrak{U} \to \mathfrak{F}$, so by virtue of lemma 1, the statement of this lemma follows.

Theorem 3. For any $\lambda \in \mathbb{R} \setminus \sigma(A)$, $\nu \in \mathbb{R}$ *and for any random variable* $\eta_0 \in \mathbf{U^1_K L}_2$ *there exists a solution to problem* (3), (4) *which is of the form* $\eta(t) = U^t\eta_0$, $t \in \mathcal{J}$.

Proof. By virtue of lemmas 3 and 4,, the stochastic system of equations (2) satisfies all the requirements of Theorem 1. The phase space has the form

$$\mathbf{U^1_K L}_2 = \begin{cases} \mathbf{U_K L}_2, & \text{if } \lambda \neq \nu_k \text{ for } k \in \mathbb{N}; \\ \eta \in \mathbf{U_K L}_2 \colon \langle\,\cdot\,, \varphi_k\rangle\varphi_k = 0, & \text{if } \lambda = \nu_k, \end{cases}$$

where ν_k is the spectrum of the operator $\tilde{A} \colon \mathbb{H}^2_\pi \to \mathbb{H}^2_\pi$, that is the contraction of the operator A onto $\mathbb{H}^2_\pi$. The resolving group can be represented as

$$U^t = \begin{pmatrix} \displaystyle\sum_{\nu_k \neq \lambda} \exp\left\{\frac{\nu\nu_k}{\nu_k - \lambda}\right\}\langle\cdot, \varphi_k\rangle\varphi_k & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} \end{pmatrix}.$$

4. EXPONENTIAL DICHOTOMIES AND STABILIZATION OF SOLUTIONS OF A STOCHASTIC SYSTEM OF EQUATIONS

The relative spectrum has the form $\sigma^L(M) = \left\{\frac{\nu\nu_k}{(\nu_k-\lambda)}\right\}$. Note that the spectrum $\sigma(\tilde{A}) = \{\nu_k\}$ is positive discrete finite and condensed to the point $+\infty$ (Solonnikov–Vorovich–Yudovich theorem). The following holds

Theorem 4. *For any $\lambda \in \mathbb{R}\backslash\sigma(A)$, $\lambda > \nu_1$ and $\nu \in \mathbb{R}\backslash\{0\}$, solutions $\eta = \eta(t)$ of the stochastic system of equations* (2) *have an exponential dichotomy.*

Proof. Let $\lambda \in \mathbb{R}\backslash\sigma(A)$ and $\lambda > \nu_1$, then $\sigma^L(M) = \sigma_1^L(M) \cup \sigma_2^L(M)$, where $\sigma_1^L(M) = \{\mu \in \sigma^L(M) : \nu_k < \lambda\}$, $\sigma_2^L(M) = \{\mu \in \sigma^L(M) : \nu_k > \lambda\}$. This spectral decomposition is accompanied by invariant spaces

$$\mathbf{I}^1 = \{\eta \in \mathbf{U}^1_\mathbf{K}\mathbf{L}_2 \colon \langle\,\cdot\,,\varphi_k\rangle\varphi_k = 0,\ \nu_k < \lambda\}, \quad \mathbf{I}^2 = \{\eta \in \mathbf{U}^1_\mathbf{K}\mathbf{L}_2 \colon \langle\,\cdot\,,\varphi_k\rangle\varphi_k = 0,\ \nu_k > \lambda\}.$$

The space $\mathbf{I}^1$ is finite-dimensional, $\dim \mathbf{I}^1 = \max\{k \colon \nu_k < \lambda\}$, , and the space $\mathbf{I}^2$ is infinite-dimensional, $\operatorname{codim} \mathbf{I}^2 = \dim \mathbf{I}^1 + \dim \ker L$.

If $\nu > 0$ ($\nu < 0$), then $\sigma^L_{1(2)}(M)$ lies in the left half-plane and $\sigma^L_{2(1)}(M)$ lies in the right half-plane of the complex plane. By virtue of Theorem 2, $\mathbf{I}^{1(2)}$ is a stable invariant space, $\mathbf{I}^{2(1)}$ is an unstable invariant space, and the solutions of the stochastic system of equations (2) have exponential dichotomy. The theorem is proved.

Corollary 2. *If $\lambda < \nu_1$ and $\nu < 0$, then the phase space of the stochastic system of equations (2) coincides with the stable invariant space. If $\lambda < \nu_1$ and $\nu > 0$, then the phase space of the stochastic system of equations (2) coincides with the unstable invariant space.*

Let us proceed to the problem of stabilization of unstable solutions. For this purpose, we will consider equation (4) in the form of the system (7)–(9). For definiteness, let us assume $\nu > 0$ and $\lambda > \nu_1$. It follows from Theorem 4 that $\mathbf{I}^s = \mathbf{I}^1$ and $\mathbf{I}^u = \mathbf{I}^2$. The space $\mathbf{I}^s$ is a stable invariant space, so for the solutions $\eta_l = \eta_l(t)$ of equation (8) the following is true

$$\lim_{t\to+\infty} \|\eta_l(t)\|_{\mathbf{U_K L}_2} = 0.$$

By virtue of Remark 1, consider the following stabilization problem. It is required to find such a stochastic process χ, so that for the solutions of Eq.

$$L_r \overset{\circ}{\eta}_r = M_r\eta_r + \chi \tag{12}$$

the following condition was satisfied

$$\lim_{t\to+\infty} \|\eta_r(t)\|_{\mathbf{U_K L}_2} = 0. \tag{13}$$

We will find χ using the inverse of $\chi = B\eta_r$, where B is some linear bounded operator. Equation (12) will take the form

$$L_r\eta_r = M_r\eta_r + B\eta_r = (M_r + B)\eta_r.$$

Let's find $m = \max \mu_k \in \sigma_2^L(M)\{\mu_k\}$ and the number n of the obtained maximum value. Let's put

$$B = -\nu(\varepsilon + \nu_n)\mathbb{I},$$

where ε can be chosen as small as desired. Then the relative spectrum

$$\sigma^{L_u}(M_u + B) = \left\{\frac{\nu\nu_k - \nu(\varepsilon + \nu_n)}{\lambda - \nu_k}\right\}$$

lies in the left half-plane, of the complex plane and by virtue of Theorem 2, equality (13) is satisfied for the solution of $\eta_r = \eta_r(t)$.

CONCLUSION

It is planned to continue studies on stability and instability of solutions for stochastic semilinear equations of Sobolev type with a relatively spectral operator. It is planned to carry out numerical experiments on finding stable and unstable solutions of the stochastic system (2) and stabilization of unstable solutions.

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CONFLICT OF INTERESTS

The author of this paper declares that she has no conflict of interests.

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PARTIAL DIFFERENTIAL EQUATIONS

EXISTENCE OF SOLUTIONS OF THE BOUNDARY VALUE PROBLEM FOR THE DIFFUSION EQUATION WITH PIECEWISE CONSTANT ARGUMENTS

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Abstract. In this paper the boundary value problem (BVP) for diffusion equation with piecewise constant arguments is studied. By using the separation of variables method, the considered BVP is reduced to the investigation of the existence conditions of solutions of initial value problems for differential equation with piecewise constant arguments. Existence conditions of infinitely many solutions or emptiness for considered differential equation are established, and explicit formulas for these solutions are obtained. Several examples are given to illustrate the obtained results.

Keywords: *diffusion equation, piecewise constant argument, periodic solution*

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1. INTRODUCTION. PROBLEM STATEMENT

Differential equations with piecewise constant arguments are encountered in the study of hybrid systems and can model certain harmonic oscillators with almost periodic effects [1, 2]. A wide review of studies devoted to ordinary equations and partial differential equations with piecewise constant arguments is given in [3, 4].

In articles [5, 6], differential equations of special kind with piecewise constant argument are studied. Periodic (solvable) problems are reduced to a system of linear algebraic equations, all conditions for the existence of its n-periodic solutions are described, by means of which explicit formulas for solutions of differential equations are found.

Partial derivative equations with piecewise constant temporal argument arise naturally in the process of approximation [7].

In [8], the existence, oscillation and asymptotic bounds of solutions of initial problems with piecewise constant lags are studied for a partial derivative equation with piecewise constant argument.

Boundary and initial problems for the diffusion equation with piecewise constant arguments were studied in [9] and [10], respectively. The equation with piecewise constant mixed arguments of the form

$$u_t(x,t) = a^2u_{xx}(x,t) + bu_{xx}(x,[t-1]) + cu(x,[t]) + du(x,[t+1])$$

was considered in [11], where the questions of existence of solutions, convergence of solutions to zero, unboundedness of solutions and their oscillations were investigated.

In the paper [12], the asymptotic behavior of the solution of the diffusion equation with piecewise constant argument of generalized form is found.

In this paper, we consider a boundary value problem for the diffusion equation with piecewise constant arguments of the form [10, 13]

$$u_t(x,t) = a^2 u_{xx}(x,t) - b u_{xx}(x,[t]) - c u_{xx}(x,[t+1]), \quad 0 < x < 1,\ t > 0, \tag{1}$$

$$u(0,t) = u(1,t) = 0, \tag{2}$$

$$u(x,0) = v(x). \tag{3}$$

Adapting the method of [10, 14], we first obtain the formal solution of the problem (1)–(3) in the form of a series. For this purpose, after the separation of variables, we study the first order differential equation with piecewise constant time argument, obtain the existence condition and the explicit formula for its solution. Then, applying the method of [5, 6, 15, 16], we will find N-periodic solutions and their explicit formulas of this differential equation. In a special case, we prove the existence of an infinite number of solutions of the differential equation with piecewise constant argument, which shows the incorrectness of the result about the uniqueness given in [13].

2. DIFFERENTIAL EQUATION WITH PIECEWISE CONSTANT ARGUMENT

Let v_j be the coefficients of the sinusoidal Fourier series for the function $v(x)$, i.e.,

$$v(x) = \sum_{j=1}^{+\infty} v_j \sin(j\pi x), \quad v_j = 2\int_0^1 v(x)\sin(j\pi x)dx.$$

The solution of the problem (1)–(3) is found in the form

$$u(x,t) = \sum_{j=1}^{+\infty} T_j(t)\sin(j\pi x). \tag{4}$$

Substituting the function (4) into equation (1) and initial conditions from (3), we obtain

$$\sum_{j=1}^{\infty} \big(T_j'(t) + a^2\pi^2 j^2 T_j(t) + b\pi^2 j^2 T_j([t]) + c\pi^2 j^2 T_j([t+1])\big)\sin(j\pi x) = 0,$$

$$u(x,0) = \sum_{j=1}^{\infty} T_j(0)\sin(j\pi x) = v(x), \quad T_j(0) = v_j.$$

Hence, taking into account orthogonality of functions $\sin(n\pi x)$, we have an infinite sequence of ordinary differential equations with piecewise constant argument

$$T_j'(t) + a^2\pi^2 j^2 T_j(t) + b\pi^2 j^2 T_j([t]) + c\pi^2 j^2 T_j([t+1]) = 0, \quad t > 0, j \in \mathbb{N}, \tag{5}$$

with the initial condition

$$T_j(0) = v_j. \tag{6}$$

Definition 1. The function $T(t)$ is called a solution to the problem (5), (6), if it satisfies the following conditions:
(i) $T(t)$ is continuous with $\mathbb{R}_+$;
(ii) the derivative of $T'(t)$ exists and is continuous with $\mathbb{R}_+$, except for points $[t] \in \mathbb{R}_+$ where one-sided derivatives exist;
(iii) $T(t)$ satisfies (5) and (6) at $\mathbb{R}_+$ with a possible exception at $[t] \in \mathbb{R}_+$.

Let's denote

$$E_j(t) = e^{-a^2\pi^2 j^2 t} - \frac{b}{a^2}(1 - e^{-a^2\pi^2 j^2 t}), \quad D_j(t) = \frac{c}{a^2}(1 - e^{-a^2\pi^2 j^2 t}), \quad j \in \mathbb{N}.$$

Theorem 1. *Let a, b, c be real numbers. If $D_j(1) \neq -1$, then the equation* (5) *has a single solution represented at the intervals $t \in [n, n+1)$, $n = 0, 1, 2, \ldots$, in the form of*

$$T_j(t) = \left(E_j(t-n) - D_j(t-n)\frac{E_j(1)}{1 + D_j(1)}\right)\frac{E_j^n(1)}{(1 + D_j(1))^n} v_j. \tag{7}$$

Theorem 2. 1. *If* $D_j(1)=-1$ *and* $E_j(1)=0$ *for* $j>0$*, then the problem* (5), (6) *has infinitely many solutions. In particular, this problem has a single one-periodic and infinitely many* N*-periodic solutions,* $N=2,3,\ldots$

2. *Let* $D_j(1)=-1$ *and* $E_j(1)\neq 0$*. Then if* $v_j\neq 0$*, then problems* (5), (6) *have no solution. If* $v_j=0$*, then this problem has a trivial solution.*

Example 1. Let $j=1$, $a\in\mathbb{R}$, $c=a^2/(e^{-a^2\pi^2j^2}-1)$, $b=-a^2e^{-a^2\pi^2j^2}/(e^{-a^2\pi^2j^2}-1)$, $v_1=1$. In this case, $D_j(1)=-1$, $E_j(1)=0$. Functions

$$F_2(t)=\begin{cases}\left(\frac{1}{1-e^{a^2\pi^2}}+\frac{e^{a^2\pi^2}}{e^{a^2\pi^2}-1}e^{-a^2\pi^2t}\right)v_1-\frac{1-e^{-a^2\pi^2t}}{e^{-a^2\pi^2}-1}T_{11}(1), & t\in[0,1),\\ \left(\frac{1}{1-e^{a^2\pi^2}}+\frac{e^{a^2\pi^2}}{e^{a^2\pi^2}-1}e^{-a^2\pi^2(t-1)}\right)T_{11}(1)-\frac{1-e^{-a^2\pi^2(t-1)}}{e^{-a^2\pi^2}-1}v_1, & t\in[1,2],\end{cases}$$

and

$$F_3(t)=\begin{cases}\left(-\frac{b}{a^2}(1-e^{-a^2\pi^2t})+e^{-a^2\pi^2t}\right)v_1-\frac{c}{a^2}(1-e^{-a^2\pi^2t})T_{11}(1), & t\in[0,1),\\ \left(-\frac{b}{a^2}(1-e^{-a^2\pi^2(t-1)})+e^{-a^2\pi^2(t-1)}\right)T_{11}(1)-\frac{c}{a^2}(1-e^{-a^2\pi^2(t-1)})T_{21}(2), & t\in[1,2),\\ \left(-\frac{b}{a^2}(1-e^{-a^2\pi^2(t-2)})+e^{-a^2\pi^2(t-2)}\right)T_{21}(2)-\frac{c}{a^2}(1-e^{-a^2\pi^2(t-2)})v_1, & t\in[2,3),\end{cases}$$

are two- and three-periodic solutions of the problem (5), (6) at $j=1$, respectively, where $T_{11}(1)$, $T_{21}(2)$ are arbitrary numbers. Having chosen these constants, we give the solutions and their graphs.

The function $F_2(t)$ at $T_{11}(1)=3$ and $a=\frac{1}{\pi}$ has the following form (Fig. 1, a)

$$F_2(t)=\begin{cases}\frac{1}{1-e}+\frac{e^{1-t}}{e-1}-\frac{3(1-e^{-t})}{e^{-1}-1}, & t\in[0,1),\\ \frac{1-e^{1-t}}{1-e^{-1}}+3\left(\frac{1}{1-e}+\frac{e^{2-t}}{e-1}\right), & t\in[1,2].\end{cases}\qquad(8)$$

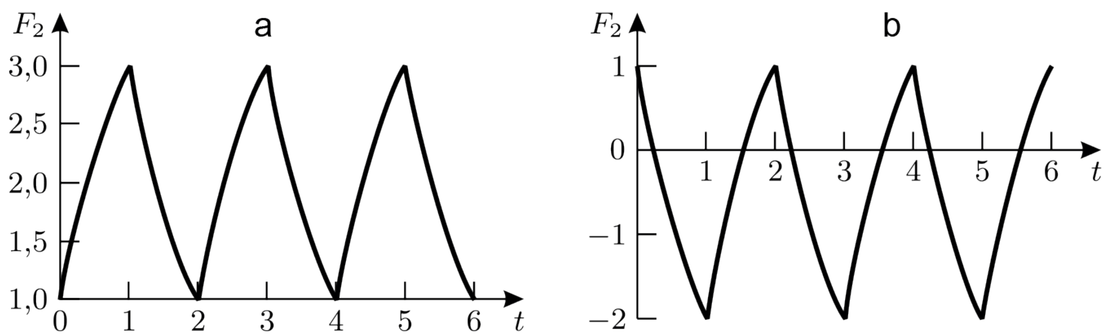

Fig. 1. Graphs of the function $F_2(t)$

and at $T_{11}(1)=-2$ and $a=\frac{1}{\pi}$ (Fig. 1, *b*)

$$F_2(t)=\begin{cases}\frac{1}{1-e}+\frac{e^{1-t}}{e-1}+\frac{2(1-e^{-t})}{e^{-1}-1}, & t\in[0,1),\\ \frac{e^{1-t}-1}{e^{-1}-1}-2\left(\frac{1}{1-e}+\frac{e^{2-t}}{e-1}\right), & t\in[1,2].\end{cases}\qquad(9)$$

The function $F_3(t)$ at $T_{11}(1)=2$, $T_{21}(2)=\frac{3}{2}$ and $a=\frac{1}{\pi}$ is represented as (Fig. 2, *a*)

$$F_3(t)=\begin{cases}\frac{1}{1-e}+\frac{e(2-e^{-t})}{e-1}, & t\in[0,1),\\ \frac{2}{1-e}+\frac{e(3+e^{1-t})}{2(e-1)}, & t\in[1,2),\\ \frac{3}{2(1-e)}+\frac{e(2+e^{2-t})}{2(e-1)}, & t\in[2,3],\end{cases}\qquad(10)$$

at $T_{11}(1)=-2$, $T_{21}(2)=-\frac{3}{2}$ and $a=\frac{1}{\pi}$ (Fig. 2, *b*)

$$F_3(t)=\begin{cases}\frac{1}{1-e}+\frac{e(3e^{-t}-2)}{e-1}, & t\in[0,1),\\ \frac{2}{e-1}-\frac{e(3+e^{1-t})}{2(e-1)}, & t\in[1,2),\\ \frac{3}{2(e-1)}-\frac{e(5e^{2-t}-2)}{2(e-1)}, & t\in[2,3],\end{cases} \tag{11}$$

and at $T_{11}(1)=3$, $T_{21}(2)=-4$ and $a=\frac{1}{\pi}$ (Fig. 2, *c*)

$$F_3(t)=\begin{cases}\frac{1}{1-e}+\frac{e(3-2e^{-t})}{e-1}, & t\in[0,1),\\ \frac{3}{1-e}+\frac{e(7e^{1-t}-4)}{e-1}, & t\in[1,2),\\ \frac{4}{e-1}-\frac{e(5e^{2-t}-1)}{e-1}, & t\in[2,3].\end{cases} \tag{12}$$

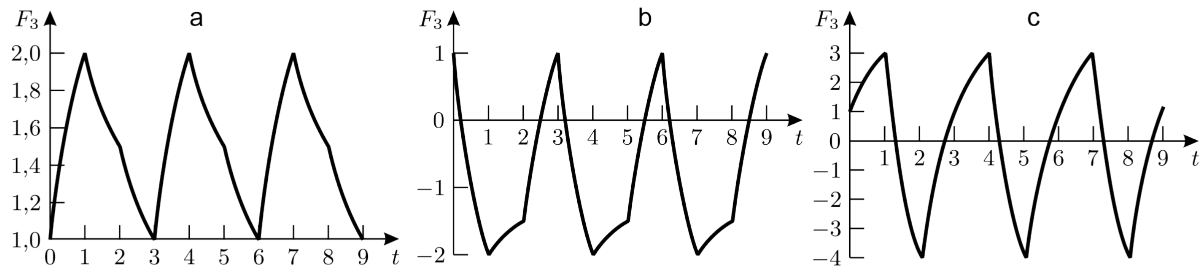

Fig. 2. Graphs of the function $F_3(t)$

Remark 1. In Example 1, the parameters of the equation satisfy the conditions of the singularity theorem from [13]. It shows the incorrectness of the results of Theorem 2 of [11], which asserts the uniqueness of the solution of the problem (5), (6).

3. PROBLEM SOLVING

Definition 2. The function $u(x,t)$ is called a solution of the problem (1)–(3), if the following conditions are satisfied:

(i) $u(x,t)$ is continuous on the set $\Omega=[0,1]\times\mathbb{R}_+$, $\mathbb{R}_+=[0,\infty)$;

(ii) the partial derivatives of u_t and u_{xx} exist and are continuous at Ω with a possible exception at points $(x,[t])\in\Omega$, where one-sided derivatives exist on the second argument;

(iii) $u(x,t)$ satisfies (1)–(3) at Ω with a possible exception at $(x,[t])\in\Omega$.

Assumption. *Let the function* $v(\cdot)$ *have continuous derivatives up to and including third order at the segment* $[0,1]$ *and satisfy the conditions* $v(0)=v(1)=v''(0)=v''(1)=0$.

Theorem 3. *Let the assumption* $c\neq -a^2$ *and* $D_j(1)\neq -1$ *at* $j\in\mathbb{N}$ *be satisfied. Then the problem* (1)–(3) *has a single solution represented as a series*

$$u(x,t)=\sum_{j=1}^{+\infty}\left(E_j(t-n)-D_j(t-n)\frac{E_j(1)}{1+D_j(1)}\right)\frac{E_j^n(1)}{(1+D_j(1))^n}v_j\sin(j\pi x),\quad t\in[n,n+1), n=0,1,2,\ldots$$

Theorem 4. 1. *Let the assumption be satisfied,* $D_{j_0}(1)=-1$ *and* $E_{j_0}(1)=0$. *Then the problem* (1)–(3) *has an infinite number of solutions represented by* $t\in[n,n+1)$, $n=0,1,2,\ldots$, *as*

$$u(x,t)=\sum_{\substack{j=1\\ j\neq j_0}}^{+\infty}\left(E_j(t-n)-D_j(t-n)\frac{E_j(1)}{1+D_j(1)}\right)\frac{E_j^n(1)}{(1+D_j(1))^n}v_j\sin(j\pi x)+T_{j_0}(t)\sin(j\pi x), \tag{12}$$

where $T_{j_0}(t)$ is an arbitrary solution of the problem (5), (6) (*see point 2 in Theorem 2).*

2. *If $D_{j_0}(1) = -1$, $E_{j_0}(1) \neq 0$ and $v_{j_0} \neq 0$ at $j = j_0$, then the problem* (1)–(3) *has no solution.*

Example 2. Let $a = 1/\pi$, $c = 2$, $b = 3$ in equation (1) and $u(x,0) = \sum_{j=1}^{5} \frac{\sin(j\pi x)}{j}$ in condition (3). Then the solution of the problem (1)–(3) has the following form (Fig. 3)

$$u(x,t) = \sum_{j=1}^{5} \left[\left(E_j(t-n) - D_j(t-n) \frac{E_j(1)}{1 + D_j(1)} \right) \frac{E_j^n(1)}{(1 + D_j(1))^n} v_j \right] \sin(j\pi x), \quad t \in [n, n+1), n = 0, 1, 2, \ldots$$

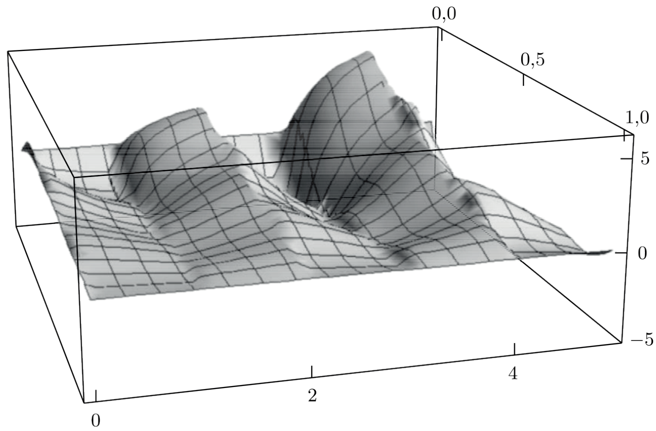

Fig. 3. Graph of the function $u(x,t)$

Example 3. Let $a \in \mathbb{R}$, $c = \frac{a^2}{e^{-a^2\pi^2 j^2} - 1}$, $b = -a^2 e^{-a^2\pi^2 j^2}/(e^{-a^2\pi^2 j^2} - 1)$, $v(x) = \sin(\pi x) + 2\sin(2\pi x)$. Then the solution of the problem (1)–(3) is defined by the formula

$$u(x,t) = T_1(t)\sin(\pi x) + 2T_2(t)\sin(2\pi x).$$

Note that $D_1(1) = -1$, $E_1(1) = 0$ and $D_2(1) = -1$, i.e., the numbers a, b and c satisfy the conditions of point 1 of Theorem 2 and Theorem 1. Therefore, according to Theorem 1, the function $T_2(t)$ has the form

$$T_2(t) = 2(E_2(t-n) - D_2(t-n)), \quad t \in [n, n+1),$$

and the function $T_1(t)$ can be defined in many ways.

Here are the graphs of $u(x,t)$ for Example 1. In the case when $T_1(t) = F_2(t)$ and $F_2(t)$ are defined by the equality (8), the graph of the function $u(x,t)$ is shown in Fig. 4, *a*; if $F_2(t)$ is defined by expression (9), then in Fig. 4, *b*. When $T_1(t) = F_3(t)$, where $F_3(t)$ is defined by equality (10), the graph of the function $u(x,t)$ is shown in Fig. 5, *a*; and if $F_3(t)$ is defined by equality (11), then in Fig. 5, *b*.

Remark 2. In Example 3, the parameters of the equation do not satisfy the conditions of Corollary 1 in [13], i.e., $a^2 + b + c = 0$. The solution of u is periodic on t. This means that the null solution of the problem (1)–(3) is not asymptotically stable. Therefore, the conditions of Corollary 1 are sufficient for the null solution to be asymptotically stable.

4. EVIDENCE FOR KEY FINDINGS

Proof of theorem 1. Let us denote by $T_{nj}(t)$ the solution of equation (5) on the interval $[n, n+1)$, i.e.

$$T_j(t) = T_{nj}(t), \quad t \in [n, n+1), \quad n = 0, 1, 2, \ldots$$

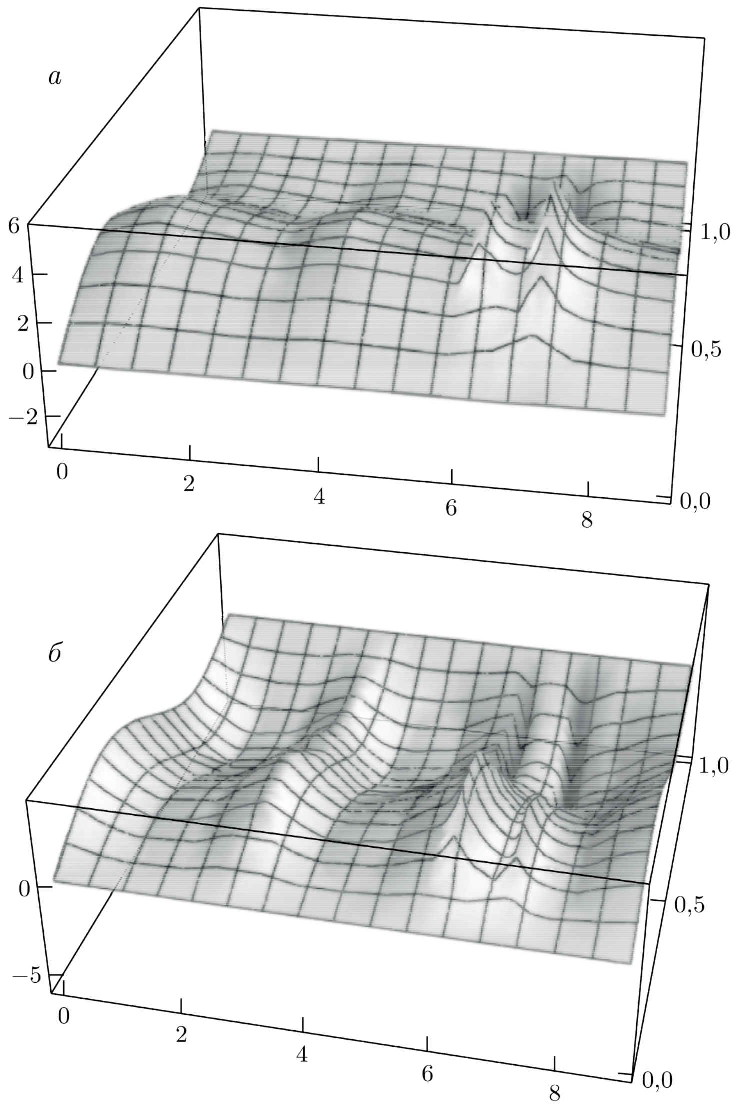

Fig. 4. Graph of the function $u(x,t)$

Then

$$T'_{nj}(t)+a^2\pi^2j^2T_{nj}(t)=-b\pi^2j^2T_{nj}(n)-c\pi^2j^2T_{nj}(n+1),\quad t\in[n,n+1). \tag{13}$$

The solution of the equation (13) is determined by the formula

$$T_{nj}(t)=-\frac{bT_{nj}(n)}{a^2}(1-e^{-a^2\pi^2j^2(t-n)})+T_{nj}(n)e^{-a^2\pi^2j^2(t-n)}-\frac{cT_{nj}(n+1)}{a^2}(1-e^{-a^2\pi^2j^2(t-n)})$$

or

$$T_{nj}(t)=E_j(t-n)T_{nj}(n)-D_j(t-n)T_{nj}(n+1),\quad t\in[n,n+1). \tag{14}$$

Putting $t=n+1$ in (14) for all $n=0,1,2,\ldots$, we get

$$T_{nj}(n+1)=E_j(1)T_{nj}(n)-D_j(1)T_{nj}(n+1).$$

Hence, taking into account $D_j(1)\neq-1$ we have

$$T_{nj}(n+1)=\frac{E_j(1)T_{nj}(n)}{1+D_j(1)}. \tag{15}$$

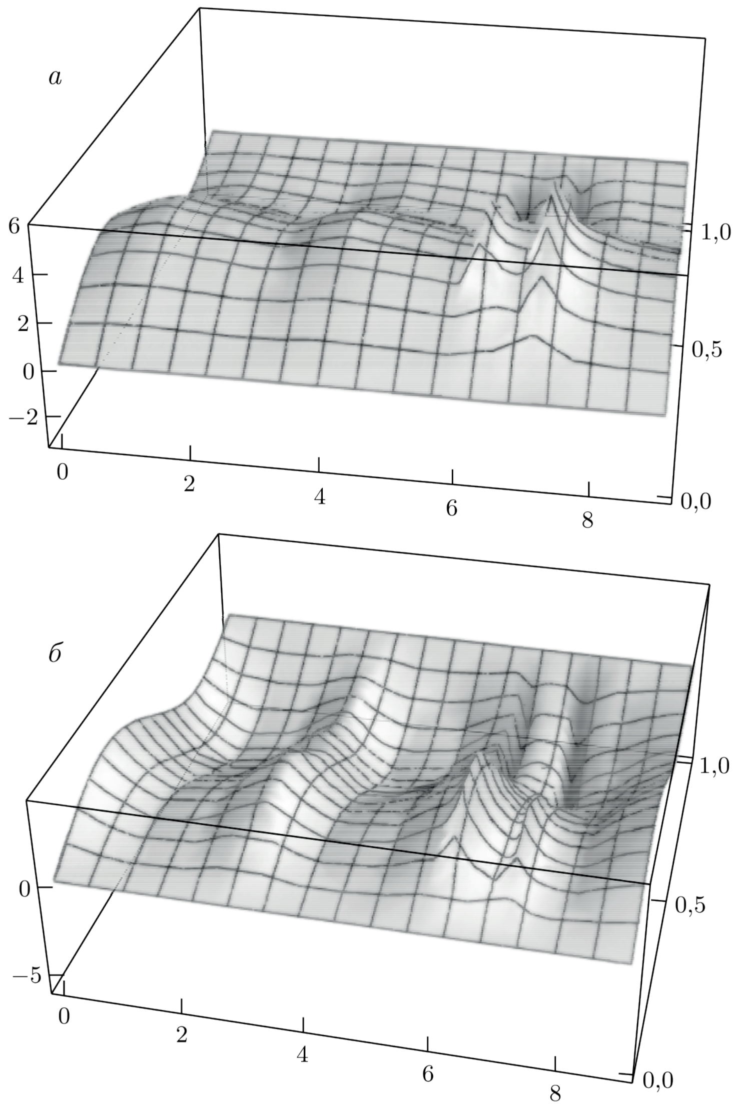

Fig. 5. Graph of the function $u(x,t)$

Then we write (14) as

$$T_{nj}(t) = E_j(t-n)T_{nj}(n) - \frac{D_j(t-n)}{1+D_j(1)}E_j(1)T_{nj}(n). \tag{16}$$

From the continuity of the function $T_j(t)$ over $t>0$ the following equations follow

$$T_{(n+1)j}(n+1) = T_j(n+1) = \lim_{t\to n+1-0} T_j(t) = T_{nj}(n+1).$$

Consequently, formula (15) can be rewritten in the form

$$T_{(n+1)j}(n+1) = \frac{E_j(1)T_{nj}(n)}{1+D_j(1)},$$

from where

$$T_{nj}(n) = \frac{E_j(1)}{1+D_j(1)}T_{(n-1)j}(n-1) = \frac{E_j^2(1)}{(1+D_j(1))^2}T_{(n-2)j}(n-2) = \dots = \frac{E_j^n(1)}{(1+D_j(1))^n}T_{0j}(0),$$

or

$$T_{nj}(n) = \frac{E_j^n(1)}{(1+D_j(1))^n} T_{0j}(0).$$

Thus, the solution $T_{nj}(t)$, defined by the formula (16), is represented only via $T_{0j}(0)$:

$$T_{nj}(t) = \left(E_j(t-n) - D_j(t-n)\frac{E_j(1)}{1+D_j(1)} \right) \frac{E_j^n(1)}{(1+D_j(1))^n} T_{0j}(0).$$

The equality $T_{0j}(0) = v_j$ completes the proof of the theorem.

Proof of theorem 2. 1. Let $D_j(1) = -1$, $E_j(1) = 0$. Construct the function $T_j(t) = T_{nj}(t)$, $t \in [n, n+1)$, $n = 0, 1, 2, \ldots$, as follows. Function

$$T_{0j}(t) = E_j(t)T_{0j}(0) - D_j(t)C_{0j}, \quad t \in [0,1),$$

satisfies the equation (5), where $T_{0j}(0) = v_j$ and C_{0j} are arbitrary numbers. Since $D_j(1) = -1$ and $E_j(1) = 0$, there is an equality $T_{0j}(1) = \lim_{t\to 1} T_{0j}(t) = C_{0j}$. It is easy to check that the function

$$T_{1j}(t) = E_j(t-1)T_{1j}(1) - D_j(t-1)C_{1j}, \quad t \in [1,2),$$

satisfies equation (5), where C_{1j} is an arbitrary number.

By virtue of continuity of the function $T_j(t)$ we have

$$T_j(1) = T_{1j}(1) = \lim_{t\to 1-0} T_{0j}(t) = T_{0j}(1).$$

The equalities $D_j(1) = -1$ and $E_j(1) = 0$ give $T_{1j}(2) = \lim_{t\to 2} T_{1j}(t) = C_{1j}$.

Function

$$T_{nj}(t) = E_j(t-n)T_{nj}(n) - D_j(t-n)C_{nj}, \quad \text{at } (n, n+1), n \in \mathbb{N},$$

satisfies the equation (5), where C_{nj} is an arbitrary number. Clearly,

$$T_j(n) = T_{nj}(n) = \lim_{t\to n-0} T_{(n-1)j}(t) = T_{(n-1)j}(n).$$

Similarly, from the equalities $D_j(1) = -1$ and $E_j(1) = 0$, we obtain $T_{nj}(n) = \lim_{t\to n+1} T_{nj}(t) = C_{nj}$. After the construction of the function

$$T_j(t) = T_{nj}(t), \quad t \in [n, n+1), n = 0, 1, 2, \ldots,$$

appears the solution of the problem (5), (6). Since the constants $C_{0j}, C_{1j}, \ldots, C_{nj}, \ldots$ are arbitrary, the problem has an infinite number of solutions.

Let $T_j(t)$ be a one-periodic solution of the problem (5), (6), then it can be represented as

$$T_j(t) = T_0 j(t) = E_j(t)T_0 j(0) - D_j(t)C_0 j, \quad t \in [0,1].$$

Since the function $T_j(t)$ is one-periodic and $T_0 j(1) = C_0 j$, then $T_0 j(0) = T_0 j(1)$, $C_0 j(1) = T_0 j(0) = v_j$. This shows the uniqueness of the one-periodic solution (5), (6).

Let $T_j(t)$ be a two-periodic solution of the problem (5), (6). Then the function $T_j(t)$ on $[0,2]$ has the form

$$T_j(t) = \begin{cases} E_j(t)T_0 j(0) - D_j(t)T_1 j(1), & t \in [0,1), \\ E_j(t-1)T_1 j(1) - D_j(t-1)C_1 j, & t \in [1,2), \end{cases}$$

where $T_0 j(0) = v_j$, $T_1 j(1)$ is an arbitrary number. From the periodicity of $T_j(t)$ it follows that $T_j(0) = T_0 j(0) = T_j(2) = C_1 j$. This shows that the problem (5), (6) has infinitely many two-periodic solutions.

Let $T_j(t)$ be the N-periodic solution of the problem (5), (6). The function $T_j(t)$ on the interval $[0, N]$ has the form

$$T_j(t) = \begin{cases} E_j(t)v_j - D_j(t)T_1 j(1), & t \in [0,1), \\ E_j(t-1)T_1 j(1) - D_j(t-1)T_2 j(2), & t \in [1,2), \\ \vdots & \\ E_j(t-N+2)T_{(N-1,j)}(N-2) - D_j(t-N+2)T_{(N-1,j)}(N-1), & t \in [N-2, N-1), \\ E_j(t-N+1)T_{(N-1,j)}(N-1) - D_j(t-N+1)v_j, & t \in [N-1, N), \end{cases}$$

where $T_1j(1), T_2j(2), \ldots, T_{(N-1,j)}(N-1)$ are arbitrary numbers.

2. Suppose that the function $T_j(t)$ is a solution of the problem (5), (6). Then, according to (14), the following equality holds

$$T_nj(t) = E_j(t-n)T_nj(n) - D_j(t-n)T_nj(n+1), \quad t \in [n, n+1).$$

Hence at $t = n+1$ taking into account $D_j(1) = -1$, we have $E_j(1)T_nj(n) = 0$ for all $n = 0, 1, 2, \ldots$ Therefore, $T_nj(n) = 0$ for all $n = 0, 1, 2, \ldots$, since $E_j(1) \neq 0$, i.e., the equation has only a trivial solution. Hence, if $T_j(0) = v_j = T_0j(0) \neq 0$, then the problem (5), (6) has no solution.

Proof of Theorem 3. First, prove uniform convergence in any closed set $\overline{\Lambda} \subset [0,1] \times \mathbb{R}_+$ of the following series:

$$\sum_{j=1}^{+\infty} T_j(t)\sin(j\pi x), \tag{17}$$

$$\sum_{j=1}^{+\infty} T_j'(t)\sin(j\pi x), \tag{18}$$

$$\sum_{j=1}^{+\infty} \pi^2 j^2 T_j(t)\sin(j\pi x), \tag{19}$$

where $T_j(t)$ is the solution of the problem (5), (6), and at $[n, n+1)$, $n = 0, 1, 2, \ldots$, the functions $T_j(t)$, $T_j'(t)$ are represented, respectively, as (7) and

$$T_j'(t) = -\left(a^2 + b + c\frac{E_j(1)}{1 + D_j(1)}\right)\pi^2 j^2 e^{-a^2\pi^2 j^2(t-n)}\frac{E_j^n(1)}{(1 + D_j(1))^n}v_j.$$

According to the assumption there is equality

$$v_j = -\frac{2v_j'''}{\pi^3 j^3}, \quad v_j''' = \int_0^1 v'''(x)\cos(j\pi x)dx, \quad j = 1, 2, \ldots$$

The continuity of the function $v'''(x)$ implies the convergence of the series $\sum_{j=1}^{+\infty}(v_j''')^2$. Hence, taking into account the Cauchy-Bunyakovsky inequality, we have

$$\left|\sum_{j=1}^{+\infty} j^2 v_j\right| = \frac{2}{\pi^3}\left|\sum_{j=1}^{+\infty}\frac{v_j'''}{j}\right| < +\infty. \tag{20}$$

Since $0 \le 1 - e^{-a^2\pi^2 j^2 t} \le 1$, the inequalities are true for all $t \in [0, \infty)$ and $j \in \mathbb{N}$:

$$|E_j(t)| \le 1 + \frac{|b|}{a^2}, \quad |D_j(t)| < \frac{|c|}{a^2}. \tag{21}$$

Note that $\lim_{j\to\infty} D_j(1) = c/a^2$, so given $D_j(1) \neq -1$ and $c \neq -a^2$ there exists a number $\rho > 0$ such that

$$|1 + D_j(1)| \ge \rho, \quad j \in \mathbb{N}. \tag{22}$$

Using inequalities (21) and (22), we obtain uniform estimates for $T_j(t)$ and $T_j'(t)$:

$$|T_j(t)| \le C_1\left(\frac{1 + \frac{|b|}{a^2}}{\rho}\right)^n |v_j|, \quad t \in [n, n+1), \tag{23}$$

$$|T_j'(t)| \le C_2\left(\frac{1 + \frac{|b|}{a^2}}{\rho}\right)^n \pi^2 j^2 |v_j|, \quad t \in [n, n+1), \tag{24}$$

where

$$C_1 = 1 + \frac{|b|}{a^2} + \frac{|c|}{a^2}\frac{1 + \frac{|b|}{a^2}}{\rho},$$

$$C_2 = a^2 + |b| + |c|\frac{1 + \frac{|b|}{a^2}}{\rho}.$$

Let $m = 1 + \sup_{(x,t)\in\overline{\Lambda}} t$. Then from (23) and (24) for all $(x,t) \in \overline{\Lambda}$, the series (17)–(19) are evaluated as follows:

$$\left|\sum_{j=1}^{+\infty} T_j(t)\sin(j\pi x)\right| \le C_1 \left(\frac{1+\frac{|b|}{a^2}}{\rho}\right)^m \sum_{j=1}^{+\infty} |v_j|,$$

$$\left|\sum_{j=1}^{+\infty} T_j'(t)\sin(j\pi x)\right| \le C_2 \left(\frac{1+\frac{|b|}{a^2}}{\rho}\right)^m \pi^2 \sum_{j=1}^{+\infty} j^2|v_j|,$$

$$\left|\sum_{j=1}^{+\infty} \pi^2 j^2 T_j(t)\sin(j\pi x)\right| \le C_1 \left(\frac{1+\frac{|b|}{a^2}}{\rho}\right)^m \pi^2 \sum_{j=1}^{+\infty} j^2|v_j|.$$

Hence and from (20), we obtain uniform convergence of series (17)–(19) in any closed set $\overline{\Lambda} \subset [0,1] \times \mathbb{R}_+$.

Thus, the function $u(x,t) = \sum_{j=1}^{+\infty} T_j(t)\sin(j\pi x)$ is continuous on the set $\Omega = [0,1] \times \mathbb{R}_+$; and the partial derivatives $u_t = \sum_{j=1}^{+\infty} T_j'(t)\sin(j\pi x)$, $u_{xx} = \sum_{j=1}^{+\infty} \pi^2 j^2 T_j(t)\sin(j\pi x)$ exist and are continuous on Ω with a possible exception at points $(x,[t]) \in \Omega$, where one-sided derivatives exist on the second argument.

Since $D_j(1) \ne -1$ for each $j \in \mathbb{N}$, then by Theorem 1 the problem (5), (6) has a single solution $T_j(t)$ for each $j \in \mathbb{N}$. Hence, the function $u(x,t)$, defined by the formula (4), satisfies the equalities (5), (6) in Ω with possible exceptions at the points $(x,[t]) \in \Omega$ and is the only solution of the problem (1)–(3).

Proof of Theorem 4. 1. Let $D_{j_0}(1) = -1$ and $E_{j_0}(1) = 0$ for some $j = j_0$. Then $D_j(1) > -1$ at $j < j_0$ and $D_j(1) < -1$ at $j > j_0$. Hence we have

$$|1 + D_j(1)| \ge \rho_1$$

for some number $\rho_1 > 0$ and for all $j \in \mathbb{N}\backslash\{j_0\}$.

By Theorem 1, the problem (5), (6) is solvable for $j \ne j_0$ and the solution of $T_j(t)$ at $j \ne j_0$ is of the form (7). Since $D_{j_0}(1) = -1$; and $E_{j_0}(1) = 0$, then by point 1 of Theorem 2, the problem (5), (6) has infinitely many solutions. Let us denote by $T_{j_0}(\cdot)$ the solution of the problem (5), (6) for $j = j_0$. Then from (4) the solution of the boundary value, problem (1)–(3) has the form (12). The uniform convergence of this series to a continuous function $u(x,t)$ in any closed set $\overline{\Lambda} \subset [0,1] \times \mathbb{R}_+$ and the existence of continuous partial derivatives of u_t and u_{xx} on Ω with a possible exception at the points $(x,[t]) \in \Omega$, where one-sided derivatives exist on the second argument, are proved similarly as in the proof of Theorem 3.

2. If $D_{j_0}(1) = -1$, $E_{j_0}(1) \ne 0$ and $v_{j_0} \ne 0$, then by Theorem 2 the problem (5), (6) has no solution at $j = j_0$. Hence, according to (4), the boundary value problem (1)–(3) has no solution.

CONFLICT OF INTERESTS

The authors of this paper declare that they have no conflict of interests.

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PARTIAL DIFFERENTIAL EQUATIONS

ON FRONT MOTION IN THE REACTION–DIFFUSION–ADVECTION PROBLEM WITH KPZ-NONLINEARITY

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Abstract. We obtain an asymptotic approximation to a moving inner layer (front) solution of an initial–boundary value problem for a singularly perturbed parabolic reaction–diffusion–advection equation with KPZ-nonlinearity. An asymptotic approximation for the velocity of the front is found. To prove the existence and uniqueness of a solution the asymptotic method of differential inequalities is used.

Keywords: *reaction–diffusion–advection equation, KPZ-nonlinearity, contrast structures, front motion, small parameter*

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1. INTRODUCTION. PROBLEM STATEMENT

In this paper, we consider the initial boundary value problem for the singularly perturbed parabolic equation, that differs from the classical singularly perturbed reaction–diffusion–advection equation (see [1, 2]) by the presence of an additional nonlinear term containing the square of the gradient of the desired function (KPZ-nonlinearities [3, 4]):

$$\varepsilon^2\frac{\partial^2 u}{\partial x^2}-\varepsilon\frac{\partial u}{\partial t}-\varepsilon^2 A(u,x)\left(\frac{\partial u}{\partial x}\right)^2-f(u,x,\varepsilon)=0,\quad x\in(-1,1), t\in(0,T],$$
$$\frac{\partial u}{\partial x}(-1,t,\varepsilon)=0,\quad \frac{\partial u}{\partial x}(1,t,\varepsilon)=0,\quad t\in[0,T],$$
$$u(x,0,\varepsilon)=u_{\text{init}}(x,\varepsilon),\quad x\in[-1,1],\tag{1}$$

where $\varepsilon\in(0,\varepsilon_0]$ is the small parameter, $\varepsilon>0$ is a given constant.

Traveling wave type solutions for quasilinear parabolic reaction–diffusion–advection equations are the subject of intensive study (see extensive monographs [5, 6]). Attention to nonlinearities of the form $A(u,x)\left(\frac{\partial u}{\partial x}\right)^2$ is due to both theoretical interest – the square is the limit of degree at which the Bernstein conditions on the growth of the nonlinearity are satisfied (see, e.g., [7–9]), and important applications where such nonlinearities are used in mathematical models, in particular, population dynamics models [10], in modeling free surface growth in polymer theory [3, 4, 11], and many others. We note the work [12], in which exact solutions of the KPZ equation are constructed for several physically justified nonlinearities. However, it is assumed there that $(u,x)=\text{const}\ f=f(x,t)$. The principal difference of problem (1) is that we consider an equation, where the nonlinear terms depend explicitly on the coordinate and the desired function. In this paper, we propose an algorithm for constructing an asymptotic approximation of the solution of the front view, with the velocity of motion being a function of the coordinate.

Stationary solutions of problem (1) with boundary and inner layers are studied in [13, 14]. The boundary-layer solutions of the Tikhonov-type system with KPZ-nonlinearities are studied in [15].

The paper is structured as follows. In (2), we construct an asymptotic approximation of the moving front solution using the method of A. B. Vasilieva [16]. Note that since problem (1) is singularly perturbed, at $\varepsilon=0$ the equation of problem (1) changes its type from parabolic to algebraic with three roots (see condition 2), two of them describe stable equilibrium positions of the system and represent the regular part of the asymptotic approximation of zero order of accuracy. However, the regular approximation does not allow us to describe a narrow region with a large gradient, in which the solution passes from one stable level to another. To describe the solution in this region and to harmonize the stable equilibrium positions among themselves, the so-called transition layer functions are constructed. In this way, a formal asymptotic approximation of the solution in the whole region under consideration is constructed. In (3), an algorithm for finding an asymptotic approximation of the front position is given. In (4), we give a justification of the formal asymptotics and prove the existence and uniqueness theorem using the asymptotic method of differential inequalities of N. N. Nefedov, that has shown its efficiency in many singularly perturbed problems [16]. The obtained results are illustrated in Section 5 by an example, that can be used to develop and verify new numerical methods for the considered class of problems (see [17]).

The results obtained in this paper develop the studies [1, 2], in which the front motion in the reaction–diffusion–advection equation with weak advection and smooth or modular (discontinuous at some value of the desired function nonlinearities) sources was considered, and transfer them to a new class of singularly perturbed problems with KPZ-nonlinearities. At the same time, as in [1, 2], the existence and uniqueness theorem of the solution having in both cases the same form of the contrast structure of the step type [16] is proved.

In the problem discussed below, it is assumed that at the initial moment of time the front is already formed. This means that the function $u_{\text{init}}(x,\varepsilon)$ has an internal transition layer in the neighborhood of some point $x_{00} \in (-1,1)$, i.e., it is close to some root $\varphi^{(-)}(x)$ of the degenerate equation $f(u,x,0) = 0$ to the left of the point x_{00} and to the root $\varphi^{(+)}(x)$ to the right of this point. In the neighborhood of x_{00} there is a sharp transition from $\varphi^{(-)}(x)$ to $\varphi^{(+)}(x)$.

We will assume that the following conditions are satisfied.

Condition 1. The functions $A(u,x)$, $f(u,x,\varepsilon)$ are sufficiently smooth in their areas of definition.

Condition 2. The derived equation $f(u,x,0) = 0$ has exactly three solutions $u = \varphi^{(\pm,0)}(x)$, with $\varphi^{(-)}(x) < \varphi^{(0)}(x) < \varphi^{(+)}(x)$, $x \in [-1,1]$, while the following inequalities are also valid

$$f_u(\varphi^{(\pm)}(x),x,0) > 0, \quad f_u(\varphi^{(0)}(x),x,0) < 0, \quad x \in [-1,1].$$

2. CONSTRUCTION OF FORMAL ASYMPTOTICS OF THE SOLUTION

The asymptotics of the solution of problem (1) is constructed by the method of boundary functions separately in each of the regions $[-1,\hat{x}]\times[0,T]$ and $[\hat{x},1]\times[0,T]$ with a moving boundary (see [16]) using the effective method developed in the scientific school of Professors A. B. Vasilieva, V. F. Butuzov, and N. N. Nefedov for constructing the asymptotics of localization of the inner layer in the form of

$$U(x,\varepsilon) = \begin{cases} U^{(-)}(x,t,\varepsilon), & (x,t,\varepsilon) \in [-1,\hat{x}] \times [0,T] \times (0,\varepsilon_0], \\ U^{(+)}(x,t,\varepsilon), & (x,t,\varepsilon) \in [\hat{x},1] \times [0,T] \times (0,\varepsilon_0]. \end{cases}$$

We will represent each of the functions $U^{(\pm)}(x,\varepsilon)$ as a sum of three summands:

$$U^{(\pm)}(x,t,\varepsilon) = \bar{u}^{(\pm)}(x,\varepsilon) + Q^{(\pm)}(\xi,t,\varepsilon) + R^{(\pm)}(\eta^{(\pm)},\varepsilon),$$

where $\bar{u}^{(\pm)}(x,\varepsilon) = \bar{u}_0^{(\pm)}(x) + \varepsilon\bar{u}_1^{(\pm)}(x) + \cdots$ is the regular part of the decomposition, functions $Q^{(\pm)}(\xi,t,\varepsilon) = Q_0^{(\pm)}(\xi,t,\varepsilon) + \varepsilon Q_1^{(\pm)}(\xi,t,\varepsilon) + \cdots$ describe the behavior of the solution in the vicinity of the transition point $\hat{x}(t,\varepsilon)$, $\xi = \frac{x-\hat{x}(t,\varepsilon)}{\varepsilon}$ is the transition layer variable: $\xi \le 0$ for functions with index $(-)$ and $\xi \ge 0$ for functions with index $(+)$; functions $R^{(\pm)}(\eta^{(\pm)},\varepsilon) = R_0^{(\pm)}(\eta^{(\pm)}) + \varepsilon R_1^{(\pm)}(\eta^{(\pm)}) + \cdots$ describe the behavior of the solution in the vicinity of the boundary points of the segment $[-1,1]$; $\eta^{(\pm)} = \frac{x\mp 1}{\varepsilon}$ are stretched variables near the points $x = \pm 1$, respectively. Since the functions $R_i^{(\pm)}(\eta^{(\pm)})$ are defined in a standard way (see, for example, [16]), we omit the procedure of their construction. Note that these functions do not depend on the variable t and thus do not participate in the description of the moving transition layer, and the functions $R_0^{(\pm)}(\eta^{(\pm)}) = 0$ by virtue of the Neumann boundary conditions.

The position of the inner transition layer is determined from the condition C^1-combining the asymptotic representations $U^{(-)}(x,t,\varepsilon)$ and $U^{(+)}(x,t,\varepsilon)$ at the transition point $\hat{x}(t,\varepsilon)$:

$$U^{(-)}(\hat{x}(t,\varepsilon),t,\varepsilon) = U^{(+)}(\hat{x}(t,\varepsilon),t,\varepsilon) = \phi^{(0)}(\hat{x}(t,\varepsilon)), \tag{2}$$

$$\varepsilon\frac{\partial}{\partial x}U^{(-)}(\hat{x}(t,\varepsilon),t,\varepsilon) = \varepsilon\frac{\partial}{\partial x}U^{(+)}(\hat{x}(t,\varepsilon),t,\varepsilon). \tag{3}$$

We will look for the transition point $x = \hat{x}(t,\varepsilon)$ in the form of expansion by powers of the small parameter ε:

$$\hat{x}(t,\varepsilon) = x_0(t) + \varepsilon x_1(t) + \cdots \tag{4}$$

The coefficients of this expansion will be determined in the process of asymptotics construction.

The regular part of the asymptotics is determined after substituting the representation for the functions $\bar{u}^{(\pm)}(x,\varepsilon)$ into the equation.

$$\varepsilon^2\frac{\partial^2\bar{u}^{(\pm)}}{\partial x^2} - \varepsilon^2 A(\bar{u}^{(\pm)},x)\left(\frac{\partial\bar{u}^{(\pm)}}{\partial x}\right)^2 - f(\bar{u}^{(\pm)},x,\varepsilon) = 0.$$

In the standard way [16], we obtain the algebraic equations for determining the functions of the regular part $\bar{u}_k^{(\pm)}(x)$, $k = 0,1,\ldots$

Taking into account condition 2, the regular zero-order functions are defined as

$$\bar{u}_0^{(\pm)}(x) = \phi^{(\pm)}(x).$$

To shorten the record, we introduce the notations

$$\bar{f}_u^{(\pm)}(x) := f_u(\phi^{(\pm)}(x),x,0).$$

Functions $\bar{u}_k^{(\pm)}(x)$ at $k = 1,2,\ldots$ are defined from equations

$$\bar{f}_u^{(\pm)}(x)\bar{u}_k^{(\pm)}(x) = \bar{h}_k^{(\pm)}(x),$$

where the functions $\bar{h}_k^{(\pm)}(x)$ are known at each k-step and are expressed recurrently through the functions $\bar{u}_k^{(\pm)}(x)$ with indices $0,1,\ldots,k-1$. The solvability of these equations follows from condition 2.

In order to obtain the equations satisfied by the transition layer functions $Q_k^{(\pm)}(\xi,t,\varepsilon)$, let us rewrite the differential operator of the problem in the variables (ξ,t). Then the equations for the functions $Q_k^{(\pm)}(\xi,t,\varepsilon)$, $k = 0,1,\ldots$, are determined in the standard way [16] by equating the coefficients at the same degrees ε in both parts of the equations:

$$\begin{aligned}\frac{\partial^2 Q^{(\pm)}}{\partial\xi^2} + \frac{\partial\hat{x}(t,\varepsilon)}{\partial t}\frac{\partial Q^{(\pm)}}{\partial\xi} + A\big(\bar{u}^{(\pm)}\big(\varepsilon\xi+\hat{x}(t,\varepsilon),\varepsilon\big),\varepsilon\xi+\hat{x}(t,\varepsilon)\big)\left(\frac{\partial\bar{u}^{(\pm)}}{\partial\xi}\right)^2 - \\ - A\big(\bar{u}^{(\pm)}\big(\varepsilon\xi+\hat{x}(t,\varepsilon),\varepsilon\big) + Q^{(\pm)}(\xi,t,\varepsilon),\varepsilon\xi+\hat{x}(t,\varepsilon)\big)\left(\frac{\partial Q^{(\pm)}}{\partial\xi} + \frac{\partial\bar{u}^{(\pm)}}{\partial\xi}\right)^2 - \varepsilon\frac{\partial Q^{(\pm)}}{\partial t} = \\ = f\big(\bar{u}^{(\pm)}\big(\varepsilon\xi+\hat{x}(t,\varepsilon),\varepsilon\big) + Q^{(\pm)}(\xi,t,\varepsilon),\varepsilon\xi+\hat{x}(t,\varepsilon),\varepsilon\big) - f\big(\bar{u}^{(\pm)}\big(\varepsilon\xi+\hat{x}(t,\varepsilon),\varepsilon\big),\varepsilon\xi+\hat{x}(t,\varepsilon),\varepsilon\big).\end{aligned} \tag{5}$$

In contrast to the approach in [2], we will not decompose by powers of ε the transition point $\hat{x}(t,\varepsilon)$. This will simplify the algorithm for constructing the asymptotics. Note that the equations from which the functions $Q_k^{(\pm)}(\xi,t,\varepsilon)$ are found contain functions depending on $\hat{x}(t,\varepsilon)$, $\frac{\partial\hat{x}(t,\varepsilon)}{\partial t}$, and that explains the presence of the argument ε at $Q_k^{(\pm)}(\xi,t,\varepsilon)$.

We require that the transition layer functions $Q_k^{(\pm)}(\xi,t,\varepsilon)$, $k = 0,1,\ldots$, satisfy the conditions of equality to zero at infinity: $Q_k^{(-)}(\xi,t,\varepsilon)\to 0$ at $\xi\to-\infty$, $Q_k^{(+)}(\xi,t,\varepsilon)\to 0$ at $\xi\to+\infty$, $k = 0,1,\ldots$, $t\in[0,T]$.

Equating the coefficients at ε^0 in the right and left parts of equations (5), we obtain equations for the function $Q_0^{(-)}(\xi,t,\varepsilon)$ at $\xi\le 0$ and the function $Q_0^{(+)}(\xi,t,\varepsilon)$ at $\xi\ge 0$:

$$\begin{aligned}\frac{\partial^2 Q_0^{(\pm)}}{\partial\xi^2} + \frac{\partial\hat{x}(t,\varepsilon)}{\partial t}\frac{\partial Q_0^{(\pm)}}{\partial\xi} - A\big(\varphi^{(\pm)}(\hat{x}(t,\varepsilon)) + Q_0^{(\pm)}(\xi,t,\varepsilon),\hat{x}(t,\varepsilon)\big)\left(\frac{\partial Q_0^{(\pm)}}{\partial\xi}\right)^2 = \\ = f\big(\varphi^{(\pm)}(\hat{x}(t,\varepsilon)) + Q_0^{(\pm)}(\xi,t,\varepsilon),\hat{x}(t,\varepsilon),0\big).\end{aligned} \tag{6}$$

We obtain the additional conditions at $\xi = 0$ from the continuous cross-linking condition (2) written in zero order at ε:

$$Q_0^{(-)}(0,t,\varepsilon) + \phi^{(-)}(\hat{x}(t,\varepsilon)) = Q_0^{(+)}(0,t,\varepsilon) + \phi^{(+)}(\hat{x}(t,\varepsilon)) = \phi^{(0)}(\hat{x}(t,\varepsilon)).$$

We also add conditions at infinity: $Q_0^{(-)}(\xi,t,\varepsilon) \to 0$ at $\xi \to -\infty$, $Q_0^{(+)}(\xi,t,\varepsilon) \to 0$ at $\xi \to +\infty$, $t \in [0,T]$.

Let's introduce the operator D, acting by the rule

$$D\hat{x} := \frac{\partial \hat{x}(t,\varepsilon)}{\partial t}, \tag{7}$$

and functions

$$\begin{gathered}
\tilde{u}^{(\pm)}(\xi,t,\varepsilon) = \phi^{(\pm)}(\hat{x}(t,\varepsilon)) + Q_0^{(\pm)}(\xi,t,\varepsilon), \\
\tilde{u}(\xi,t,\varepsilon) = \begin{cases} \phi^{(-)}(\hat{x}(t,\varepsilon)) + Q_0^{(-)}(\xi,t,\varepsilon), & \text{if } \xi \le 0, \\ \phi^{(+)}(\hat{x}(t,\varepsilon)) + Q_0^{(+)}(\xi,t,\varepsilon), & \text{if } \xi \ge 0, \end{cases} \\
\tilde{v}^{(-)}(\xi,t,\varepsilon) = \frac{\partial \tilde{u}}{\partial \xi}(\xi,t,\varepsilon), \quad \xi \le 0, \\
\tilde{v}^{(+)}(\xi,t,\varepsilon) = \frac{\partial \tilde{u}}{\partial \xi}(\xi,t,\varepsilon), \quad \xi \ge 0.
\end{gathered} \tag{8}$$

Remark. It follows from the form of equations (6), that in the functions $Q_0^{(\pm)}(\xi,t,\varepsilon)$, $\tilde{u}(\xi,t,\varepsilon)$, $\tilde{u}^{(\pm)}(\xi,t,\varepsilon)$, $\tilde{v}^{(\pm)}(\xi,t,\varepsilon)$, we can switch to another set of arguments – $(\xi,\hat{x})$. In the future, we will use both sets, choosing the most convenient for each particular case.

Let us rewrite equations (6), as well as the additional conditions, using (8):

$$\begin{gathered}
\frac{\partial^2 \tilde{u}^{(\pm)}}{\partial \xi^2} + D\hat{x}\frac{\partial \tilde{u}^{(\pm)}}{\partial \xi} - A(\tilde{u}^{(\pm)},\hat{x})\left(\frac{\partial \tilde{u}^{(\pm)}}{\partial \xi}\right)^2 = f(\tilde{u}^{(\pm)},\hat{x},0), \\
\tilde{u}^{(\pm)}(0,\hat{x}) = \phi^{(0)}(\hat{x}), \quad \tilde{u}^{(\pm)}(\pm\infty,\hat{x}) = \phi^{(\pm)}(\hat{x}).
\end{gathered} \tag{9}$$

Along with the problems (9), let us consider the problem

$$\frac{\partial^2 \hat{u}}{\partial \xi^2} + W\frac{\partial \hat{u}}{\partial \xi} - A(\hat{u},\hat{x})\left(\frac{\partial \hat{u}}{\partial \xi}\right)^2 = f(\hat{u},\hat{x},0), \quad \hat{u}(0,\hat{x}) = \phi^{(0)}(\hat{x}), \quad \hat{u}(\pm\infty,\hat{x}) = \phi^{(\pm)}(\hat{x}). \tag{10}$$

Let us formulate and prove the existence result of the solution of problem (10) in the form of a lemma.

Lemma. *For each $\hat{x} \in (-1,1)$, there exists a single value W such that the problem* (10) *has a single smooth monotone solution $\hat{u}(\xi,\hat{x})$, satisfying the estimation*

$$|\hat{u}(\xi,\hat{x}) - \phi^{(\pm)}(\hat{x})| < C\exp\{-\kappa|\xi|\},$$

where C and κ are some positive constants. In this case, the dependence $W(\hat{x})$ is defined as

$$\begin{aligned}
W(\hat{x}) = \int_{\phi^{(-)}(\hat{x})}^{\phi^{(+)}(\hat{x})} f(u,\hat{x},0)\exp\left\{-2\int_{\phi^{(-)}(\hat{x})}^{u} A(y,\hat{x})dy\right\}du \,\times \\
\times\left[\int_{-\infty}^{+\infty}\left(\frac{\partial \hat{u}}{\partial \xi}(\xi,\hat{x})\right)^2 \exp\left\{-2\int_{\phi^{(-)}(\hat{x})}^{\tilde{u}(\xi,\hat{x})} A(y,\hat{x})dy\right\}d\xi\right]^{-1}.
\end{aligned}$$

The smoothness of the function $W(\hat{x})$ coincides with the smoothness of the functions $f(u,\hat{x},0)$ and $A(u,\hat{x})$.

Proof. In order to use the known result from [18], we make a monotonic transformation proposed by A. V. Bitsadze in [19]:

$$z(\xi,\hat{x}) := z(\hat{u}(\xi,\hat{x}),\hat{x}) = \int_{\phi^{(-)}(\hat{x})}^{\hat{u}(\xi,\hat{x})} \exp\left\{-\int_{\phi^{(-)}(\hat{x})}^{y} A(r,\hat{x})dr\right\}dy, \quad (\hat{u},\hat{x}) \in [\phi^{(-)}(\hat{x}),\phi^{(+)}(\hat{x})] \times [-1,1].$$

Let's introduce the notations

$$z^{(\pm,0)}(\hat{x}) = \int_{\phi^{(-)}(\hat{x})}^{\phi^{(\pm,0)}(\hat{x})} \exp\left\{-\int_{\phi^{(-)}(\hat{x})}^{y} A(r,\hat{x})dr\right\} dy.$$

Due to the monotonicity of the transformation $z(\hat{u},\hat{x})$ by $\hat{u}$ we can define the inverse function

$$\hat{u}(\xi,\hat{x}) = h(z(\xi,\hat{x}),\hat{x}), \quad (z,\hat{x}) \in [0, z^{(+)}(\hat{x})] \times [-1,1].$$

Thus, the problem (10) transforms into the problem

$$\frac{\partial^2 z}{\partial \xi^2} + W\frac{\partial z}{\partial \xi} - f(h(z,\hat{x}),\hat{x},0)\exp\left\{-\int_{\phi^{(-)}(\hat{x})}^{h(z,\hat{x})} A(r,\hat{x})dr\right\} = 0,$$

$$z(-\infty,\hat{x}) = 0, \quad z(0,\hat{x}) = z^{(0)}(\hat{x}), \quad z(+\infty,\hat{x}) = z^{(+)}(\hat{x}), \tag{11}$$

for which, by virtue of conditions 1 and 2, the following statements are true [18].

1. For each $\hat{x} \in (-1,1)$, there exists a single value W, such that the problem (11) has a single smooth monotone solution $\hat{z}(\xi,\hat{x})$, satisfying the estimation

$$|z(\xi,\hat{x}) - z^{(\pm)}(\hat{x})| < C\exp\{-\kappa|\xi|\},$$

where C and κ are some positive constants.

2. The dependence $W(\hat{x})$ is defined as

$$W(\hat{x}) = \int_0^{z^{(+)}(\hat{x})} f(h(z,\hat{x}),\hat{x},0)\exp\left\{-\int_{\phi^{(-)}(\hat{x})}^{h(z,\hat{x})} A(r,\hat{x})dr\right\} dz \left[\int_{-\infty}^{+\infty}\left(\frac{\partial \hat{z}}{\partial \xi}(\xi,\hat{x})\right)^2 d\xi\right]^{-1}. \tag{12}$$

The smoothness of the function $W(\hat{x})$ coincides with the smoothness of the functions $f(u,\hat{x},0)$ and $A(u,\hat{x})$.

Finally, returning to the function $\hat{u}(\xi,\hat{x})$ using the transformation $\hat{u}(\xi,\hat{x}) = h(z(\xi,\hat{x}),\hat{x})$ and recalculating the integrals in expression (13), we have the statement of the lemma. The lemma is proved.

Let's condition.

Condition 3. Task

$$\frac{dx}{dt} = W(x), \quad x(0) = x_{00} \tag{13}$$

has a solution $x = x_0(t)$, such that $x_0(t) \in (-1,1)$ at $t \in [0,T]$; $W(x) > 0$ for all $x \in [-1,1]$.

The inequality $W(x) > 0$ in condition 3 guarantees the absence of stationary solutions for problem (13). Let us denote by (9a) the problems (9) in which we replace $\hat{x}$ by $x_0(t)$, or, otherwise, in which we put $\varepsilon = 0$.

It follows from the lemma and condition 3, that problems (9a) are singularly solvable, since the condition $D\hat{x}_0 = W(x_0)$ is satisfied. Thus

$$\frac{\partial \tilde{u}^{(+)}}{\partial \xi}(0,x_0(t)) - \frac{\partial \tilde{u}^{(-)}}{\partial \xi}(0,x_0(t)) = 0.$$

By virtue of the assumed smoothness of the functions $f(u,\hat{x},0)$, $A(u,\hat{x})$ (see condition 1), problems (9) are regular perturbations of problems (9a), so they are also uniquely solvable. Note that by virtue of the representation (4)

$$\frac{\partial \tilde{u}^{(+)}}{\partial \xi}(0,\hat{x}(t,\varepsilon)) - \frac{\partial \tilde{u}^{(-)}}{\partial \xi}(0,\hat{x}(t,\varepsilon)) = O(\varepsilon).$$

Thus, the construction of the zero-order transition layer functions is completed.

The first-order transition layer functions are found from the following problems:

$$\frac{\partial^2 Q_1^{(\pm)}}{\partial \xi^2} + D\hat{x}\frac{\partial Q_1^{(\pm)}}{\partial \xi} - 2\tilde{A}(\xi,t)\tilde{v}^{(\pm)}(\xi,\hat{x})\frac{\partial Q_1^{(\pm)}}{\partial \xi} - \left(\tilde{A}_u(\xi,t)(\tilde{v}^{(\pm)}(\xi,\hat{x}))^2 + \tilde{f}_u(\xi,t)\right)Q_1^{(\pm)} = r_1^{(\pm)}(\xi,t,\varepsilon),$$

$$Q_1^{(\pm)}(0,t,\varepsilon) + \bar{u}_1^{(\pm)}(\hat{x}) = 0, \quad Q_1^{(\pm)}(\pm\infty,t,\varepsilon) = 0, \tag{14}$$

where the notations are defined

$$\tilde{f}_u(\xi,t) = f_u(\tilde{u}(\xi,\hat{x}),\hat{x},0), \quad \tilde{A}(\xi,t) = A_u(\tilde{u}(\xi,\hat{x}),\hat{x}), \quad \tilde{A}_u(\xi,t) = A_u(\tilde{u}(\xi,\hat{x}),\hat{x}) \tag{15}$$

and

$$r_1^{(\pm)}(\xi,t,\varepsilon) = \frac{\partial Q_0^{(\pm)}}{\partial t}(\xi,t,\varepsilon) + 2\tilde{A}(\xi,t)\tilde{v}^{(\pm)}(\xi,\hat{x})\frac{d\varphi^{(\pm)}}{dx}(\hat{x}) +$$
$$+ \left(\bar{u}_1^{(\pm)}(\hat{x}) + \xi\frac{d\varphi^{(\pm)}}{dx}(\hat{x})\right)\big(\tilde{f}_u(\xi,t) + \tilde{A}_u(\xi,t)\big(\tilde{v}^{(\pm)}(\xi,\hat{x})\big)^2\big) + \xi\big(\tilde{f}_x(\xi,t) + \tilde{A}_x(\xi,t)\big(\tilde{v}^{(\pm)}(\xi,\hat{x})\big)^2\big) + \tilde{f}_\varepsilon(\xi,t).$$

Here, the derivatives of $\tilde{f}_x(\xi,t)$, $\tilde{f}_\varepsilon(\xi,t)$ are computed at the same point as the derivative of $\tilde{f}_u(\xi,t)$ in (15). Similarly, $\tilde{A}_x(\xi,t)$ is computed at the same point as $\tilde{A}_u(\xi,t)$. In all the notations introduced here, the argument ε is implied, but we omit it for brevity. The problem for the function $Q_1^{(-)}(\xi,t,\varepsilon)$ will be solved on the semi-straight $\xi \le 0$, and for the function $Q_1^{(+)}(\xi,t,\varepsilon)$ – on the semi-straight $\xi \ge 0$. The solutions of problems (14) are written in explicit form:

$$Q_1^{(\pm)}(\xi,t,\varepsilon) = -\bar{u}_1^{(\pm)}(\hat{x})\frac{\tilde{v}^{(\pm)}(\xi,\hat{x})}{\tilde{v}^{(\pm)}(0,\hat{x})} +$$
$$+ \tilde{v}^{(\pm)}(\xi,\hat{x})\int_0^\xi \frac{e^{-(D\hat{x})\eta}}{(\tilde{v}^{(\pm)}(\eta,\hat{x}))^2 p^{(\pm)}(\eta,\hat{x})}\int_{\pm\infty}^{\eta} \tilde{v}^{(\pm)}(\sigma,\hat{x})p^{(\pm)}(\sigma,\hat{x})e^{(D\hat{x})\sigma} r_1^{(\pm)}(\sigma,t,\varepsilon)d\sigma d\eta, \tag{16}$$

where

$$p^{(\pm)}(\xi,\hat{x}) = \exp\left\{-2\int_0^\xi A(\tilde{u}^{(\pm)}(y,\hat{x}),\hat{x})\tilde{v}^{(\pm)}(y,\hat{x})dy\right\}.$$

It follows from the expression for the functions $r_1^{(\pm)}(\xi,t,\varepsilon)$, that they have exponential valuations [16], and from (16) we deduce in the standard way that similar valuations are true for functions $Q_1^{(\pm)}(\xi,t,\varepsilon)$.

Similarly to the first approximation, one can find for any $k = 2,3,\ldots$ transition layer functions $Q_k^{(\pm)}(\xi,t,\varepsilon)$: they are determined from boundary value problems with the same differential operator as in problems (14).

3. ASYMPTOTIC APPROXIMATION OF FRONT POSITION

Let us describe the algorithm for finding an asymptotic approximation of the front position. The unknown coefficients $x_i(t)$, $i \in \mathbb{N}$, of the expansion are determined from the crossing conditions (3) of the derivatives of the asymptotic approximations. Let us introduce the function

$$H(\varepsilon,t) := \varepsilon\left(\frac{dU^{(+)}}{dx}(\hat{x},t,\varepsilon) - \frac{dU^{(-)}}{dx}(\hat{x},t,\varepsilon)\right) = H_0(\varepsilon,t) + \varepsilon H_1(\varepsilon,t) + \varepsilon^2 H_2(\varepsilon,t) + \cdots, \tag{17}$$

where

$$H_0(\varepsilon,t) = \frac{\partial Q_0^{(+)}}{\partial\xi}(0,\hat{x}) - \frac{\partial Q_0^{(-)}}{\partial\xi}(0,\hat{x}),$$
$$H_1(\varepsilon,t) = \frac{d\phi^{(+)}}{dx}(\hat{x}) - \frac{d\phi^{(-)}}{dx}(\hat{x}) + \left(\frac{\partial Q_1^{(+)}}{\partial\xi}(0,t,\varepsilon) - \frac{\partial Q_1^{(-)}}{\partial\xi}(0,t,\varepsilon)\right)$$

etc.

The C^1-linking condition (3) is expressed by the equality $H(\varepsilon,t) = 0$. By virtue of the lemma and condition 3, taking into account the decomposition of the transition point (4), this equality is satisfied in the order ε^0.

The analysis of problems (9), (10) shows that the function $H_0(\varepsilon,t)$ can be represented as

$$H_0(\varepsilon,t) = (D\hat{x} - W(\hat{x}))\left[\frac{1}{\tilde{v}^{(\pm)}(0,\hat{x})}\int_0^{\pm\infty}(\tilde{v}^{(\pm)}(\xi,\hat{x}))^2 e^{(D\hat{x})\xi}p^{(\pm)}(\xi,\hat{x})d\xi\right]_-^+ + O(\varepsilon^2). \tag{18}$$

Hereinafter, $[\,]_-^+$ means the difference between the expressions labeled $+$ and $-$.

As follows from the decomposition (17) and the representation (18), the higher order terms $x_i(t)$, $i \geq 1$, in (4) can be found from the following Cauchy problems:

$$\frac{dx_i}{dt} - W'(x_0(t))x_i(t) = G_i(t), \quad x_i(0) = 0,$$

where $G_i(t)$ are known functions.

4. JUSTIFICATION OF FORMAL ASYMPTOTICS

Let's say

$$X_n(t,\varepsilon) = \sum_{i=0}^{n+1} \varepsilon^i x_i(t), \quad \xi = \frac{x - X_n(t,\varepsilon)}{\varepsilon}.$$

The curve $X_n(t,\varepsilon)$ divides the area $\overline{D} : (x,t) \in [-1,1] \times [0,T]$ into two sub-areas:

$$\bar{D}_n^{(-)} : (x,t) \in [-1, X_n(t,\varepsilon)] \times [0,T] \quad \text{and} \quad \bar{D}_n^{(+)} : (x,t) \in [X_n(t,\varepsilon), 1] \times [0,T].$$

Let's define the functions

$$U_n^{(-)}(x,t,\varepsilon) = \sum_{i=0}^{n} \varepsilon^i \left(\bar{u}_i^{(-)}(x) + Q_i^{(-)}(\xi,t,\varepsilon) + R_i^{(-)}(\eta^{(-)})\right), \quad (x,t) \in \bar{D}_n^{(-)},$$
$$U_n^{(+)}(x,t,\varepsilon) = \sum_{i=0}^{n} \varepsilon^i \left(\bar{u}_i^{(+)}(x) + Q_i^{(+)}(\xi,t,\varepsilon) + R_i^{(+)}(\eta^{(+)})\right), \quad (x,t) \in \bar{D}_n^{(+)},$$

where $\hat{x}(t,\varepsilon)$, included in the expressions for the transition layer functions, are replaced by $X_n(t,\varepsilon)$, and denoted by

$$U_n(x,t,\varepsilon) = \begin{cases} U_n^{(-)}(x,t,\varepsilon), & (x,t) \in \bar{D}_n^{(-)}, \\ U_n^{(+)}(x,t,\varepsilon), & (x,t) \in \bar{D}_n^{(+)}. \end{cases} \tag{19}$$

To prove the existence and uniqueness of the moving front solution, we use the asymptotic method of differential inequalities [16]. Let us construct continuous functions $\alpha(x,t,\varepsilon)$, $\beta(x,t,\varepsilon)$ in such a way that they satisfy the following conditions.

1. Ordering condition:

$$\alpha(x,t,\varepsilon) \leq \beta(x,t,\varepsilon), \quad x \in [-1,1], t \in [0,T], \varepsilon \in (0,\varepsilon_0]. \tag{20}$$

2. Action of the differential operator on upper and lower solutions:

$$L[\beta] := \varepsilon^2 \frac{\partial^2 \beta}{\partial x^2} - \varepsilon \frac{\partial \beta}{\partial t} - \varepsilon^2 A(\beta,x)\left(\frac{\partial \beta}{\partial x}\right)^2 - f(\beta,x,\varepsilon) \leq 0 \leq$$
$$\leq L[\alpha] := \varepsilon^2 \frac{\partial^2 \alpha}{\partial x^2} - \varepsilon \frac{\partial \alpha}{\partial t} - \varepsilon^2 A(\alpha,x)\left(\frac{\partial \alpha}{\partial x}\right)^2 - f(\alpha,x,\varepsilon) \tag{21}$$

for all $x \in (-1,1)$ and $t \in [0,T]$, except those $x(t)$, in which the functions $\alpha(x,t,\varepsilon)$ and $\beta(x,t,\varepsilon)$ are nonsmooth.

3. Boundary conditions:

$$\frac{\partial \alpha}{\partial x}(-1,t,\varepsilon) \geq 0 \geq \frac{\partial \beta}{\partial x}(-1,t,\varepsilon), \quad \frac{\partial \alpha}{\partial x}(+1,t,\varepsilon) \leq 0 \leq \frac{\partial \beta}{\partial x}(+1,t,\varepsilon), \quad t \in [0,T], \varepsilon \in (0,\varepsilon_0]. \tag{22}$$

4. Conditions on the initial function:

$$\alpha(x,0,\varepsilon) \leq u_{\text{init}}(x,\varepsilon) \leq \beta(x,0,\varepsilon), \quad x \in [-1,1], \varepsilon \in (0,\varepsilon_0]. \tag{23}$$

5. Conditions on the jump of derivatives:

$$\frac{\partial\beta}{\partial x}(\overline{x}(t)-0,t,\varepsilon) \geq \frac{\partial\beta}{\partial x}(\overline{x}(t)+0,t,\varepsilon), \tag{24}$$

where $\overline{x}(t)$ is the point at which the upper solution is nonsmooth;

$$\frac{\partial\alpha}{\partial x}(\underline{x}(t)-0,t,\varepsilon) \leq \frac{\partial\alpha}{\partial x}(\underline{x}(t)+0,t,\varepsilon), \tag{25}$$

where $\underline{x}(t)$ is the point at which the lower solution is nonsmooth.

It is known (see [20]) that if the conditions (20)–(25) are satisfied, there exists a single solution of problem (1) for which the inequalities are satisfied

$$\alpha(x,t,\varepsilon) \leq u(x,t,\varepsilon) \leq \beta(x,t,\varepsilon), \quad (x,t) \in [-1,1]\times[0,T].$$

Let us prove the following existence and uniqueness theorem.

Theorem. *When conditions* 1–3 *are satisfied for any sufficiently smooth initial function* $u_{init}(x)$*, lying between upper and lower solutions*

$$\alpha(x,0,\varepsilon) \leq u_{\text{init}}(x,\varepsilon) \leq \beta(x,0,\varepsilon),$$

there exists a single solution $u(x,t,\varepsilon)$ *of problem* (1)*, that at any* $t\in[0,T]$ *is enclosed between these upper and lower solutions, and for which the function* $U_n(x,t,\varepsilon)$ *is a uniform in the domain* $[-1,1]\times[0,T]$ *asymptotic approximation with accuracy* $O(\varepsilon^{n+1})$.

Proof. The upper and lower solutions of the problem will be constructed as a modification of the asymptotic series (19). Set the function

$$x_\beta(t,\varepsilon) = X_{n+1}(t) - \varepsilon^{n+1}\delta(t),$$

and the positive function $\delta(t) > 0$ will be defined below. Let us construct the upper solution of the problem in each of the regions $\overline{D}_\beta^{(-)}: (x,t)\in[-1,x_\beta(t,\varepsilon)]\times[0,T]$ and $\overline{D}_\beta^{(+)}: (x,t)\in[x_\beta(t,\varepsilon),1]\times[0,T]$:

$$\beta(x,t,\varepsilon) = \begin{cases} \beta^{(-)}(x,t,\varepsilon), & (x,t)\in\overline{D}_\beta^{(-)}, \\ \beta^{(+)}(x,t,\varepsilon), & (x,t)\in\overline{D}_\beta^{(+)}. \end{cases}$$

We will connect the functions $\beta^{(-)}(x,t,\varepsilon)$ and $\beta^{(+)}(x,t,\varepsilon)$ at the point $x_\beta(t,\varepsilon)$ in such a way, that the following equality is satisfied

$$\beta^{(-)}(x_\beta(t,\varepsilon),t,\varepsilon) = \beta^{(+)}(x_\beta(t,\varepsilon),t,\varepsilon) = \phi^{(0)}(x_\beta(t,\varepsilon)).$$

Note that the function $\beta(x,t,\varepsilon)$ is not smooth. Let us introduce a stretched variable

$$\xi_\beta = \frac{x - x_\beta(t,\varepsilon)}{\varepsilon}.$$

Let us construct the functions $\beta^{(\pm)}(x,t,\varepsilon)$ as modifications of the formal asymptotics (19):

$$\beta^{(-)}(x,t,\varepsilon) = U_{n+1}^{(-)}|_{\xi_\beta} + \varepsilon^{n+1}(\mu + q_\beta^{(-)}(\xi_\beta,t,\varepsilon)) + \varepsilon^{n+1}R_\beta^{(-)}(\eta^{(-)}),$$
$$(x,t)\in D_\beta^{(-)}, \xi_\beta \leq 0, \eta^{(-)} \geq 0;$$
$$\beta^{(+)}(x,t,\varepsilon) = U_{n+1}^{(+)}|_{\xi_\beta} + \varepsilon^{n+1}(\mu + q_\beta^{(+)}(\xi_\beta,t,\varepsilon)) + \varepsilon^{n+1}R_\beta^{(+)}(\eta^{(+)}),$$
$$(x,t)\in D_\beta^{(+)}, \xi_\beta \geq 0, \eta^{(+)} \leq 0.$$

Here under the notation $U_{n+1}^{(\pm)}|_{\xi_\beta}$ we understand the functions from (19), where the argument ξ of the transition layer functions is replaced by ξ_β, and X_{n+1} – by x_β.

The positive value μ is chosen, so that conditions (20) and (21) are satisfied. The functions $R_\beta^{(\pm)}(\eta^{(\pm)})$ are chosen, so that condition (22) is satisfied (their construction is not considered in this paper). The functions

$q_\beta^{(\pm)}(\xi_\beta, t, \varepsilon)$ are needed to eliminate the inconsistencies that arise, when the operator acts on the upper solution. Let us define them from the following problems:

$$\frac{\partial^2 q_\beta^{(\pm)}}{\partial \xi_\beta^2} + Dx_\beta \frac{\partial q_\beta^{(\pm)}}{\partial \xi_\beta} - 2\tilde{A}(\xi_\beta, t)\tilde{v}^{(\pm)}(\xi_\beta, x_\beta)\frac{\partial q_\beta^{(\pm)}}{\partial \xi_\beta} -$$

$$- \big(\tilde{A}_u(\xi_\beta, t)\big(\tilde{v}^{(\pm)}(\xi_\beta, x_\beta)\big)^2 + \tilde{f}_u(\xi_\beta, t)\big)q_\beta^{(\pm)} - qf^{(\pm)}(\xi_\beta, t, \varepsilon) = 0,$$

$$q_\beta^{(\pm)}(0, t, \varepsilon) + \mu = 0, \quad q_\beta^{(\pm)}(\pm\infty, t, \varepsilon) = 0, \tag{26}$$

where $qf^{(\pm)}(\xi_\beta, t, \varepsilon) = \mu\big(\tilde{A}_u(\xi_\beta, t)\big(\tilde{v}^{(\pm)}(\xi_\beta, x_\beta)\big)^2 + \tilde{f}_u(\xi_\beta, t) - \bar{f}_u^{(\pm)}(x_\beta)\big)$.

Explicit expressions for these functions can be obtained

$$q_\beta^{(\pm)}(\xi_\beta, t, \varepsilon) = -\mu\frac{\tilde{v}^{(\pm)}(\xi, x_\beta)}{\tilde{v}^{(\pm)}(0, x_\beta)} +$$

$$+ \tilde{v}^{(\pm)}(\xi_\beta, x_\beta)\int_0^{\xi_\beta} \frac{e^{-(Dx_\beta)\eta}}{(\tilde{v}^{(\pm)}(\eta, x_\beta))^2 p^{(\pm)}(\eta, x_\beta)} \int_{\pm\infty}^{\eta} \tilde{v}^{(\pm)}(\sigma, x_\beta)e^{(Dx_\beta)\sigma}p^{(\pm)}(\sigma, x_\beta)qf^{(\pm)}(\sigma, t, \varepsilon)\, d\sigma\, d\eta. \tag{27}$$

The functions $q^{(\pm)}(\xi_\beta, t, \varepsilon)$ have exponential estimates [16].

We can simplify expressions (27) as follows:

$$q_\beta^{(\pm)}(\xi_\beta, t, \varepsilon) =$$

$$= -\mu - \mu\bar{f}_u^{(\pm)}(x_\beta)\tilde{v}^{(\pm)}(\xi_\beta, x_\beta)\int_0^{\xi_\beta} \frac{e^{-(Dx_\beta)\eta}}{(\tilde{v}^{(\pm)}(\eta, x_\beta))^2 p^{(\pm)}(\eta, x_\beta)} \int_{\pm\infty}^{\eta} \tilde{v}^{(\pm)}(\sigma, x_\beta)e^{(Dx_\beta)\sigma}p^{(\pm)}(\sigma, x_\beta)\, d\sigma\, d\eta.$$

Using a similar algorithm, we construct the lower solution. Set the function

$$x_\alpha(t, \varepsilon) = X_{n+1}(t) + \varepsilon^{n+1}\delta(t),$$

where $\delta(t)$ is the same function as in the construction of the upper solution.

Let's construct the lower solution of the problem in each of the regions $\overline{D}_\alpha^{(-)} : (x, t) \in [-1, x_\alpha(t, \varepsilon)] \times [0, T]$ and $\overline{D}_\alpha^{(+)} : (x, t) \in [x_\alpha(t, \varepsilon), 1] \times [0, T]$:

$$\alpha(x, t, \varepsilon) = \begin{cases} \alpha^{(-)}(x, t, \varepsilon), & (x, t) \in \overline{D}_\alpha^{(-)}, \\ \alpha^{(+)}(x, t, \varepsilon), & (x, t) \in \overline{D}_\alpha^{(+)}. \end{cases}$$

We will merge the functions $\alpha^{(-)}(x, t, \varepsilon)$ and $\alpha^{(+)}(x, t, \varepsilon)$ at the point $x_\alpha(t, \varepsilon)$ in such a way that the equality is satisfied

$$\alpha^{(-)}(x_\alpha(t, \varepsilon), t, \varepsilon) = \alpha^{(+)}(x_\alpha(t, \varepsilon), t, \varepsilon) = \phi^{(0)}(x_\alpha(t, \varepsilon)).$$

Note that the function $\alpha(x, t, \varepsilon)$ is not smooth. Let us introduce a stretched variable

$$\xi_\alpha = \frac{x - x_\alpha(t, \varepsilon)}{\varepsilon}.$$

Let us construct the functions $\alpha^{(\pm)}(x, t, \varepsilon)$ as modifications of the formal asymptotics (19):

$$\alpha^{(-)}(x, t, \varepsilon) = U_{n+1}^{(-)}|_{\xi_\alpha} - \varepsilon^{n+1}(\mu + q_\alpha^{(-)}(\xi_\alpha, t, \varepsilon)) + \varepsilon^{n+1}R_\alpha^{(-)}(\eta^{(-)}),$$
$$(x, t) \in D_\alpha^{(-)}, \xi_\alpha \le 0, \eta^{(-)} \ge 0;$$
$$\alpha^{(+)}(x, t, \varepsilon) = U_{n+1}^{(+)}|_{\xi_\alpha} - \varepsilon^{n+1}(\mu + q_\alpha^{(+)}(\xi_\alpha, t, \varepsilon)) + \varepsilon^{n+1}R_\alpha^{(+)}(\eta^{(+)}),$$
$$(x, t) \in D_\alpha^{(+)}, \xi_\alpha \ge 0, \eta^{(+)} \le 0.$$

Here $\mu > 0$ is the same value as in the expression for the upper solution, and $q_\alpha^{(\pm)}(\xi_\alpha, t, \varepsilon)$ are determined from problems (26), in which the stretched variable ξ_β is replaced by ξ_α, and x_β is replaced by x_α.

Let us make sure that the constructed functions $\alpha(x,t,\varepsilon)$ and $\beta(x,t,\varepsilon)$ satisfy the differential inequalities (20)–(25). The ordering condition (20) can be checked similarly as it was done in [2].

Let us show that inequality (21) holds. From the way of constructing the upper and lower solutions the following equations follow

$$L[\alpha^{(\pm)}] = \varepsilon^{n+1}\bar{f}_u^{(\pm)}(x_\alpha)\mu + O(\varepsilon^{n+2}), \quad L[\beta^{(\pm)}] = -\varepsilon^{n+1}\bar{f}_u^{(\pm)}(x_\beta)\mu + O(\varepsilon^{n+2}).$$

The inequalities near the boundary (22) are fulfilled due to a standard modification of the boundary-layer functions [16] (their verification is not intended for this paper).

Let's check the jump condition of the derivative (24)

$$\varepsilon\left(\left.\frac{\partial\beta^{(+)}}{\partial x}\right|_{x=x_\beta} - \left.\frac{\partial\beta^{(-)}}{\partial x}\right|_{x=x_\beta}\right) = -\varepsilon^{n+1}\frac{1}{\tilde{v}(0,x_0)}\left(L(x_0)\frac{d\delta}{dt} - L(x_0)W'(x_0(t))\delta(t) + F(x_0)\right) + O(\varepsilon^{n+2}),$$

where

$$F(x_0) = \mu\left[\bar{f}_u^{(\pm)}(x_0)\int_{\pm\infty}^{0} p(\sigma,x_0)\tilde{v}(\sigma,x_0)e^{(Dx_0)\sigma}d\sigma\right]_-^+,$$

$$L(x_0) = \int_{-\infty}^{+\infty}\tilde{v}^2(\xi,x_0)e^{(Dx_0)\xi}p(\xi,x_0)d\xi > 0.$$

Here, the index at the functions $\tilde{v}(\xi,x_0)$, $p(\xi,x_0)$ is omitted due to their smoothness at $\xi = 0$.

Let's define the function $\delta(t)$ as a solution to the problem

$$L(x_0)\frac{d\delta}{dt} - L(x_0)W'(x_0(t))\delta(t) + F(x_0) = \sigma, \quad \delta(0) = \delta_0,$$

where σ is a sufficiently large positive value and $\delta_0 > 0$. In this case, the solution to the problem $\delta(t)$ is a positive function. Thus,

$$\varepsilon\left(\left.\frac{\partial\beta^{(+)}}{\partial x}\right|_{x=x_\beta} - \left.\frac{\partial\beta^{(-)}}{\partial x}\right|_{x=x_\beta}\right) = -\varepsilon^{n+1}\frac{\sigma}{\tilde{v}(0,x_0)} + O(\varepsilon^{n+2}).$$

The expression in the right-hand side is negative due to $\sigma > 0$. With the same choice of function $\delta(t)$, the derivative jump inequality will be satisfied for the lower solution $\alpha(x,t,\varepsilon)$. The theorem is proved.

5. EXAMPLE

Consider the initial boundary value problem

$$\varepsilon^2\frac{\partial^2 u}{\partial x^2} - \varepsilon\frac{\partial u}{\partial t} - \varepsilon^2\left(\frac{\partial u}{\partial x}\right)^2 = e^u(1-e^{-u})\left(\frac{1}{2} - e^{-u}\right)(1-\phi^{(0)}(x) - e^{-u}), \quad x\in(-1,1), t\in(0,T],$$

$$\frac{\partial u}{\partial x}(-1,t,\varepsilon) = 0, \quad \frac{\partial u}{\partial x}(1,t,\varepsilon) = 0, \quad t\in[0,T],$$

$$u(x,0,\varepsilon) = u_{\text{init}}(x,\varepsilon), \quad x\in[-1,1].$$

We will assume that for all $x\in[-1,1]$, the inequality $1/4 < \phi^{(0)}(x) < 1/2$ is satisfied. The members of the regular part of zero order are easily determined:

$$\bar{u}_0^{(-)}(x) = 0, \quad \bar{u}_0^{(+)}(x) = \ln 2.$$

The problem for the function $\tilde{u}(\xi,x_0)$ has the following form:

$$\frac{\partial^2\tilde{u}}{\partial\xi^2} + W\frac{\partial\tilde{u}}{\partial\xi} - \left(\frac{\partial\tilde{u}}{\partial\xi}\right)^2 = e^{\tilde{u}}(1-e^{-\tilde{u}})\left(\frac{1}{2} - e^{-\tilde{u}}\right)(1-\phi^{(0)}(x_0) - e^{-\tilde{u}}),$$

$$\tilde{u}(0,x_0) = -\ln(1-\phi^{(0)}(x_0)), \quad \tilde{u}(-\infty,x_0)=0, \quad \tilde{u}(\infty,x_0)=\ln 2. \tag{28}$$

By replacing $z(\xi,x_0) := z(\tilde{u}(\xi,x_0)) = 1-e^{-\tilde{u}(\xi,x_0)}$ the problem (28) is transformed to the form

$$\frac{\partial^2 z}{\partial \xi^2} + W\frac{\partial z}{\partial \xi} = z\left(z-\frac{1}{2}\right)(z-\phi^0(x_0)), \quad z(-\infty,x_0)=0, \quad z(\infty,x_0)=\frac{1}{2}. \tag{29}$$

The solution of problem (29) is determined by the formula

$$z = \left(2+\left(\frac{1}{\phi(x_0)}-2\right)\exp\left\{-\frac{\xi}{2\sqrt{2}}\right\}\right)^{-1}.$$

Making the inverse substitution, we obtain the expression for the solution of the original problem (28):

$$\tilde{u}(\xi,x_0) = -\ln\left(1-\left(2+\left(\frac{1}{\phi(x_0)}-2\right)\exp\left\{-\frac{\xi}{2\sqrt{2}}\right\}\right)^{-1}\right).$$

The initial problem for determining the front position in the zero approximation has the form

$$\frac{dx_0}{dt} = \sqrt{2}\left(\phi^{(0)}(x_0)-\frac{1}{4}\right), \quad x_0(0)=x_{00}. \tag{30}$$

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PARTIAL DIFFERENTIAL EQUATIONS

DIRICHLET PROBLEM FOR A TWO-DIMENSIONAL WAVE EQUATION IN A CYLINDRICAL DOMAIN

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Abstract. In this work, the first boundary value problem is studied for a two-dimensional wave equation in a cylindrical domain. A uniqueness criterion has been established. The solution is constructed as the sum of an orthogonal series. When justifying the convergence of a series, the problem of small denominators from two natural arguments arose for the first time. An estimate for separation from zero with the corresponding asymptotics was established, which made it possible to prove the convergence of the series in the class of regular solutions and the stability of the solution.

Keywords: *wave equation, Dirichlet problem, uniqueness criterion, existence, stability, series, small denominators*

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1. INTRODUCTION. PROBLEM STATEMENT

Consider the wave equation

$$Lu \equiv u_{tt} - a^2(u_{xx} + u_{yy}) - bu = 0 \tag{1}$$

in the cylinder $Q = \{(x, y, t) : (x, y) \in D, 0 < t < T\}$, where $D = \{(x, y) : x^2 + y^2 < l^2\}$; $a > 0$, b, $T > 0$ and $l > 0$ are given real constants, and we set the first boundary value problem.

It is required to find the function $u(x, y, t)$, satisfying the following conditions:

$$u(x, y, t) \in C^1(\overline{Q}) \cap C^2(Q); \tag{2}$$

$$Lu(x, y, t) \equiv 0, \quad (x, y, t) \in Q; \tag{3}$$

$$u(x, y, t)|_{x^2+y^2=l^2} = 0, \quad 0 \le t \le T; \tag{4}$$

$$u(x, y, 0) = \tau(x, y), \quad u(x, y, T) = \psi(x, y), \quad (x, y) \in \overline{D}, \tag{5}$$

where $\tau(x, y)$ and $\psi(x, y)$ are given sufficiently smooth functions satisfying the matching conditions with the boundary condition (4).

It is known that the Dirichlet problem for hyperbolic type equations is incorrectly posed. S L. Sobolev showed [1], that the study of unstable oscillations (resonances of oscillations in the liquid inside thin-walled rocket tanks with natural oscillations) is closely related to the Dirichlet problem for the wave equation. In a better known form, this connection is shown in the book by V. I. Arnold [2, p. 132]. A rather complete review of the works devoted to the study of the Dirichlet problem for hyperbolic equations is given in the monograph by B. I. Ptashnik [3, pp. 89–95] and in the works [4; 5, pp. 112–118] by the author.

The works of R. Denchev [6–8] are devoted to the study of the Dirichlet problem for equation (1) at $b = 0$, $a = 1$ with a non-zero right part and homogeneous conditions on the boundary of the region Ω, when Ω is an ellipsoid, a cylinder with formations parallel to the axis t, and a parallelepiped. They also establish the criterion of singularity and existence of the solution of the problem in the Sobolev space $W_2^1(\Omega)$ under certain conditions on the right part related to the convergence of numerical series, while the arising small denominators are not studied.

In [9], for a multidimensional equation with a wave operator in the cylindrical domain $D\times(0,T)$, the conditions $\sqrt{\lambda_k}T \neq m\pi$, where $k, m \in \mathbb{N}$, under which the uniqueness theorem of the solution of the Dirichlet problem takes place, were found. Here, λ_k are the eigenvalues of the corresponding spectral problem in the domain D.

In the monograph by B. I. Ptashnik [3, pp. 95–101], the Dirichlet problem in $(p+1)$-dimensional parallelepiped $Q = [0,T] \times \Pi$, where $\Pi = \{x \in \mathbb{R}^p : 0 \leq x_r \leq \pi, r = \overline{1,p}\}$, for a strictly hyperbolic equation of even order $2n$ with constant coefficients is also studied. The solution of the problem is determined by p-dimensional Fourier series. A criterion for the uniqueness of the solution in $C^{2n}(Q)$ is established. For a series of inequalities expressing the evaluation of small denominators with the corresponding asymptotics, the justification of convergence of the series in the specified class is given. It is not shown for what numbers of the form π/T these estimates take place, only it is noted that the set of numbers π/T, for which they are not fulfilled, is the set of zero Lebesgue measure.

In the paper by V. P. Bursky [10], a necessary and sufficient condition for the trivial solvability of the homogeneous Dirichlet problem in a unit ball B centered at the origin of coordinates in space $C^2(\overline{B})$ for an equation with complex is obtained:

$$u_{xx} + u_{yy} - a^2 u_{zz} = 0.$$

In the works of S. A. Aldashev [11–14], the Dirichlet problem and the problem with mixed boundary conditions in the cylindrical domain Q (where $l = 1, T = \alpha$) for multidimensional hyperbolic equations with a wave operator are studied; the solutions of the problems are constructed as a sum of Fourier series in the spherical coordinate system. But because of the arising small denominators, one cannot assume that these series converge in the space $C^1(\overline{Q}) \cap C^2(Q)$. When proving the singularity theorems, questions also arise about the uniform convergence of the series used, since they contain small denominators.

In this paper, in the class of regular solutions of equation (1), i.e., satisfying conditions (2) and (3), the criterion of uniqueness of the solution of problem (2)–(5) is established and the solution itself is constructed in explicit form – sums of Fourier series. When justifying the convergence of the series, the problem of small denominators arose, as in the well-known works of V. I. Arnold [15, 16] and V. V. Kozlov [17], but from two natural arguments. In this connection, we establish estimates of the separability from zero of small denominators, on the basis of which we prove the convergence of the series in the class of functions $C^2(\overline{Q})$ under some conditions concerning the functions $\tau(x,y)$ and $\psi(x,y)$ and also obtain estimates of the stability of the solution.

2. UNIQUENESS CRITERION FOR THE SOLUTION OF THE DIRICHLET PROBLEM

In the cylindrical coordinate system $x = r\cos\varphi$, $y = r\sin\varphi$, $t = t$, $0 \leq r < l$, $0 \leq \varphi \leq 2\pi$, equation (1) will take the following form

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\varphi\varphi} + \frac{b}{a^2}u = \frac{1}{a^2}u_{tt}. \tag{6}$$

Dividing the variables $u(r,\varphi,t) = v(r,\varphi)T(t)$ in equation (6), we obtain the following spectral problem with respect to the function $v(r,\varphi)$:

$$v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\varphi\varphi} + \lambda^2 v = 0, \tag{7}$$

$$v(l,\varphi) = 0, \tag{8}$$

$$|v(0,\varphi)| < +\infty, \quad v(r,\varphi) = v(r,\varphi+2\pi), \tag{9}$$

where $\lambda^2 = \frac{b}{a^2} + \mu^2$, μ is the variable separation constant.

The solution of the problem (7)–(9) is similar [18, p. 215]: we will look for in the form of $v(r,\varphi) = R(r)\Phi(\varphi)$ and obtain two one-dimensional spectral problems:

$$\Phi''(\varphi) + p^2\Phi(\varphi) = 0, \quad 0 \leq \varphi \leq 2\pi, \tag{10}$$

$$\Phi(\varphi) = \Phi(\varphi+2\pi), \quad \Phi'(\varphi) = \Phi'(\varphi+2\pi); \tag{11}$$

$$R''(r) + \frac{1}{r}R'(r) + \left(\lambda^2 - \frac{p^2}{r^2}\right)R(r) = 0, \quad 0 < r < l, \tag{12}$$

$$|R(0)| < +\infty, \quad R(l) = 0. \tag{13}$$

Nonzero periodic solutions of the problem (10) and (11) exist only at the whole $p = n$ and are defined by the formula

$$\Phi_n(\varphi) = a_n \cos(n\varphi) + b_n \sin(n\varphi),$$

where a_n, b_n are arbitrary constants, $n = 0, 1, 2, \ldots$ At $p = n$, the general solution of equation (12) has the form

$$R_n(r) = c_n J_n(\lambda r) + d_n Y_n(\lambda r),$$

here c_n and d_n are arbitrary constants, $J_n(\lambda r)$ and $Y_n(\lambda r)$ are cylindrical functions of the first and second kind, respectively. From the first condition in (13) it follows that $d_n = 0$, and the second condition gives the equation

$$J_n(q) = 0, \quad q = \lambda l,$$

that, as it is known, has a countable set of positive roots q_{nm}, $n = 0, 1, 2, \ldots$, $m = 1, 2, \ldots$, and eigenvalues corresponding to them

$$\lambda_{nm} = \frac{q_{nm}}{l}, \quad m = 1, 2, \ldots, \quad n = 0, 1, 2, \ldots,$$

and eigenfunctions

$$\tilde{R}_{nm}(r) = J_n(\lambda_{nm} r) = J_n\left(\frac{q_{nm}}{l} r\right)$$

of the spectral problem (12), (13).

Thus, the spectral problem (10), (11) has a system of eigenfunctions

$$\Phi_n(\varphi) = \left\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos(n\varphi), \frac{1}{\sqrt{\pi}}\sin(n\varphi)\right\}, \tag{14}$$

orthonormalized, complete and forming a basis in the space $L_2(0, 2\pi)$, and the spectral problem (12), (13) — a system of eigenfunctions

$$R_{nm}(r) = \frac{J_n(\lambda_{nm} r)}{\|J_n(\lambda_{nm} r)\|_{L_2(0,l)}} = \frac{\sqrt{2}}{l}\frac{J_n(\lambda_{nm} r)}{|J_{n+1}(q_{nm})|}, \tag{15}$$

complete and an orthonormalized basis in $L_2(0, l)$ with weight r.

Then, the spectral problem (7)–(9) has eigenvalues $\lambda_{nm}^2 = \frac{b}{a^2} + \mu_{nm}^2 = \left(\frac{q_{nm}}{l}\right)^2$, and the system of eigenfunctions corresponds to them, taking into account (14) and (15)

$$v_{nm}(r, \varphi) = \left\{\frac{1}{\sqrt{2\pi}}R_{0m}(r), \frac{1}{\sqrt{\pi}}R_{nm}(r)\cos(n\varphi), \frac{1}{\sqrt{\pi}}R_{nm}(r)\sin(n\varphi)\right\}, \tag{16}$$

that is complete and forms an orthonormalized basis in the space $L_2(D)$ with weight r.

Further, we will assume that $b \geq 0$, because if $b < 0$, then, starting from some numbers $n > n_0$ or $m > m_0$, the right part of $\lambda_{nm}^2 = \frac{b}{a^2} + \mu_{nm}^2$, takes only positive values, i.e., the sign of the coefficient b, essentially does not affect the obtained results.

Let $u(r, \varphi, t)$ be the solution of problem (2)–(5). Based on the system (16) we introduce the functions

$$A_{0m}(t) = \frac{1}{\sqrt{2\pi}}\iint_D u(r, \varphi, t) R_{0m}(r) r\, dr\, d\varphi, \tag{17}$$

$$A_{nm}(t) = \frac{1}{\sqrt{\pi}}\iint_D u(r, \varphi, t) R_{nm}(r)\cos(n\varphi) r\, dr\, d\varphi, \tag{18}$$

$$B_{nm}(t) = \frac{1}{\sqrt{\pi}}\iint_D u(r, \varphi, t) R_{nm}(r)\sin(n\varphi) r\, dr\, d\varphi. \tag{19}$$

Differentiating equality (18) by t twice and considering equation (6), we obtain

$$A''_{nm}(t) = \frac{1}{\sqrt{\pi}} \iint_D u_{tt}(r,\varphi,t) R_{nm}(r)\cos(n\varphi) r\,dr\,d\varphi =$$

$$= \frac{a^2}{\sqrt{\pi}} \iint_D \left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\varphi\varphi}\right) R_{nm}(r)\cos(n\varphi) r\,dr\,d\varphi + bA_{nm}(t) = J_1 + J_2 + bA_{nm}(t), \tag{20}$$

where

$$J_1 = \frac{a^2}{\sqrt{\pi}} \iint_D \left(u_{rr} + \frac{1}{r}u_r\right) R_{nm}(r)\cos(n\varphi) r\,dr\,d\varphi = \frac{a^2}{\sqrt{\pi}} \int_0^{2\pi} \cos(n\varphi) \int_0^l (ru_r)'_r R_{nm}(r)\,dr\,d\varphi, \tag{21}$$

$$J_2 = \frac{a^2}{\sqrt{\pi}} \iint_D \frac{1}{r} u_{\varphi\varphi} R_{nm}(r)\cos(n\varphi)\,dr\,d\varphi = \frac{a^2}{\sqrt{\pi}} \int_0^l \frac{1}{r} R_{nm}(r) \int_0^{2\pi} u_{\varphi\varphi}\cos(n\varphi)\,d\varphi\,dr. \tag{22}$$

Let us calculate the internal integrals in the right-hand sides of the equalities (21) and (22):

$$\int_0^l (ru_r)'_r R_{nm}(r)\,dr = ru_r R_{nm}(r)\Big|_0^l - \int_0^l u_r r R'_{nm}(r)\,dr = -\int_0^l u_r r R'_{nm}(r)\,dr =$$

$$= ruR'_{nm}(r)\Big|_0^l + \int_0^l u(rR'_{nm}(r))'\,dr = -\lambda_{nm}^2 \int_0^l urR_{nm}(r)\,dr + n^2 \int_0^l u\frac{R_{nm}(r)}{r}\,dr,$$

$$\int_0^{2\pi} u_{\varphi\varphi}\cos(n\varphi)\,d\varphi = -n^2 \int_0^{2\pi} u\cos(n\varphi)\,d\varphi.$$

Substituting these values in (21) and (22), and then (21) and (22) into equality (20), we obtain

$$A''_{nm}(t) + a^2\mu_{nm}^2 A_{nm}(t) = 0. \tag{23}$$

The general solution of equation (23) is determined by the formula

$$A_{nm}(t) = a_{nm}\cos(a\mu_{nm}t) + b_{nm}\sin(a\mu_{nm}t), \tag{24}$$

where a_{nm} and b_{nm} are arbitrary constants. For their determination we will use the boundary conditions (5):

$$A_{nm}(0) = \frac{1}{\sqrt{\pi}} \iint_D u(r,\varphi,0) R_{nm}(r)\cos(n\varphi) r\,dr\,d\varphi = \frac{1}{\sqrt{\pi}} \iint_D \tau(r,\varphi) R_{nm}(r)\cos(n\varphi) r\,dr\,d\varphi =: \tau_{nm}, \tag{25}$$

$$A_{nm}(T) = \frac{1}{\sqrt{\pi}} \iint_D u(r,\varphi,T) R_{nm}(r)\cos(n\varphi) r\,dr\,d\varphi = \frac{1}{\sqrt{\pi}} \iint_D \psi(r,\varphi) R_{nm}(r)\cos(n\varphi) r\,dr\,d\varphi =: \psi_{nm}. \tag{26}$$

Subordinating the general solution (24) to the boundary conditions (25) and (26), we find

$$a_{nm} = \tau_{nm}, \quad b_{nm} = \frac{1}{\sin(a\mu_{nm}T)}(\psi_{nm} - \tau_{nm}\cos(a\mu_{nm}T))$$

provided that

$$\Delta_{nm}(T) = \sin(a\mu_{nm}T) \neq 0 \quad \text{at all} \quad n, m \in \mathbb{N}. \tag{27}$$

Then

$$A_{nm}(t) = \tau_{nm}\frac{\sin(a\mu_{nm}(T-t))}{\sin(a\mu_{nm}T)} + \psi_{nm}\frac{\sin(a\mu_{nm}t)}{\sin(a\mu_{nm}T)}. \tag{28}$$

Having differentiated equality (19) twice by t taking into account equation (6), we obtain

$$B''_{nm}(t) + a^2\mu_{nm}^2 B_{nm}(t) = 0.$$

From here (by analogy with the function $A_{nm}(t)$), we will find under condition (27)

$$B_{nm}(t) = \tilde{\tau}_{nm}\frac{\sin(a\mu_{nm}(T-t))}{\sin(a\mu_{nm}T)} + \tilde{\psi}_{nm}\frac{\sin(a\mu_{nm}t)}{\sin(a\mu_{nm}T)}, \tag{29}$$

where

$$\tilde{\tau}_{nm} = \frac{1}{\sqrt{\pi}} \iint_D \tau(r,\varphi) R_{nm}(r) \sin(n\varphi) r\, dr\, d\varphi, \tag{30}$$

$$\tilde{\psi}_{nm} = \frac{1}{\sqrt{\pi}} \iint_D \psi(r,\varphi) R_{nm}(r) \sin(n\varphi) r\, dr\, d\varphi. \tag{31}$$

Now let us differentiate equality (17) twice by t and, similarly, on the basis of equation (6) we obtain that the function $A_{0m}(t)$ is a solution of the differential equation

$$A''_{0m}(t) + a^2\mu^2_{0m} A_{0m}(t) = 0.$$

From here (by analogy with the function $A_{nm}(t)$), we find

$$A_{0m}(t) = \tau_{0m} \frac{\sin(a\mu_{0m}(T-t))}{\sin(a\mu_{0m}T)} + \psi_{0m} \frac{\sin(a\mu_{0m}t)}{\sin(a\mu_{0m}T)} \tag{32}$$

provided $\sin(\mu_{0m}T) \neq 0$ for all $m \in \mathbb{N}$, where

$$\tau_{0m} = \frac{1}{\sqrt{2\pi}} \iint_D \tau(r,\varphi) R_{0m}(r) r\, dr\, d\varphi, \tag{33}$$

$$\psi_{nm} = \frac{1}{\sqrt{2\pi}} \iint_D \psi(r,\varphi) R_{0m}(r) r\, dr\, d\varphi. \tag{34}$$

Now let us prove the uniqueness of the solution of problem (2)–(5). Let $\tau(x,y) = \psi(x,y) \equiv 0$ and conditions (27) be satisfied for all $m \in \mathbb{N}$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Then, by virtue of equations (25), (26), (30), (31), (33) and (34) all $\tau_{nm} = 0$, $\tilde{\tau}_{nm} = 0$, $\psi_{nm} = 0$, $\tilde{\psi}_{nm} = 0$, at $n = 0,1,2,\ldots$, $m = 1,2,\ldots$ Hence and on the basis of formulas (32), (29), (28) and (17)–(19) we have the following equations

$$\iint_D u(r,\varphi,t) R_{nm}(r) \cos(n\varphi) r\, dr\, d\varphi = 0, \quad \iint_D u(r,\varphi,t) R_{nm}(r) \sin(n\varphi) r\, dr\, d\varphi = 0$$

at all $n = 0,1,2,\ldots$, $m = 1,2,\ldots$, $t \in [0,T]$. From these equalities, based on the completeness of the system of functions (16) in the space $L_2(D)$ with weight r, it follows that $u(r,\varphi,t) = 0$ is almost everywhere in $\overline{D}$ at any $t \in [0,T]$. Since by virtue of (2) the function $u(r,\varphi,t)$ is continuous in $\overline{Q}$, then $u(r,\varphi,t) \equiv 0$ in $\overline{Q}$.

Suppose for some $n = n_0$ or $m = m_0$ the expression $\Delta_{n_0m}(T) = 0$ or $\Delta_{nm_0}(T) = 0$. For definiteness, suppose that $\Delta_{n_0m}(T) = 0$. Then the homogeneous problem (2)–(5) ($\tau(x,y) = \psi(x,y) \equiv 0$) has a nonzero solution

$$u_{n_0m}(r,\varphi,t) = \sin(a\mu_{n_0m}t) \left(a_{0m}R_{0m}(r) + a_{n_0m}R_{n_0m}(r)\cos(n_0\varphi) + b_{n_0m}R_{n_0m}(r)\sin(n_0\varphi)\right), \tag{35}$$

where a_{0m}, a_{n_0m} and b_{n_0m} are arbitrary constants.

Consider the zeros of the expression $\Delta_{nm}(T)$. Equality

$$\Delta_{nm}(T) = \sin(a\mu_{nm}T) = 0$$

only takes place when

$$T = \frac{\pi k}{a\mu_{nm}}, \quad k \in \mathbb{N}. \tag{36}$$

So, $\Delta_{nm}(T)$ goes to zero when T is determined by formula (36).

Thus, the criterion of uniqueness of the solution of problem (2)–(5) is established.

Theorem 1. *If there exists a solution of problem* (2)–(5), *then it is singular if and only if conditions* (27) *are satisfied at all* n *and* m.

3. EXISTENCE OF A SOLUTION TO THE PROBLEM

If the conditions (27) are satisfied, the solution of the problem (2)–(5) is defined by the sum of the series

$$u(r,\varphi,t)=\frac{1}{\sqrt{2\pi}}\sum_{m=1}^{\infty}A_{0m}(t)R_{0m}(r)+\frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(A_{nm}(t)\cos(n\varphi)+B_{nm}(t)\sin(n\varphi)\right)R_{nm}(r), \tag{37}$$

where the coefficients $A_{0m}(t)$, $A_{nm}(t)$, and $B_{nm}(t)$ are found by formulas (32), (28) and (29), respectively. Since $\Delta_{nm}(T)$ is the denominator of the coefficients of the series (37) and, as shown above, equation $\sin(a\mu_{nm}T)=0$ has a countable set of zeros (36), the problem of small denominators arises. In this regard, estimates about separability from zero should be established. For simplicity, in what follows we assume that $b=0$. The expression $\Delta_{nm}(T)$ at $b=0$ is represented in the following form:

$$\Delta_{nm}(\nu)=\sin(\nu q_{nm}),\quad \nu=\frac{aT}{l}. \tag{38}$$

Lemma 1. *If one of the following conditions is met:*

1) *the number $\nu/2=p$ is natural and odd;*

2) *the number $\nu/2=p/q$ is fractional-rational and the relation $(2r-p)/(2q)$ is not an integer where $r\in\mathbb{N}_0$ and $0\le r<q$,*

then there exist positive constants C_0 and m_0 ($m_0\in\mathbb{N}$) such that for all $m>m_0$ the evaluation is valid.

$$|\Delta_{nm}(\nu)|\ge C_0>0. \tag{39}$$

Proof. For zeros q_{nm} of the Bessel function $J_n(q)$ at large values $m>m_0$, where m_0 is a sufficiently large natural number, the asymptotic formula [19, p. 241] is valid.

$$q_{nm}=\frac{\pi}{2}(2m+n-1/2)+O\left((4m+2n-1)^{-1}\right). \tag{40}$$

Substitution (40) into (38) gives

$$\Delta_{nm}(\nu)=\sin\left(\frac{\nu\pi}{2}(2m+n-1/2)\right)+O\left((4m+2n-1)^{-1}\right), \tag{41}$$

since

$$\sin O\left((4m+2n-1)^{-1}\right)\approx O\left((4m+2n-1)^{-1}\right),\quad \cos O\left((4m+2n-1)^{-1}\right)\approx 1+O\left((4m+2n-1)^{-1}\right)$$

at large $m>m_0$.

Let the number $\nu/2=p\in\mathbb{N}$ odd. Then, from equality (41) for all $m>m_0$ and $n\in\mathbb{N}_0$ we obtain

$$\begin{aligned}|\Delta_{nm}(\nu)|&\ge\left|\sin\left(\pi p(2m+n)-\frac{p\pi}{2}\right)\right|-\left|O\left((4m+2n-1)^{-1}\right)\right|=\\&=\left|\sin\frac{p\pi}{2}\right|-\left|O\left((4m+2n-1)^{-1}\right)\right|=1-\left|O\left((4m+2n-1)^{-1}\right)\right|>\frac{1}{2}\end{aligned} \tag{42}$$

by virtue of

$$\left|O\left((4m+2n-1)^{-1}\right)\right|<C_1<\frac{1}{2}$$

at large m.

Let $\nu/2=p/q$, $p,q\in\mathbb{N}$, $(p,q)=1$, $p/q\notin\mathbb{N}$. In this case, let us divide $p(2m+n)$ by q with remainder: $p(2m+n)=qs+r$, $s,r\in\mathbb{N}_0$, $0\le r<q$. Then the relation (41) will take the form

$$\Delta_{nm}(\nu)=\sin\left(s\pi+\frac{r\pi}{q}-\frac{p\pi}{2q}\right)+O\left((4m+2n-1)^{-1}\right)=(-1)^s\sin\left(\pi\frac{2r-p}{2q}\right)+O\left((4m+2n-1)^{-1}\right).$$

If $r = 0$, then we have case 1) of the lemma. Then $1 \le r \le q-1$. Hence (since the relation $(2r-p)/(2q)$ is not an integer) it follows that

$$|\Delta_{nm}(\nu)| \ge \left|\sin\left(\pi\frac{2r-p}{2q}\right)\right| - \left|O\left((4m+2n-1)^{-1}\right)\right| \ge \left|\sin\left(\pi\frac{2r-p}{2q}\right)\right| - C_1 \ge C_2 - C_1 > 0, \qquad (43)$$

where

$$C_2 = \min_{1\le r\le q-1} |\sin(\pi(2r-p)/2q)|.$$

Then, from (42) and (43) under the condition $C_1 < C_2$, follows the validity of the estimate (39).

Lemma 2. *Let one of the conditions of Lemma 1 be satisfied, then for all $m > m_0$, $n \in \mathbb{N}_0$ and any $t \in [0, T]$ the following estimates are valid*

$$|A_{nm}(t)| \le M_1\big(|\tau_{nm}| + |\psi_{nm}|\big), \qquad (44)$$

$$|B_{nm}(t)| \le M_1\big(|\widetilde{\tau}_{nm}| + |\widetilde{\psi}_{nm}|\big), \qquad (45)$$

$$|A'_{nm}(t)| \le M_2\mu_{nm}\big(|\tau_{nm}| + |\psi_{nm}|\big), \quad |B'_{nm}(t)| \le M_2\mu_{nm}\big(|\widetilde{\tau}_{nm}| + |\widetilde{\psi}_{nm}|\big),$$

$$|A''_{nm}(t)| \le M_3\mu^2_{nm}\big(|\tau_{nm}| + |\psi_{nm}|\big), \quad |B''_{nm}(t)| \le M_3\mu^2_{nm}\big(|\widetilde{\tau}_{nm}| + |\widetilde{\psi}_{nm}|\big),$$

hereafter M_i are positive constants depending on T, a and l.

The fairness of these estimates follows directly from formulas (28) and (29) on the basis of inequalities (39).

Now formally from the series (37) at $b = 0$ by postal differentiation, we obtain the series

$$u_{tt} = \frac{1}{\sqrt{2\pi}}\sum_{m=1}^{\infty} A''_{0m}(t)R_{0m}(r) + \frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\big(A''_{nm}(t)\cos(n\varphi) + B''_{nm}(t)\sin(n\varphi)\big)R_{nm}(r),$$

$$u_{\varphi\varphi} = -\frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} n^2\big(A_{nm}(t)\cos(n\varphi) + B_{nm}(t)\sin(n\varphi)\big)R_{nm}(r),$$

$$u_{rr} = \frac{1}{\sqrt{2\pi}}\sum_{m=1}^{\infty} A_{0m}(t)R''_{0m}(r) + \frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\big(A_{nm}(t)\cos(n\varphi) + B_{nm}(t)\sin(n\varphi)\big)R''_{nm}(r),$$

which at any $(r, \varphi, t) \in \overline{Q}$ are majorized respectively by numerical series

$$\frac{4M_3}{\sqrt{2\pi}}\sum_{m>m_0}^{\infty} \mu^2_{0m}\big(|\tau_{0m}| + |\psi_{0m}|\big)|R_{0m}(r)| +$$

$$+ \frac{M_3}{\sqrt{\pi}}\sum_{n=1}^{\infty}\sum_{m>m_0}^{\infty} \mu^2_{nm}\big(|\tau_{nm}| + |\psi_{nm}| + |\widetilde{\tau}_{nm}| + |\widetilde{\psi}_{nm}|\big)|R_{nm}(r)|, \qquad (46)$$

$$\frac{M_1}{\sqrt{\pi}}\sum_{n=1}^{\infty}\sum_{m>m_0}^{\infty} n^2\big(|\tau_{nm}| + |\psi_{nm}| + |\widetilde{\tau}_{nm}| + |\widetilde{\psi}_{nm}|\big)|R_{nm}(r)|, \qquad (47)$$

$$\frac{M_1}{\sqrt{2\pi}}\sum_{m>m_0}^{\infty} \big(|\tau_{0m}| + |\psi_{0m}|\big)|R''_{0m}(r)| +$$

$$+ \frac{M_1}{\sqrt{\pi}}\sum_{n=1}^{\infty}\sum_{m>m_0}^{\infty} \big(|\tau_{nm}| + |\psi_{nm}| + |\widetilde{\tau}_{nm}| + |\widetilde{\psi}_{nm}|\big)|R''_{nm}(r)|. \qquad (48)$$

Lemma 3. *Let $0 < r_0 \le r \le l$, where r_0 is a small positive fixed constant. Then at $m > m_0$ and any fixed $n \in \mathbb{N}_0$ there are the following estimates*

$$|R_{nm}(r)| \le M_4, \qquad (49)$$

$$|R'_{nm}(r)| \le M_5\mu_{nm}, \qquad (50)$$

$$|R''_{nm}(r)| \le M_6\mu^2_{nm}. \qquad (51)$$

Proof. Based on the asymptotic formula for the Bessel function of the first kind $J_\nu(z)$ at large values of the argument z [20, p. 98]

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}}\left[\cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) - \frac{1}{2z}\sin\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\right] + O(z^{-5/2}) \tag{52}$$

we have

$$|J_n(\mu_{nm}r)| \le \sqrt{\frac{2}{\pi r_0\mu_{nm}}}\left(1 + \frac{1}{2r_0\mu_{nm}}\right) \le 2\sqrt{\frac{2}{\pi r_0\mu_{nm}}}, \tag{53}$$

as $1(2r_0\mu_{nm}) < 1$ at large m.

Similarly, we obtain the estimates

$$|J_{n+1}(q_{nm})| = |J_{n+1}(l\mu_{nm})| \le 2\sqrt{\frac{2}{\pi l\mu_{nm}}}, \tag{54}$$

from which follows the estimation (49).

Now find the derivative

$$R'_{nm}(r) = \frac{\sqrt{2}}{l|J_{n+1}(q_{nm})|}\mu_{nm}J'_n(z), \quad z = \mu_{nm}r. \tag{55}$$

Using the equality

$$J'_\nu(z) = \frac{1}{2}[J_{\nu-1}(z) - J_{\nu+1}(z)] \tag{56}$$

and formula (52), we obtain the asymptotic formula for $J'_n(z)$ at large z

$$J'_n(z) = \frac{1}{2}\sqrt{\frac{2}{\pi z}}\left[\cos\left(z - \frac{(n-1)}{2}\pi - \frac{\pi}{4}\right) - \cos\left(z - \frac{(n+1)}{2}\pi - \frac{\pi}{4}\right)\right] + O(z^{-3/2}) =$$
$$= \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{n\pi}{2} + \frac{\pi}{4}\right) + O(z^{-3/2}),$$

on the basis of which, similarly to estimates (53) and (54), we find

$$|J'_n(\mu_{nm}r)| \le 2\sqrt{\frac{2}{\pi r_0\mu_{nm}}}. \tag{57}$$

Then from equality (55) by virtue of estimates (57) and (54), follows estimate (50).

From (12) we calculate the second derivative

$$J''_n(\mu_{nm}r) = -\frac{1}{r}J'_n(\mu_{nm}r) + \left(\frac{n^2}{r^2} - \mu_{nm}^2\right)J_n(\mu_{nm}r). \tag{58}$$

Hence, taking into account estimates (53) and (57), we have

$$|J''_n(\mu_{nm}r)| \le \frac{1}{r_0}2\sqrt{\frac{2}{\pi r_0\mu_{nm}}} + \frac{n^2}{r_0^2}2\sqrt{\frac{2}{\pi r_0\mu_{nm}}} + \mu_{nm}^2 2\sqrt{\frac{2}{\pi r_0\mu_{nm}}}.$$

From this inequality, by virtue of (54), we verify the validity of the estimate (51).

Lemma 4. *Let $0 < r_0 \le r \le l$. Then for large n and any fixed $m \in \mathbb{N}$, the following estimates are valid*

$$|R_{nm}(r)| \le M_7, \tag{59}$$
$$|R'_{nm}(r)| \le M_8 n, \tag{60}$$
$$|R''_{nm}(r)| \le M_9 n^2. \tag{61}$$

Proof. To obtain these estimates, let us use Langer's asymptotic formula at large values of order p of the Bessel function [20, p. 103]

$$J_p(t) = \frac{1}{\pi}\sqrt{1 - \frac{\operatorname{arctg}\omega}{\omega}}K_{1/3}(z) + O(p^{-4/3}), \tag{62}$$

where

$$\omega = \sqrt{1 - \left(\frac{t}{p}\right)^2}, \quad t < p, \quad z = p(\operatorname{Arth}\omega - \omega),$$

$K_{1/3}(z)$ – McDonald's function.

Using a power series expansion of the function

$$arctg\,\omega = \omega - \frac{\omega^3}{3} + \frac{\omega^5}{5} - \frac{\omega^7}{7} + \ldots,$$

evaluate the expression

$$\frac{\omega^2}{3}\left(1 - \frac{3}{5}\omega^2\right) < 1 - \frac{arctg\,\omega}{\omega} < \frac{\omega^2}{3}.$$

Hence at $0 < \omega < 1$ we have

$$\sqrt{\frac{2}{15}}\omega < \left(1 - \frac{arctg\,\omega}{\omega}\right)^{1/2} < \frac{\omega}{\sqrt{3}}. \tag{63}$$

Then from the formula (62), taking into account the estimation (63), we obtain

$$|J_p(t)| \le \frac{\omega}{\pi\sqrt{3}} K_{1/3}(z), \tag{64}$$

$$|J_p(t)| > \sqrt{\frac{2}{15}}\frac{\omega}{\pi} K_{1/3}(z). \tag{65}$$

Now on the basis of estimates (64) and (65) we have

$$|J_n(\mu_{nm} r)| \le \frac{\omega_1}{\pi\sqrt{3}} K_{1/3}(z_1), \tag{66}$$

$$|J_n(q_{nm})| \ge \sqrt{\frac{2}{15}}\frac{\omega_2}{\pi} K_{1/3}(z_2), \tag{67}$$

where

$$\omega_1 = \sqrt{1 - \left(\frac{q_{nm} r}{nl}\right)^2}, \quad z_1 = n(\operatorname{Arth}\omega_1 - \omega_1),$$

$$\omega_2 = \sqrt{1 - \left(\frac{q_{nm}}{n+1}\right)^2}, \quad z_2 = (n+1)(\operatorname{Arth}\omega_2 - \omega_2).$$

From inequalities (66) and (67), estimate (59) follows, since $\omega_1 \approx \omega_2$ at large n.

Based on formulas (55) and (56), we estimate the derivative $R'_{nm}(r)$:

$$|R'_{nm}(r)| \le \frac{q_{nm}}{\sqrt{2} l^2 |J_{n+1}(q_{nm})|}(|J_{n-1}(\mu_{nm} r)| + |J_{n+1}(\mu_{nm} r)|).$$

Hence, taking into account estimates (66) and (67), we obtain (60).

By virtue of equality (58) on the basis of (59) and (60), we are convinced of the fairness of the estimate (61).

Remark. Note that the function $R_{nm}(r)$ and its derivatives $R'_{nm}(r)$, $R''_{nm}(r)$, starting from some number n, tend to zero at $r \to 0$. Therefore, in Lemmas 3 and 4 the estimates (49)–(51) and (59)–(61) are obtained at $r \ge r_0 > 0$.

By virtue of lemmas 3 and 4, rows (46)–(48) are majorized by the combination of rows

$$M_{10} \sum_{m>m_0}^{\infty} m^2(|\tau_{0m}| + |\psi_{0m}|), \quad M_{11} \sum_{n=1}^{\infty}\sum_{m>m_0}^{\infty} n^2(|\tau_{0m}| + |\psi_{0m}|),$$

$$M_{12} \sum_{n=1}^{\infty}\sum_{m>m_0}^{\infty} \mu_{nm}^2(|\tau_{nm}| + |\psi_{nm}| + |\tilde{\tau}_{nm}| + |\tilde{\psi}_{nm}|). \tag{68}$$

Let us denote by $C^{4,4}(\overline{D})$ the set of functions $f(r,\varphi)$, that have continuous mixed derivatives on r and φ up to and including fourth order in the closed region $\overline{D}$.

Lemma 5. *Let* $\tau(r,\varphi)$, $\psi(r,\varphi)\in C^{4,4}(\overline{D})$, and $\tau^{(0,i)}(r,0)=\tau^{(0,i)}(r,2\pi)$, $i=\overline{0,3}$, $\tau^{(k,4)}(0,\varphi)=0$, $k=\overline{0,3}$, $\psi^{(0,i)}(r,0)=\psi^{(0,i)}(r,2\pi)$, $i=\overline{0,3}$, $\psi^{(k,4)}(0,\varphi)=0$, $k=\overline{0,3}$. *Then the coefficients of* τ_{nm}, $\widetilde{\tau}_{nm}$, ψ_{nm}, $\widetilde{\psi}_{nm}$ *at* $\mu_{nm}\to+\infty$ *have estimates of*

$$\tau_{nm}=O\left(\frac{1}{n\mu_{nm}^4}\right),\quad \widetilde{\tau}_{nm}=O\left(\frac{1}{n\mu_{nm}^4}\right),\quad \psi_{nm}=O\left(\frac{1}{n\mu_{nm}^4}\right),\quad \widetilde{\psi}_{nm}=O\left(\frac{1}{n\mu_{nm}^4}\right).$$

Proof. Consider the coefficients $\tau_{nm},\psi_{nm},\tilde{\tau}_{nm}$, and $\tilde{\psi}_{nm}$ defined by formulas (25), (26), (30), and (31), respectively. Let us represent τ_{nm} in the following form:

$$\tau_{nm}=\frac{1}{\sqrt{\pi}}\int_0^l R_{nm}(\mu_{nm}r)I(r)r\,dr, \tag{69}$$

where

$$I(r)=\int_0^{2\pi}\tau(r,\varphi)\cos(n\varphi)\,d\varphi.$$

By the condition $\tau'_\varphi(r,0)=\tau'_\varphi(r,2\pi)$ and $\tau'''_\varphi(r,0)=\tau'''_\varphi(r,2\pi)$, then the integral $I(r)$ can be transformed by fourfold integration by parts into the form

$$I(r)=\frac{1}{n^4}\int_0^{2\pi}\tau_\varphi^{(4)}(r,\varphi)\cos(n\varphi)\,d\varphi. \tag{70}$$

Now let us write the integral (69), taking into account the representation (70), as

$$\tau_{nm}=\frac{\sqrt{2}}{l\sqrt{\pi}|J_{n+1}(q_{nm})|n^4}\int_0^{2\pi}J(\varphi)\cos(n\varphi)\,d\varphi, \tag{71}$$

where

$$J(\varphi)=\int_0^l\tau_\varphi^{(4)}(r,\varphi)J_n(\mu_{nm}r)r\,dr. \tag{72}$$

Note that the function $X_n(r)=r^{-n}J_n(\xi)$, $\xi=\mu_{nm}r$ is a solution of the differential equation

$$X''_n(r)+\frac{2n+1}{r}X'_n(r)+\mu_{nm}^2X_n(r)=0. \tag{73}$$

Then the integral (72), taking into account equation (73), is transformed as follows:

$$\begin{aligned}
J(\varphi)&=\int_0^l\tau_\varphi^{(4)}(r,\varphi)X_n(r)r^{n+1}\,dr=\\
&=-\frac{1}{\mu_{nm}^2}\int_0^l\tau_\varphi^{(4)}(r,\varphi)\left[X''_n(r)+\frac{2n+1}{r}X'_n(r)\right]r^{n+1}\,dr=\\
&=-\frac{1}{\mu_{nm}^2}\int_0^l\tau_\varphi^{(4)}(r,\varphi)\left[(r^{n+1}X'_n(r))'+nr^nX'_n(r)\right]dr=\\
&=-\frac{1}{\mu_{nm}^2}\int_0^l\tau_{r,\varphi}^{(2,4)}(r,\varphi)r^{n+1}X_n(r)\,dr-\\
&\quad-\frac{1}{\mu_{nm}^2}\int_0^l\tau_{r,\varphi}^{(1,4)}(r,\varphi)r^nX_n(r)\,dr+\\
&\quad+\frac{n^2}{\mu_{nm}^2}\int_0^l\tau_{r,\varphi}^{(0,4)}(r,\varphi)r^{n-1}X_n(r)\,dr=\\
&=-\frac{1}{\mu_{nm}^2}J_1-\frac{1}{\mu_{nm}^2}J_2+\frac{n^2}{\mu_{nm}^2}J_3,
\end{aligned} \tag{74}$$

where

$$J_1 = \int_0^l \tau_{r,\varphi}^{(2,4)}(r,\varphi) r^{n+1} X_n(r)\, dr,$$
$$J_2 = \int_0^l \tau_1(r,\varphi) r^{n+1} X_n(r)\, dr,$$
$$J_3 = \int_0^l \tau_2(r,\varphi) r^{n+1} X_n(r)\, dr,$$
$$\tau_1(r,\varphi) = \frac{\tau_{r,\varphi}^{(1,4)}(r,\varphi)}{r}, \quad \tau_2(r,\varphi) = \frac{\tau_{r,\varphi}^{(0,4)}(r,\varphi)}{r^2}.$$

Similarly to the integral $J(\varphi)$ by formula (74), we transform the integrals J_i, $i = 1, 2$:

$$J_i = -\frac{1}{\mu_{nm}^2} J_{i1} - \frac{1}{\mu_{nm}^2} J_{i2} + \frac{n^2}{\mu_{nm}^2} J_{i3}, \tag{75}$$

where

$$J_{11} = \int_0^l \tau_{r,\varphi}^{(4,4)}(r,\varphi) r^{n+1} X_n(r)\, dr = \int_0^l \tau_{r,\varphi}^{(4,4)}(r,\varphi) J_n(\mu_{nm} r) r\, dr,$$
$$J_{12} = \int_0^l \tau_{r,\varphi}^{(3,4)}(r,\varphi) r^{n} X_n(r)\, dr = \int_0^l \frac{\tau_{r,\varphi}^{(3,4)}(r,\varphi)}{r} J_n(\mu_{nm} r) r\, dr,$$
$$J_{13} = \int_0^l \tau_{r,\varphi}^{(2,4)}(r,\varphi) r^{n-1} X_n(r)\, dr = \int_0^l \frac{\tau_{r,\varphi}^{(2,4)}(r,\varphi)}{r^2} J_n(\mu_{nm} r) r\, dr,$$
$$J_{21} = \int_0^l \tau_{1r}''(r,\varphi) r^{n+1} X_n(r)\, dr = \int_0^l \tau_{1r}''(r,\varphi) J_n(\mu_{nm} r) r\, dr,$$
$$J_{22} = \int_0^l \tau_{1r}'(r,\varphi) r^{n} X_n(r)\, dr = \int_0^l \frac{\tau_{1r}'(r,\varphi)}{r} J_n(\mu_{nm} r) r\, dr,$$
$$J_{23} = \int_0^l \tau_1(r,\varphi) r^{n-1} X_n(r)\, dr = \int_0^l \frac{\tau_1(r,\varphi)}{r^2} J_n(\mu_{nm} r) r\, dr.$$

We transform the integral J_3 as follows:

$$J_3 = \int_0^l \tau_{r,\varphi}^{(0,4)}(r,\varphi) r^{-1} J_n(\mu_{nm} r)\, dr =$$
$$= \int_0^l \tau_{r,\varphi}^{(0,4)}(r,\varphi) r^{-n-2} r^{n+1} J_n(\mu_{nm} r)\, dr =$$
$$= \frac{\tau_{r,\varphi}^{(0,4)}(r,\varphi)}{r} J_{n+1}(\mu_{nm} r)\bigg|_0^l - \frac{1}{\mu_{nm}} \int_0^l d\left[r^{-n-2} \tau_{r,\varphi}^{(0,4)}(r,\varphi)\right] r^{n+1} J_{n+1}(\mu_{nm} r)\, dr =$$
$$= -\frac{1}{\mu_{nm}} \int_0^l \tau_{r,\varphi}^{(1,4)}(r,\varphi) r^{-1} J_{n+1}(\mu_{nm} r)\, dr + \frac{n+2}{\mu_{nm}} \int_0^l \tau_{r,\varphi}^{(0,4)}(r,\varphi) r^{-2} J_{n+1}(\mu_{nm} r)\, dr =$$
$$= -\frac{1}{\mu_{nm}} J_{31} + \frac{n+2}{\mu_{nm}} J_{32}. \tag{76}$$

After substituting (75) and (76) into equality (74), we obtain

$$J(\varphi) = \frac{1}{\mu_{nm}^4}(J_{11} + J_{12} + J_{21} + J_{22}) - \frac{n^2}{\mu_{nm}^4}(J_{13} + J_{23}) - \frac{n^2}{\mu_{nm}^3} J_{31} + \frac{n^2(n+2)}{\mu_{nm}^3} J_{32}. \tag{77}$$

If $\tau_{r,\varphi}^{(0,4)}(r,\varphi)\in C^4[0,l]$ and $\tau_{r,\varphi}^{(k,4)}(0,\varphi)=0$, $k=\overline{0,3}$, then the representations are fair

$$\tau_{r,\varphi}^{(0,4)}(r,\varphi)=\frac{\tau_{r,\varphi}^{(4,4)}(\theta,\varphi)r^4}{4!},\quad 0<\theta<r,$$
$$\tau_{r,\varphi}^{(1,4)}(r,\varphi)=\frac{\tau_{r,\varphi}^{(4,4)}(\theta,\varphi)r^3}{3!},$$
$$\tau_{r,\varphi}^{(2,4)}(r,\varphi)=\frac{\tau_{r,\varphi}^{(4,4)}(\theta,\varphi)r^2}{2!},$$
$$\tau_{r,\varphi}^{(3,4)}(r,\varphi)=\tau_{r,\varphi}^{(4,4)}(\theta,\varphi)r.$$

By virtue of this in the integrals J_{31} and J_{32}, the functions $\tau_{r,\varphi}^{(0,4)}(r,\varphi)r^{-5/2}$, $\tau_{r,\varphi}^{(1,4)}(r,\varphi)r^{-3/2}$ are continuously differentiable on $[0,l]$, so on this interval they have complete bounded variation, i.e., finite variation. Taking into account the theorem from [21, p. 653], the integrals J_{31} and J_{32} at $\mu_{nm}\to\infty$ have the following evaluation

$$J_{31}=O(\mu_{nm}^{-3/2}),\quad J_{32}=O(\mu_{nm}^{-3/2}). \tag{78}$$

In the integrals J_{1i}, $i=1,2,3$, the integrand functions $\tau_{r,\varphi}^{(4,4)}(r,\varphi)$, $\tau_{r,\varphi}^{(3,4)}(r,\varphi)r^{-1}$, and $\tau_{r,\varphi}^{(2,4)}(r,\varphi)r^{-2}$ are continuous on the segment $[0,l]$. Then by virtue of Young's theorem [21, p. 654], these integrals at $\mu_{nm}\to\infty$ have the following evaluation

$$J_{1i}=O(\mu_{nm}^{-1/2}). \tag{79}$$

Now consider the integrals J_{2i}, $i=1,2,3$. In them, the functions $\tau_{1r}''(r,\varphi)$, $\tau_{1r}'(r,\varphi)r^{-1}$ and $\tau_{1r}(r,\varphi)r^{-2}$ are also continuous on the segment $[0,l]$, so the estimates are valid

$$J_{2i}=O(\mu_{nm}^{-1/2}),\quad \mu_{nm}\to\infty. \tag{80}$$

Then from the representation (71), taking into account equality (77) and estimates (78)–(80), we obtain

$$\tau_{nm}=O\left(\frac{1}{n\mu_{nm}^4}\right).$$

Similarly, from formulas (26), (30), and (31), the rest of the estimates follow. The lemma is proved.

Numerical series (68), by virtue of formula (40), are majorized by convergent series, respectively

$$M_{13}\sum_{m>m_0}^{\infty}\frac{1}{m^2},\quad M_{14}\sum_{n=1}^{\infty}\sum_{m>m_0}^{\infty}\frac{n}{(4m+2n-1)^4},\quad M_{15}\sum_{n=1}^{\infty}\sum_{m>m_0}^{\infty}\frac{1}{n(4m+2n-1)^2}.$$

If for the numbers ν from lemma 1, for some $m=m_1,m_2,\dots,m_s\le m_0$, where $1\le m_1<m_2<\dots<m_s$, $\Delta_{nm_i}(\nu)=0$, then it is necessary and sufficient for the solvability of problem (2)–(5) that the conditions are satisfied

$$\tau_{nm_i}=\psi_{nm_i}=0,\quad \tilde\tau_{nm_i}=\tilde\psi_{nm_i}=0,\quad i=\overline{1,s}. \tag{81}$$

In this case, the solution of the problem (2)–(5) is defined as a sum of series:

$$\begin{aligned}u(r,\varphi,t)={}&\frac{1}{\sqrt{2\pi}}\left(\sum_{m=1}^{m_1-1}+\sum_{m=m_1+1}^{m_2-1}+\dots+\sum_{m=m_{s-1}+1}^{m_s-1}+\sum_{m=m_s+1}^{\infty}\right)A_{0m}(t)R_{0m}(r)+\\&+\frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty}\left(\sum_{m=1}^{m_1-1}+\sum_{m=m_1+1}^{m_2-1}+\dots+\sum_{m=m_{s-1}+1}^{m_s-1}+\sum_{m=m_s+1}^{\infty}\right)\times\\&\times(A_{nm}(t)\cos(n\varphi)+B_{nm}(t)\sin(n\varphi))R_{nm}(r)+\\&+\sum_{i=1}^{s}C_{nm_i}u_{nm_i}(r,\varphi,t),\end{aligned} \tag{82}$$

here $u_{nm_i}(r,\varphi,t)$ are determined by formula (35), where m_0 should be replaced by m_i, C_{nm_i} are arbitrary constants; if in the finite sums in the right-hand side of (82), the upper limit is less than the lower limit, they should be considered as zeros.

Thus, the following has been proved

Theorem 2. *Let the conditions of lemmas* 1 *and* 5 *be satisfied. Then if* $\Delta_{nm}(\nu)\neq 0$ *at all* $m=\overline{1,m_0}$*, then problem* (2)–(5) *is uniquely solvable, and this solution is defined by row* (37); *if* $\Delta_{nm}(\nu)=0$ *at some* $m=m_1,m_2,\dots,m_s\leq m_0$*, then problem* (2)–(5) *is solvable only when conditions* (81) *are satisfied, and the solution is defined by row* (82).

Note that the fulfillment of the condition $\Delta_{nm}(\nu)\neq 0$ at $m=\overline{1,m_0}$ can be achieved if $\nu\neq \pi k/q_{nm}$ (by virtue of formula (36) at $b=0$).

4. STABILITY OF THE PROBLEM SOLUTION

Consider the following norms:

$$\|u(r,\varphi,t)\|_{L_2(D)}=\iint_D u^2(r,\varphi,t)r\,dr\,d\varphi,$$
$$\|u(r,\varphi,t)\|_{C(\overline{Q})}=\max_{r,\varphi,t\in\overline{Q}}|u(r,\varphi,t)|,$$
$$\|f^{(2,2)}_{r,\varphi}(r,\varphi)\|_{L_2(D)}=\iint_D (f^{(2,2)}_{r,\varphi}(r,\varphi))^2 r\,dr\,d\varphi,$$
$$\|g^{(2,2)}_{r,\varphi}(r,\varphi)\|^2_{C(\overline{D})}=\max_{r,\varphi\in\overline{D}}|g^{(2,2)}_{r,\varphi}(r,\varphi)|.$$

Theorem 3. *Let the conditions of Theorem* 2 *and* $\Delta_{nm}(\nu)\neq 0$ *be satisfied at* $m=\overline{1,m_0}$. *Then for the solution* (37) *of the problem* (2)–(5), *the following estimates are valid*

$$\|u(r,\varphi,t)\|_{L_2(D)}\leq M_{16}(\|\tau(r,\varphi)\|_{L_2(D)}+\|\psi(r,\varphi)\|_{L_2(D)}),\tag{83}$$
$$\|u(r,\varphi,t)\|_{C(\overline{Q})}\leq M_{17}(\|\tau^{(2,2)}_{r,\varphi}(r,\varphi)\|_{C(\overline{D})}+\|\psi^{(2,2)}_{r,\varphi}(r,\varphi)\|_{C(\overline{D})}).\tag{84}$$

Proof. The constructed system of eigenfunctions (16) is orthonormalized in the space $L_2(D)$ with weight r. Then from formula (37) on the basis of estimates (44), (45), and (49), we will have

$$\|u(r,\varphi,t)\|^2_{L_2(D)}=\sum_{m=1}^{\infty}A^2_{0m}(t)+\sum_{n,m=1}^{\infty}A^2_{nm}(t)+B^2_{nm}(t)\leq$$
$$\leq 2M_1^2M_4^2\left[\sum_{m=1}^{\infty}\left(|\tau_{0m}|^2+|\psi_{0m}|^2\right)+\sum_{n,m=1}^{\infty}\left(|\tau_{nm}|^2+|\widetilde{\tau}_{nm}|^2+|\psi_{nm}|^2+|\widetilde{\psi}_{nm}|^2\right)\right]=$$
$$=2M_1^2M_4^2\left(\|\tau(r,\varphi)\|^2_{L_2(D)}+\|\psi(r,\varphi)\|^2_{L_2(D)}\right).$$

Hence we obtain the estimate (83).

Let (r,φ,t) be an arbitrary point $\overline{Q}$. Then from formula (37), taking into account estimates (44), (45) and (49), we have

$$|u(r,\varphi,t)|\leq M_1M_4\left[\sum_{m=1}^{\infty}(|\tau_{0m}|+|\psi_{0m}|)+\sum_{n,m=1}^{\infty}(|\tau_{nm}|+|\psi_{nm}|+|\tilde{\tau}_{nm}|+|\tilde{\psi}_{nm}|)\right].\tag{85}$$

Further, based on the reasoning given in the proof of Lemma 5, we will represent the coefficient τ_{nm} as

$$\tau_{nm}=-\frac{\sqrt{2}}{l\sqrt{\pi}|J_{n+1}(q_{nm})|n^2}\int_0^{2\pi}J(\varphi)\cos(n\varphi)\,d\varphi,$$

where

$$J(\varphi)=\int_0^l \tau_{r,\varphi}^{(0,2)}(r,\varphi)J_n(\mu_{nm}r)r\,dr=-\frac{1}{\mu_{nm}^2}(J_1'+J_2'-n^2J_3'),$$
$$J_1'=\int_0^l \tau_{r,\varphi}^{(2,2)}(r,\varphi)J_n(\mu_{nm}r)r\,dr,$$
$$J_2'=\int_0^l \frac{\tau_{r,\varphi}^{(1,2)}(r,\varphi)}{r}J_n(\mu_{nm}r)r\,dr,$$
$$J_3'=\int_0^l \frac{\tau_{r,\varphi}^{(0,2)}(r,\varphi)}{r^2}J_n(\mu_{nm}r)r\,dr.$$

If $\tau_{r,\varphi}^{(0,2)}(r,\varphi)\in C^2[0,l]$ and $\tau_{r,\varphi}^{(0,2)}(0,\varphi)=\tau^{(1,2)}(0,\varphi)=0$, then the functions $\tau_{r,\varphi}^{(1,2)}(r,\varphi)r^{-1}=\tau_{r,\varphi}^{(2,2)}(\theta,\varphi)$, $\tau_{r,\varphi}^{(0,2)}(r,\varphi)=\tau_{r,\varphi}^{(2,2)}(\theta,\varphi)/2$, $0<\theta<r$ are continuous on the segment $[0,l]$, then

$$|\tau_{nm}|\le\frac{M_{18}}{\mu_{nm}^2}|\tau_{nm}^{(2,2)}|,$$

where

$$\tau_{nm}^{(2,2)}=\frac{1}{\sqrt{\pi}}\iint_D \tau_{r,\varphi}^{(2,2)}(r,\varphi)\cos(n\varphi)R_{nm}(r)r\,dr\,d\varphi. \tag{86}$$

Similarly, we obtain the estimates

$$|\tilde{\tau}_{nm}|\le\frac{M_{18}}{\mu_{nm}^2}|\tilde{\tau}_{nm}^{(2,2)}|,$$
$$\tilde{\tau}_{nm}^{(2,2)}=\frac{1}{\sqrt{\pi}}\iint_D \tau_{r,\varphi}^{(2,2)}(r,\varphi)\sin(n\varphi)R_{nm}(r)r\,dr\,d\varphi, \tag{87}$$
$$|\psi_{nm}|\le\frac{M_{18}}{\mu_{nm}^2}|\psi_{nm}^{(2,2)}|,$$
$$|\tilde{\psi}_{nm}|\le\frac{M_{18}}{\mu_{nm}^2}|\tilde{\psi}_{nm}^{(2,2)}|,$$

where $\psi_{nm}^{(2,2)}$ and $\tilde{\psi}_{nm}^{(2,2)}$ are defined according to formulas (86) and (87), but with the replacement of $\tau(r,\varphi)$ with $\psi(r,\varphi)$.

Now, continuing the estimation (85), we have

$$|u(r,\varphi,t)|\le M_{19}\left[\sum_{m=1}^{\infty}\frac{1}{\mu_{0m}^2}(|\tau_{0m}^{(2,2)}|+|\psi_{0m}^{(2,2)}|)+\sum_{n,m=1}^{\infty}\frac{1}{\mu_{nm}^2}(|\tau_{nm}^{(2,2)}|+|\tilde{\tau}_{nm}^{(2,2)}|+|\psi_{nm}^{(2,2)}|+|\tilde{\psi}_{nm}^{(2,2)}|)\right].$$

Hence, using Bunyakovsky's inequality, we obtain

$$|u(r,\varphi,t)|\le M_{20}\left\{\left(\sum_{m=1}^{\infty}\frac{1}{\mu_{0m}^4}\right)^{1/2}\left[\left(\sum_{m=1}^{\infty}|\tau_{0m}^{(2,2)}|^2\right)^{1/2}+\left(\sum_{m=1}^{\infty}|\psi_{0m}^{(2,2)}|^2\right)^{1/2}\right]+\right.$$
$$\left.+\left(\sum_{n,m=1}^{\infty}\frac{1}{\mu_{nm}^4}\right)^{1/2}\left[\left(2\sum_{n,m=1}^{\infty}\left(|\tau_{nm}^{(2,2)}|^2+|\tilde{\tau}_{nm}^{(2,2)}|^2\right)\right)^{1/2}+\left(2\sum_{n,m=1}^{\infty}\left(|\psi_{nm}^{(2,2)}|^2+|\tilde{\psi}_{nm}^{(2,2)}|^2\right)\right)^{1/2}\right]\right\}\le$$
$$\le M_{21}\left[\left(\sum_{m=1}^{\infty}|\tau_{0m}^{(2,2)}|^2\right)^{1/2}+\left(\sum_{n,m=1}^{\infty}\left(|\tau_{nm}^{(2,2)}|^2+|\tilde{\tau}_{nm}^{(2,2)}|^2\right)\right)^{1/2}+\right.$$

$$+\left(\sum_{m=1}^{\infty}|\psi_{0m}^{(2,2)}|^2\right)^{1/2}+\left(\sum_{n,m=1}^{\infty}\left(|\psi_{nm}^{(2,2)}|^2+|\widetilde{\psi}_{nm}^{(2,2)}|^2\right)\right)^{1/2}\Bigg]\le$$

$$\le M_{21}\sqrt{2}\Bigg[\left(\sum_{m=1}^{\infty}|\tau_{0m}^{(2,2)}|^2+\sum_{n,m=1}^{\infty}\left(|\tau_{nm}^{(2,2)}|^2+|\widetilde{\tau}_{nm}^{(2,2)}|^2\right)\right)^{1/2}+$$

$$+\left(\sum_{m=1}^{\infty}|\psi_{0m}^{(2,2)}|^2+\sum_{n,m=1}^{\infty}\left(|\psi_{nm}^{(2,2)}|^2+|\widetilde{\psi}_{nm}^{(2,2)}|^2\right)\right)^{1/2}\Bigg]=$$

$$=\sqrt{2}M_{21}\left(\|\tau^{(2,2)}(r,\varphi)\|_{L_2(D)}+\|\psi^{(2,2)}(r,\varphi)\|_{L_2(D)}\right)\le M_{22}\left(\|\tau^{(2,2)}(r,\varphi)\|_{C(\overline{D})}+\|\psi^{(2,2)}(r,\varphi)\|_{C(\overline{D})}\right).$$

From the last inequality, the estimate (84) follows directly.

CONFLICT OF INTERESTS

The author of this paper declares that he has no conflict of interests.

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PARTIAL DIFFERENTIAL EQUATIONS

BLOW-UP OF THE SOLUTION AND GLOBAL SOLVABILITY OF THE CAUCHY PROBLEM FOR THE EQUATION OF VIBRATIONS OF A HOLLOW ROD

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Abstract. For a nonlinear partial differential equation of Sobolev type, generalizing the equation of oscillations of a hollow flexible rod, the Cauchy problem is studied in the space of continuous functions defined on the entire numerical axis and for which there are limits at infinity. The conditions for the existence of a global classical solution and the blow-up of the solution to the Cauchy problem on a finite time interval are considered.

Keywords: *equation of vibrations of a hollow flexible rod, nonlinear equation of Sobolev type, global solution, blow-up of the solution*

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1. INTRODUCTION. PROBLEM STATEMENT

The vibrations of a hollow flexible rod [1, Ch. 8, formula (8.230)] are modeled by a nonlinear differential equation of Sobolev type [2]

$$\delta u_{tt} - u_{ttxx} - \alpha_2 u_{txx} - \alpha_1 u_{tx} + \beta_2 u_{xxxx} + \beta_1 u_{xx} + \gamma u = u_{xx} f'(u_x), \tag{1}$$

where $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$, $\mathbb{R}_+ = (0,+\infty)$, $\mathbb{R} = (-\infty,+\infty)$; the dash in the equation denotes differentiation by $u_x = \partial_x u = \partial u/\partial x$; the coefficients $\alpha_i, \beta_i, i = 1,2, \gamma, \delta$ are non-negative constants; the nonlinearity f is a twice continuously differentiable function $f(r)$, $r \in \mathbb{R}$, for which the modulus $|f(r)|$ at $r \geq 0$ is a non-decreasing function and the estimates are valid

$$\sup_{x\in\mathbb{R}} |f^{(i)}(g(x))| \leq \Big|f^{(i)}\Big(\sup_{x\in\mathbb{R}} |g(x)|\Big)\Big|, \quad i = 0,1, \quad g(x) \in C[\mathbb{R}],$$
$$|f(\xi r)| \leq \chi(\xi)|f(r)|, \quad \xi > 0, \quad r \geq 0, \tag{2}$$

χ – a continuous non-decreasing function (its simplest example is the power function, for other non-trivial examples see [3]).

We assume that the rod is infinite. This idealization is acceptable [4], if there are optimal damping devices at the rod boundaries, i.e., the parameters of the boundary clamping are such, that the perturbations falling on it are not reflected.

The Cauchy problem for equation (1) is investigated in the space $C[\mathbb{R}]$ [5, Ch. 8, § 1] of continuous functions $g = g(x)$, for which both limits exist at $x \to \pm\infty$ and the norm is $\|g\|_C = \sup_{x\in\mathbb{R}} |g(x)|$, with initial conditions

$$u|_{t=0} = \varphi(x), \quad u_t|_{t=0} = \psi(x), \quad x \in \mathbb{R}. \tag{3}$$

The sought classical solution $u = u(t,x)$, $(t,x) \in \overline{\mathbb{R}}_+ \times \mathbb{R}$, $\overline{\mathbb{R}}_+ = [0,+\infty)$, and its partial derivatives included in equation (1), for all values of the temporary variable t on the variable x belong to the space $C[\mathbb{R}]$. (By a *classical* *solution of the equation* we mean a sufficiently smooth function having all continuous derivatives of the desired order and satisfying the equation at every point in the domain of its setting.)

By $C^{(k)}[\mathbb{R}] = \{g(x) \in C[\mathbb{R}] : g'(x), \dots, g^{(k)}(x) \in C[\mathbb{R}]\}$, $k = 1, 2, \dots$, we denote subsets of differentiable functions in $C[\mathbb{R}]$.

Recall [5, Chap. 8, § 1; 6, § 2] that in the space $C[\mathbb{R}]$ the differential operator ∂_x with domain of definition $D(\partial_x) = C^{(1)}[\mathbb{R}]$ generates a compressive strongly continuous group $U(\tau; \partial_x)g(x) = g(x+\tau)$, $\tau \in \mathbb{R}$, of left shifts, and the operator ∂_x^2 with domain of definition $D(\partial_x^2) = C^{(2)}[\mathbb{R}]$ is the derivative operator of the strongly continuous semigroup $U(t; \partial_x^2)g(x) = (2\sqrt{\pi t})^{-1} \int_0^{+\infty} e^{-\xi^2/(4t)} g(x+\xi)d\xi$, $t \in \overline{\mathbb{R}}_+$; and for the resolvents $(\lambda I - \partial_x)^{-1}$, $(\lambda I - \partial_x^2)^{-1}$ the estimates $\|(\lambda I - \partial_x)^{-1}\| \le 1/\lambda$ and $\|(\lambda I - \partial_x^2)^{-1}\| \le 1/\lambda$ are valid at $\lambda > 0$.

Let us investigate the Cauchy problem (1), (3) according to the following plan.

1. Let us make sure that the formulation of the Cauchy problem (1), (3) is correct and its classical solution exists locally in time. For this purpose, we find the solution of the Cauchy problem for the linear homogeneous equation corresponding to (1).

2. Let us introduce an auxiliary Cauchy problem

$$\delta v_{tt} - v_{ttxx} - \alpha_2 v_{txx} - \alpha_1 v_{tx} + \beta_2 v_{xxxx} + \beta_1 v_{xx} + \gamma v = \partial_x^2 f(v), \tag{4}$$

$$v|_{t=0} = \varphi'(x), \quad v_t|_{t=0} = \psi'(x), \quad x \in \mathbb{R}, \tag{5}$$

for which we find the time interval $[0, t_1]$ of existence and uniqueness of its classical solution and estimate the norm in $C[\mathbb{R}]$ of this local solution.

3. Let us establish the relation between the solutions of equations (1) and (4) by assuming that on the segment $[0, t_1$, the solution $u = u(t,x)$ at the variable x belongs to the intersection of the subset $C^{(4)}[\mathbb{R}] \subset C[\mathbb{R}]$ with the Sobolev space $W_2^4(\mathbb{R})$, and the temporary partial derivatives $u_t = u_t(t,x)$ and $u_{tt} = u_{tt}(t,x)$ belong to the intersection $C^{(2)}[\mathbb{R}] \cap W_2^2(\mathbb{R})$.

4. Let us find sufficient conditions for the existence of a single classical global ($t \ge 0$) solution and destruction on a finite time interval of the solution of the Cauchy problem (1), (3).

2. CAUCHY PROBLEM FOR A LINEAR HOMOGENEOUS EQUATION

Consider the linear homogeneous equation corresponding to (1):

$$(\delta I - \partial_x^2)u_{tt} - (\alpha_2 \partial_x^2 + \alpha_1 \partial_x)u_t + (\beta_2 \partial_x^4 + \beta_1 \partial_x^2 + \gamma I)u = 0. \tag{6}$$

Let's introduce in (6) a new unknown function

$$v(t,x) = \delta u(t,x) - u_{xx}(t,x), \tag{7}$$

assuming that the partial derivatives of u_{xx}, u_{txx} are continuous at $t \in \overline{\mathbb{R}}_+$. From substitution (7), provided that the initial functions $\varphi(x), \psi(x)$ belong to $C^{(2)}[\mathbb{R}]$, we can uniquely determine the initial values of the function $v = v(t,x)$:

$$v|_{t=0} = v_0(x) = \delta\varphi(x) - \varphi''(x), \quad v_t|_{t=0} = v_1(x) = \delta\psi(x) - \psi''(x),$$

and, using the membership of the positive semi-axis to the resolvent set of the differential operator ∂_x^2, express the solution $u(t,x)$ of equation (6) through the new unknown function $v(t,x)$:

$$u(t,x) = (\delta I - \partial_x^2)^{-1} v(t,x) = \frac{1}{2\sqrt{\delta}} \int_{-\infty}^{+\infty} e^{-|s|\sqrt{\delta}} v(t, x+s)ds. \tag{8}$$

As a result of substitution (7) we obtain the equivalent (6) integro-differential equation

$$v_{tt} + A_1 v_t + A_2 v = 0, \tag{9}$$

in which the operator coefficients are

$$A_1 = \alpha_2 I - (\alpha_2\sqrt{\delta} - \alpha_1)\sqrt{\delta}(\delta I - \partial_x^2)^{-1} - \alpha_1(\sqrt{\delta} I - \partial_x)^{-1}, \quad D(A_1) = C[\mathbb{R}],$$

$$A_2 = -\beta_2\partial_x^2 - (\beta_2\delta + \beta_1)I + (\beta_2\delta^2 + \beta_1\delta + \gamma)(\delta I - \partial_x^2)^{-1}, \quad D(A_2) = C^{(2)}[\mathbb{R}].$$

The bounded operator A_1 generates a uniformly continuous group $U(\tau; A_1)$, $\tau \in \mathbb{R}$, represented by a degree series

$$U(\tau; A_1) = \sum_{n=0}^{+\infty} \frac{\tau^n}{n!} A_1^n,$$

uniformly converging on τ at each finite segment from $\mathbb{R}$, and by virtue of the permutation of operators $(\sqrt{\delta} I - \partial_x)^{-1}$ and $(\delta I - \partial_x^2)^{-1}$, the representation is true

$$U(\tau; A_1) = e^{\alpha_2 \tau} U\big(-\alpha_1 \tau; (\sqrt{\delta} I - \partial_x)^{-1}\big) U\big(-(\alpha_2 \sqrt{\delta} - \alpha_1)\sqrt{\delta}\tau; (\delta I - \partial_x^2)^{-1}\big) =$$
$$= e^{\alpha_2 \tau} \left(\sum_{n=0}^{+\infty} \frac{(-1)^n \alpha_1^n \tau^n}{n!} (\sqrt{\delta} I - \partial_x)^{-n} \right) \left(\sum_{m=0}^{+\infty} \frac{(-1)^m (\alpha_2 \sqrt{\delta} - \alpha_1)^m \delta^{m/2} \tau^m}{m!} (\delta I - \partial_x^2)^{-m} \right),$$

as well as the evaluation

$$\|U(t; A_1)\| \le e^{(\alpha_2 + \alpha_1/\sqrt{\delta} + |\alpha_2 - \alpha_1/\sqrt{\delta}|)t}, \quad t \in \overline{R}_+.$$

In equation (9) we substitute the unknown function

$$w(t, x) = U(t/2; A_1) v(t, x), \tag{10}$$

then we can uniquely determine the initial values of the function $w(t, x)$:

$$\begin{aligned} w|_{t=0} &= w_0(x) = v_0(x), \\ w_t|_{t=0} &= w_1(x) = \frac{A_1 v_0(x)}{2} + v_1(x) = \\ &= \frac{\alpha_2 v_0(x)}{2} - \frac{\alpha_2 \sqrt{\delta} - \alpha_1}{4} \int_{-\infty}^{+\infty} e^{-|s|\sqrt{\delta}} v_0(x+s)\,ds - \\ &\quad - \frac{\alpha_1}{2} \int_0^{+\infty} e^{-r\sqrt{\delta}} v_0(x+r)\,dr + v_1(x) \end{aligned}$$

and express the solution $v(t, x)$ of equation (9) through a new unknown function $w(t, x)$:

$$v(t, x) = U(-t/2; A_1) w(t, x). \tag{11}$$

As a result of substitution (10), we obtain the integro-differential equation equivalent to (9)

$$w_{tt} = \left(\frac{A_1^2}{4} - A_2 \right) w, \tag{12}$$

in which the operator coefficient

$$\frac{A_1^2}{4} - A_2 = B = B_0 + B_1, \quad D(B) = C^{(2)}[\mathbb{R}],$$

where $B_0 = \beta_2 \partial_x^2$ and

$$B_1 = \left(\beta_2 \delta + \beta_1 + \frac{\alpha_2^2}{4} \right) I - \left(b_2 \delta^2 + \beta_1 \delta + \gamma + \frac{\alpha_2(\alpha_2 \sqrt{\delta} - \alpha_1)}{2} \sqrt{\delta} \right) (\delta I - \partial_x^2)^{-1} -$$
$$- \frac{\alpha_2 \alpha_1}{2} (\sqrt{\delta} I - \partial_x)^{-1} + \frac{1}{4} \Big(\alpha_1 (\sqrt{\delta} I - \partial_x)^{-1} + (\alpha_2 \sqrt{\delta} - \alpha_1) \sqrt{\delta} (\delta I - \partial_x^2)^{-1} \Big)^2.$$

Equation (12) can be written as an abstract ordinary differential equation

$$W_{tt} = BW, \quad t \in \mathbb{R}_+, \tag{13}$$

where $W = W(t) : t \to w(t, x)$ is the sought vector-function defined for $t \in \overline{\mathbb{R}}_+$ with values in the space $C[\mathbb{R}]$.

For equation (13), we consider an abstract Cauchy problem with initial conditions

$$W|_{t=0} = W_0, \quad W'|_{t=0} = W_1, \tag{14}$$

where $W_0 = w_0(x)$, $W_1 = w_1(x)$ are elements of the space $C[\mathbb{R}]$.

The Cauchy problem (13), (14) is uniformly correct [6, § 1.4], only when the operator B is the producing operator of a strongly continuous cosine operator-function $C(\tau; B)$, $\tau \in \mathbb{R}$.

In the space $C[\mathbb{R}]$, the operator B_0 is the derivative operator of the strongly continuous cosine operator-function $C(\tau; B_0)$, $\tau \in \mathbb{R}$ [6, § 1.5]:

$$C(\tau; B_0)g(x) = 2^{-1}[U(\tau\sqrt{\beta_2}; \partial_x) + U(-\tau\sqrt{\beta_2}; \partial_x)]g(x) = 2^{-1}[g(x + \tau\sqrt{\beta_2}) + g(x - \tau\sqrt{\beta_2})],$$

for which the estimate of the norm is fair

$$\|C(t; B_0)\| \le 1, \quad t \in \overline{R}_+.$$

The corresponding sine operator-function $S(\tau; B_0)$, $\tau \in \mathbb{R}$, has the form

$$S(\tau; B_0)g(x) = \int_0^\tau C(s; B_0)g(x)ds = \frac{1}{2\sqrt{\beta_2}} \int_{x-\tau\sqrt{\beta_2}}^{x+\tau\sqrt{\beta_2}} g(\xi)d\xi$$

and the norm estimation is valid for it

$$\|S(t; B_0)\| \le t, \quad t \in \overline{R}_+.$$

The bounded operator B_1 generates a strongly continuous cosine operator-function $C(\tau; B_1)$, for which the representation [6, §§ 1.4, 4.2] is valid on an arbitrary element $g(x) \in C[\mathbb{R}]$

$$C(\tau; B_1)g(x) = \sum_{n=0}^{+\infty} \frac{\tau^{2n}}{(2n)!} B_1^n g(x), \quad \tau \in \mathbb{R},$$

and the power series converges uniformly on τ on each finite segment from $\mathbb{R}$. Note that the operator-valued function $C(\tau; B_1)$ is continuous in the uniform operator topology, and the norm estimate is valid for it

$$\|C(t; B_1)\| \leqslant \sum_{n=0}^{+\infty} \frac{t^{2n}}{(2n)!} \|B_1\|^n \leqslant \operatorname{ch}(c_1 t), \quad t \in \overline{R}_+,$$

where $c_1^2 = 2\beta_2\delta + 2\beta_1 + \gamma/\delta + (\alpha_2\sqrt{\delta} + \alpha_1 + |\alpha_2\sqrt{\delta} - \alpha_1|)^2/(4\delta)$.

The operator B is obtained by perturbing the unbounded operator B_0 by the bounded operator B_1, but the perturbation by the bounded operator preserves [6, § 8.2] the ability of the operator B_0 to generate the cosine operator-function, so $B = B_0 + B_1$ is the derivative operator of the strongly continuous cosine operator-function $C(\tau; B)$, $\tau \in \mathbb{R}$, and hence the abstract Cauchy problem (13), (14) is uniformly correct.

The solution of the Cauchy problem (13), (14) for any initial data $W_0 \in D(B)$ and $W_1 \in C_1[\mathbb{R}]$ is defined by the formula

$$W(t) = C(t; B)W_0 + S(t; B)W_1,$$

where $S(t; B)$ is the sine operator-function associated with $C(t; B)$:

$$S(t; B)g = \int_0^t C(\tau; B)g d\tau, \quad g \in C[\mathbb{R}],$$

$C_1[\mathbb{R}] = \{g \in C[\mathbb{R}] : C(t; B)g \in C^{(1)}(\mathbb{R}, C[\mathbb{R}])\}$ is a linear manifold. It is obvious that $D(B) = C^{(2)}[\mathbb{R}] \subset C_1[\mathbb{R}]$.

In order to derive an estimate of the norm of the solution of equation (13) that is the abstract function $W(t)$, we find estimates of the norms of the cosine and sine of the operator functions generated by the operator B, for which we obtain a representation of the operator-valued function $C(t; B)$ via $C(t; B_0)$ and $C(t; B_1)$.

Considering the derivative operator B as the result of perturbing the derivative operator B_0 by the operator B_1, that in turn gives rise to the cosine operator-function, for $g(x) \in D(B_0) \cap D(B_1) = C^{(2)}[\mathbb{R}]$, we obtain [6, § 8.2] the representation of

$$C(t; B)g(x) = C(t; B_0)g(x) + \frac{t^2}{2} \int_0^1 j_1(t\sqrt{1 - s^2}, B_0)C(ts; B_1)g(x)ds,$$

where $j_1(t, B_0)g(x) = \frac{4}{\pi}\int_0^1 \sqrt{1-r^2}C(tr; B_0)g(x)dr$.

For $t \in \mathbb{R}_+$ we obtain estimates of the norms: $\|j_1(t, B_0)\| \le \frac{4}{\pi}\int_0^1 \sqrt{1-r^2}dr = 1$ and

$$\|C(t; B)\| \le 1 + \frac{t^2}{2}\int_0^1 \text{ch}\,(c_1 ts)\,ds = 1 + \frac{t}{2c_1}\,\text{sh}(c_1 t) = \sigma_1(t), \tag{15}$$

$$\|S(t; B)\| \le t + \frac{1}{2c_1}\int_0^t \tau\,\text{sh}(c_1\tau)\,d\tau \le t\left(1 + \frac{\text{ch}\,(c_1 t)}{2c_1^2}\right) = \sigma_2(t). \tag{16}$$

Using formulas (11) and (8) of inverse substitutions we have

$$u(t, x) = (\delta I - \partial_x^2)^{-1}v(t, x) = (\delta I - \partial_x^2)^{-1}U(-t/2; A_1)w(t, x). \tag{17}$$

Then, using the permutability of the resolvent $(\delta I - \partial_x^2)^{-1}$ and the semigroup $U(-t/2; A_1)$ both among themselves and with the cosine operator-function generated by the operator B, we find a solution of the Cauchy problem for equation (6):

$$u(t, x) = U(-t/2; A_1)\big[C(t; B)\varphi(x) + S(t; B)(A_1\varphi(x)/2 + \psi(x))\big]. \tag{18}$$

Thus, there is

Theorem 1. *Let the initial functions $\varphi(x)$ and $\psi(x)$ belong to the subset $C^{(4)}[\mathbb{R}]$ of the space $C[\mathbb{R}]$, then the Cauchy problem for the linear homogeneous equation* (6) *is uniformly correct, the classical solution is given by the formula* (18) *and the evaluation is valid for it*

$$\sup_{x\in\mathbb{R}} |u(t, x)| \le e^{-(\alpha_2 - \alpha_1/\sqrt{\delta} - |\alpha_2 - \alpha_1/\sqrt{\delta}|)t/2} \times$$

$$\times \left[\sigma_1(t)\sup_{x\in\mathbb{R}}|\varphi(x)| + \sigma_2(t)\left(\sup_{x\in\mathbb{R}}|\psi(x)| + \frac{\alpha_2\sqrt{\delta} + \alpha_1 + |\alpha_2\sqrt{\delta} - \alpha_1|}{2\sqrt{\delta}}\sup_{x\in\mathbb{R}}|\varphi(x)|\right)\right], \quad t \in \overline{R}_+.$$

Remark 1. The classical solution $W(t)$ of the abstract Cauchy problem (13), (14) belongs to $C^{(2)}(\overline{R}_+, C[\mathbb{R}])$ and for it $BW(t) \in C(\overline{R}_+, C[\mathbb{R}])$, hence $w(t, x) = U(t/2; A_1) \times (\delta I - \partial_x^2)u(t, x) \in C^{2,2}(\overline{\mathbb{R}}_+, \mathbb{R})$. By virtue of (17), the solution of the Cauchy problem (6), (3) is $u(t, x) \in C^{2,4}(\overline{R}_+, \mathbb{R})$.

3. LOCAL SOLUTION OF THE CAUCHY PROBLEM FOR THE NONLINEAR EQUATION (4)

Equation (4) is obtained from equation (1) through differentiating both parts by the variable x and then substituting $u_x = v$ (the left parts of these equations coincide).

Let's act on both parts of equation (4) by the operator $(\delta I - \partial_x^2)^{-1}$ and obtain the equivalent equation

$$v_{tt} + A_1 v_t + A_2 v = f_1(v), \tag{19}$$

in which the nonlinearity $f_1(u) = [\delta(\delta I - \partial_x^2)^{-1} - I]f(u)$, and the operators A_1 and A_2 are the same as in equation (9).

Equation (19) is reduced to an abstract semi-linear equation by substituting $v(t, x) = U(-t/2; A_1)w(t, x)$

$$W_{tt} = BW + f_2(t, U(-t/2; A_1)W), \tag{20}$$

where the operator B is the same as in (13) and the nonlinear operator f_2 is defined by the formula

$$f_2(t, \cdot) = U(t/2; A_1)[\delta(\delta I - \partial_x^2)^{-1} - I]f(\cdot),$$

here $f(\cdot)$ is the superposition operator: $f(g) = f(g(x))$, $g(x) \in C[\mathbb{R}]$.

Given $t \in \overline{\mathbb{R}}_+$, it is fair to estimate the norm of the operator $f_2(t,\cdot)$ in the space $C[\mathbb{R}]$:

$$\|F(t,g)\|_C \le 2e^{(\alpha_2+\alpha_1/\sqrt{\delta}+|\alpha_2-\alpha_1/\sqrt{\delta}|)t/2} f(\|g\|_C). \tag{21}$$

For equation (20) we consider an abstract Cauchy problem with initial conditions

$$W|_{t=0} = W_0', \quad W'|_{t=0} = W_1', \tag{22}$$

where $W_0' = (w_0(x))'$ and $W_1' = (w_1(x))'$ are elements of the space $C[\mathbb{R}]$.

From the continuous differentiability of the superposition operator in the space of continuous functions and boundedness of the operators $U(t/2;A_1)$ and $(\delta I - \partial_x^2)^{-1}$, the continuous Fréchet differentiability of the operator $f_2(t,\cdot)$ in the space $C[\mathbb{R}]$ follows and, consequently, there exists an interval $[0,t_0)$, within which the abstract Cauchy problem (20), (22) has [7, § 3] the only classical solution $W = W(t)$ (provided that the initial data W_0', W_1' belong to the domain of definition of the operator B) that satisfies the integral equation

$$W(t) = C(t;B)W_0' + S(t;B)W_1' + \int_0^t S(t-\tau;B) f_2(\tau, U(-\tau/2;A_1)W)\,d\tau. \tag{23}$$

From equation (23), using estimates (15), (16), (21), and (2), we derive the integral inequality

$$\|W(t)\|_C \le \sigma_1(t)\|W_0'\|_C + \sigma_2(t)\|W_1'\|_C +$$
$$+ 2\int_0^t \sigma_2(t-\tau) e^{(\alpha_2+\alpha_1/\sqrt{\delta}+|\alpha_2-\alpha_1/\sqrt{\delta}|)\tau/2} \chi\big(e^{-(\alpha_2-\alpha_1/\sqrt{\delta}-|\alpha_2-\alpha_1/\sqrt{\delta}|)\tau/2}\big) f(\|W(\tau)\|_C)\,d\tau, \tag{24}$$

where

$$\|W_0'\|_C = \|(w_0(x))'\|_C = \|(v_0(x))'\|_C = \sup_{x\in\mathbb{R}} |\delta\varphi'(x) - \varphi'''(x)|,$$
$$\|W_1'\|_C = \|(w_1(x))'\|_C = \|(v_1(x))'\|_C = \|(A_1 v_0(x)/2 + v_1(x))'\|_C \le$$
$$\le \frac{\alpha_2\sqrt{\delta} + \alpha_1 + |\alpha_2\sqrt{\delta} - \alpha_1|}{2\sqrt{\delta}} \sup_{x\in\mathbb{R}} |\delta\varphi'(x) - \varphi'''(x)| + \sup_{x\in\mathbb{R}} |\delta\psi'(x) - \psi'''(x)|.$$

Denoting

$$\sigma_3(t) = \sigma_1(t)\|W_0'\|_C + \sigma_2(t)\|W_1'\|_C,$$
$$\sigma_4(\tau) = e^{(\alpha_2+\alpha_1/\sqrt{\delta}+|\alpha_2-\alpha_1/\sqrt{\delta}|)\tau/2} \chi(e^{-(\alpha_2-\alpha_1/\sqrt{\delta}-|\alpha_2-\alpha_1/\sqrt{\delta}|)\tau/2})$$

and using the inequality

$$\sigma_5(t) = t(1+\operatorname{ch}(c_1 t)/(2c_1^2) \geqslant (t-\tau)(1+\operatorname{ch}(c_1(t-\tau))/(2c_1^2) = \sigma_2(t-\tau), \quad t \geqslant \tau \geqslant 0,$$

let us write the integral inequality (24) in the form

$$\|W(t)\|_C \le \sigma_3(t) + 2\sigma_5(t)\int_0^t \sigma_4(\tau) f(\|W(\tau)\|_C)\,d\tau. \tag{25}$$

From inequality (25), we derive [3] an estimate of the norm in the space $C[\mathbb{R}]$ of the solution of equation (20) on the segment $[0,t_1]$:

$$\|W(t)\|_C \le \sigma_3(t)\Phi^{-1}(\Psi(t)) = \sigma_6(t),$$

where

$$\Psi(t) = \Phi(1) + 2\sigma_5(t)\int_0^t \sigma_4(\tau)\frac{\chi(\sigma_3(\tau))}{\sigma_3(\tau)}\,d\tau,$$

$\Phi(\xi) = \int_{\xi_0}^{\xi} |f(s)|^{-1} ds$ for $\xi_0, \xi > 0$; Φ^{-1} is the inverse function to Φ, the segment $[0,t_1] \subset [0,t_0)$ is defined by those values t for which the values of the function $\Psi(t)$ belong to the region of existence $\operatorname{Dom}(\Phi^{-1})$ of the inverse function Φ^{-1}.

Thus, there is

Theorem 2. *Let the function f satisfy the conditions* (2), *and the initial functions $\varphi(x)$, $\psi(x)$ of the Cauchy problem* (4), (5) *belong to the space $C[\mathbb{R}]$ together with their derivatives up to the fifth order inclusive, then on the segment $[0,t_1]$ there exists a single classical solution $u=u(t,x)$ of this problem in the space $C[\mathbb{R}]$, for which the estimation is valid*

$$\sup_{x\in\mathbb{R}}|v(t,x)| = \sup_{x\in\mathbb{R}}|u_x(t,x)| \le e^{-(\alpha_2-\alpha_1/\sqrt{\delta}-|\alpha_2-\alpha_1/\sqrt{\delta}|)t/2}\sigma_6(t) = \sigma_7(t), \quad t\in[0,t_1].$$

4. RELATIONSHIP BETWEEN SOLUTIONS OF EQUATIONS (1) AND (4)

Further, we will assume that the solution of equation (1) belongs to the intersection of the space $C[\mathbb{R}]$ with the space $L_2(\mathbb{R})$ of functions with integrable square.

Recall that the scalar product and norm in $L_2(\mathbb{R})$ are defined by the formulas $(\varphi,\psi)=\int_{-\infty}^{+\infty}\varphi(x)\psi(x)dx$ and $\|\varphi\|_2=\left(\int_{-\infty}^{+\infty}|\varphi(x)|^2dx\right)^{1/2}$, respectively, and that for functions $g(x)$ belonging to the intersection of the space of continuous bounded functions $C(\mathbb{R})$ with the Sobolev space $W_2^1(\mathbb{R})$, the following estimate is valid

$$\|g\|_C \le \|g\|_{W_2^1} = \left(\int_{-\infty}^{+\infty}[(g(x))^2+(g'(x))^2]dx\right)^{1/2}, \tag{26}$$

and if $g(x)\in C^{(2)}(\mathbb{R})$, then [8] the limits of the functions $g(x), g'(x)$ at $x\to\pm\infty$ are zero.

Lemma. *From the existence of a local classical solution $v=v(t,x)$, $t\in[0,t_1]$, of equation* (4) *follows the existence of a corresponding solution of*

$$u=u(t,x)=\lim_{x_0\to-\infty}\int_{x_0}^{x}v(t,s)ds=\int_{-\infty}^{x}v(t,s)ds \tag{27}$$

of equation (1) *on the same time interval $[0,t_1]$ if the conditions are fulfilled*

$$u(t,x)\in C^{(4)}[\mathbb{R}]\cap W_2^4(\mathbb{R}), \quad u_t(t,x), u_{tt}(t,x)\in C^{(2)}[\mathbb{R}]\cap W_2^2(\mathbb{R}), \quad t\in[0,t_1]. \tag{28}$$

Proof. First of all, we note that from conditions (28), the limit equalities follow

$$\begin{aligned}&\lim_{x\to\pm\infty}\partial_x^k u(t,x)=0, \quad k=\overline{0,4};\\ &\lim_{x\to\pm\infty}\partial_t^n\partial_x^m u(t,x)=0, \quad n=1,2, \quad m=\overline{0,2}; \quad t\in[0,t_1].\end{aligned} \tag{29}$$

Let $v=v(t,x)$ be the classical solution of equation (4) on the time segment $[0,t_1]$. Then, using relations (29), we obtain the equations

$$\int_{-\infty}^{x}\partial_t^i\partial_s^j v(t,s)ds=\int_{-\infty}^{x}(\partial_t^i\partial_s^j u(t,s))_s ds=\partial_t^i\partial_x^j u(t,x)-\lim_{s\to-\infty}\partial_t^i\partial_s^j u(t,s)=\partial_t^i\partial_x^j u(t,x).$$

Further, by virtue of continuity of the function f', we have

$$\int_{-\infty}^{x}\partial_s^2 f(v(t,s))ds=(f(u_x(t,x)))_x-f'\left(\lim_{x_0\to-\infty}u_x(t,x_0)\right)\lim_{x_0\to-\infty}u_{xx}(t,x_0)=u_{xx}(t,x)f'(u_x(t,x)).$$

Now, using the obtained representations and substituting function (27) into equation (1), we obtain the identity equality on the segment $[0,t_1]$, whence it follows that function (27) is a solution of equation (1). The lemma is proved.

Remark 2. From the conditions (28) for the solution of the Cauchy problem (1), (3) $u=u(t,x)$, the conditions that the initial functions must satisfy are required to follow:

$$\varphi(x)\in C^{(4)}[\mathbb{R}]\cap W_2^4(\mathbb{R}), \quad \psi(x)\in C^{(2)}[\mathbb{R}]\cap W_2^2(\mathbb{R}). \tag{30}$$

5. EXISTENCE OF A GLOBAL SOLUTION OF THE CAUCHY PROBLEM FOR EQ. (1)

Consider the so-called energy integral for equation (1):

$$y(t) = \delta(u,u) + (u_x,u_x) = \int_{-\infty}^{+\infty} (\delta u^2 + u_x^2)dx, \quad t \in [0,t_1]. \tag{31}$$

Applying the Cauchy-Bunyakovsky inequality $|(\varphi,\psi)| \leq \|\varphi\|_2\|\psi\|_2$ to the derivative of the energy integral $y'(t) = 2(\delta(u_t,u) + (u_{tx},u_x))$, we derive an auxiliary estimate on the segment $t \in [0,t_1]$:

$$y'(t) \leq y(t) + z(t), \tag{32}$$

where

$$z(t) = \delta(u_t,u_t) + (u_{tx},u_{tx}) = \int_{-\infty}^{+\infty} (\delta u_t^2 + u_{tx}^2)dx, \quad t \in [0,t_1], \tag{33}$$

is the second integral of energy for equation (1).

Theorem 3. *Let the conditions of lemma and theorem* 2 *be satisfied and let the parameters* $\alpha_i, \beta_i, i = 1,2, \gamma, \delta$ *of equation* (1), *the nonlinearity* f *and the initial functions* $\varphi(x), \psi(x)$ *satisfy conditions* (30) *and*

$$E_0 = \delta\|\psi\|_2^2 + \|\psi'\|_2^2 + \beta_2\|\varphi''\|_2^2 + \gamma\|\varphi\|_2^2 + 2\int_{-\infty}^{+\infty} F(\varphi'(x))dx - \beta_1\|\varphi'\|_2^2 \geq 0;$$

$$F(\eta) = \int_0^\eta f(s)ds \geq 0, \quad \eta \in \mathbb{R}; \quad F(\varphi'(x)) \in L(\mathbb{R}).$$

Then, there exists a single global solution of the Cauchy problem (1), (3) *and for it the estimation is valid*

$$\sup_{x\in\mathbb{R}} |u(t,x)| \leq \begin{cases} \sqrt{c_2/\delta}e^{(1+\beta_1)t/2}, & 0 < \delta < 1, \\ \sqrt{c_2}e^{(1+\beta_1)t/2}, & \delta \geq 1, \end{cases} \quad t \geq 0,$$

where

$$c_2 = \big(E_0 + (1+\beta_1)(\delta\|\varphi\|_2^2 + \|\varphi'\|_2^2)\big)/(1+\beta_1).$$

Proof. Multiply both parts of equation (1) by the partial time derivative $u_t = u_t(t,x)$ and integrate from $-\infty$ to $+\infty$. Then, integrating by parts and taking into account, by virtue of (29), the equality to zero outside the integral summands, we obtain

$$\frac{\delta}{2}\frac{d}{dt}\|u_t\|_2^2 + (u_{ttx},u_{tx}) + \alpha_2(u_{tx},u_{tx}) - \frac{\alpha_1}{2}\int_{-\infty}^{+\infty} (u_t^2)_x dx +$$
$$+ \beta_2(u_{xx},u_{txx}) - \beta_1(u_x,u_{tx}) + \frac{\gamma}{2}\frac{d}{dt}\|u\|_2^2 + (f(u_x),u_{tx}) = 0. \tag{34}$$

Let us introduce the potential $F(\eta) = \int_0^\eta f(s)ds$, generated by the nonlinearity f of equation (1), and, taking into account that $\int_{-\infty}^{+\infty}(u_t^2)_x dx = u_t^2|_{-\infty}^{+\infty} = 0$, we rewrite the equality (34) as

$$\frac{1}{2}\frac{d}{dt}E(t) = 0, \tag{35}$$

where

$$E(t) = \delta\|u_t\|_2^2 + \|u_{tx}\|_2^2 + \beta_2\|u_{xx}\|_2^2 - \beta_1\|u_x\|_2^2 + \gamma\|u\|_2^2 +$$
$$+ 2\int_{-\infty}^{+\infty} F(u_x)dx + 2\alpha_2\int_0^t \|u_{sx}\|_2^2 d\tau$$

is the energy functional of equation (1).

From relation (35) it follows that the energy functional $E(t)$ does not depend on time, then, integrating both parts of (35), we obtain the conservation law

$$E(t) = E(0) \equiv E_0, \tag{36}$$

where

$$E_0 = \delta\|\psi\|_2^2 + \|\psi'\|_2^2 + \beta_2\|\varphi''\|_2^2 - \beta_1\|\varphi'\|_2^2 + \gamma\|\varphi\|_2^2 + 2\int_{-\infty}^{+\infty} F(\varphi'(x))dx$$

is the initial energy.

Let us require that the initial energy is non-negative: $E_0 \geq 0$, i.e., the inequality

$$\delta\|\psi\|_2^2 + \|\psi'\|_2^2 + \beta_2\|\varphi''\|_2^2 + \gamma\|\varphi\|_2^2 + 2\int_{-\infty}^{+\infty} F(\varphi'(x))dx \geq \beta_1\|\varphi'\|_2^2,$$

where the function $F(\varphi'(x))$ belongs to the space $L(\mathbb{R})$ of functions absolutely integrable on $\mathbb{R}$.

From the conservation law (36) we deduce

$$\begin{aligned}&\delta\|u_t\|_2^2 + \|u_{tx}\|_2^2 + \beta_2\|u_{xx}\|_2^2 + \gamma\|u\|_2^2 + \\ &+ 2\int_{-\infty}^{+\infty} F(u_x)dx + 2\alpha_2\int_0^t \|u_{sx}\|_2^2 ds = E_0 + \beta_1\|u_x\|_2^2.\end{aligned} \tag{37}$$

Suppose that

$$F(\eta) \geq 0, \quad \eta \in \mathbb{R}, \tag{38}$$

then from equality (37), reducing the left part, we obtain

$$z(t) \leq E_0 + \beta_1(\delta\|u\|_2^2 + \|u_x\|_2^2) = E_0 + \beta_1 y(t), \quad t \in [0, t_1]. \tag{39}$$

From inequalities (32) and (39), the integral inequality follows

$$y(t) \leq E_0 t + y(0) + (1+\beta_1)\int_0^t y(s)ds, \quad t \in [0, t_1]. \tag{40}$$

Applying to (40) Gronwall's lemma [9, § 1, formula (1.10)], we obtain an estimate of the first energy integral

$$y(t) \leq \left(\frac{E_0}{1+\beta_1} + y(0)\right) e^{(1+\beta_1)t} = \sigma_8(t), \tag{41}$$

true on the entire positive semi-axis of $t \in \overline{\mathbb{R}}_+$, and hence the classical solution of $u = u(t,x)$ at $t \in \overline{\mathbb{R}}_+$ belongs to the Sobolev space $W_2^1(\mathbb{R})$:

$$\|u\|_{W_2^1}^2 = \|u\|_2^2 + \|u_x\|_2^2 \leq \begin{cases} \left(1 + \frac{1-\delta}{\delta}\right) y(t) \leq \frac{1}{\delta}\sigma_8(t), & 0 < \delta < 1, \\ \delta\|u\|_2^2 + \|u_x\|_2^2 = y(t) \leq \sigma_8(t), & \delta \geq 1. \end{cases}$$

Now, using inequalities (26) and (41), we obtain an estimate of the solution $u = u(t,x)$, $t \in \overline{\mathbb{R}}_+$ of the Cauchy problem (1), (3) in the space $C[\mathbb{R}]$:

$$\|u\|_C = \sup_{x\in\mathbb{R}} |u(t,x)| \leq \|u\|_{W_2^1} \leq \begin{cases} \sqrt{\frac{\sigma_8(t)}{\delta}}, & 0 < \delta < 1, \\ \sqrt{\sigma_8(t)}, & \delta \geq 1, \end{cases}$$

ensuring the existence of a global solution. The theorem is proved.

6. DECOMPOSITION OF THE SOLUTION OF THE CAUCHY PROBLEM FOR EQ. (1)

Let us find sufficient conditions for the occurrence of a gap of the second kind for the energy integral (31) on the segment $[0, t_2] \subseteq [0, t_1]$, that we choose so that the inequality $y(t) > 0$, following from the initial condition $y(0) = \delta\|\varphi\|_2^2 + \|\varphi'\|_2^2 > 0$, holds.

Applying the Cauchy-Bunyakovsky inequality to the square of the derivative of the energy integral $y(t)$ on the segment $t \in [0, t_2]$, we have

$$[y'(t)]^2 \leq 4y(t)z(t).$$

Let us derive an estimate of the square of the norm of the partial derivative u_{tt}, using the representation of equation (1) in an equivalent form

$$u_{tt} = -A_1u_t - A_2u + (\delta I - \partial_x^2)^{-1}u_{xx}f'(u_x),$$

obtained by acting on both parts of equation (1) by a linear bounded operator $(\delta I - \partial_x^2)^{-1}$. For this purpose, we obtain auxiliary estimates

$$\|u_{xx}f'(u_x)\|_2^2 \leq \sup_{x\in}(f'(u_x))^2 \int\limits_{-\infty}^{+\infty} u_{xx}^2\,dx \leq \Big(f'\Big(\sup_{x\in}|u_x|\Big)\Big)^2\|u_{xx}\|_2^2 \leq \sigma_9(t)\|u_{xx}\|_2^2,$$

where $\sigma_9(t) = (f'(\sigma_7(t)))^2$ — is a continuous function on the segment $[0, t_1]$;

$$\|A_1u_t\|_2^2 \leq \big\|\alpha_2u_t - (\alpha_2\sqrt{\delta} - \alpha_1)\sqrt{\delta}(\delta I - \partial_x^2)^{-1}u_t - \alpha_1(\sqrt{\delta}I - \partial_x)^{-1}u_t\big\|_2^2 \leq$$

$$\leq \left(\alpha_2\|u_t\|_2 + \left|\alpha_2 - \frac{\alpha_1}{\sqrt{\delta}}\right|\|u_t\|_2 + \frac{\alpha_1}{\sqrt{\delta}}\|u_t\|_2\right)^2 \leq c_3\|u_t\|_2^2 \leq c_3z(t),$$

where $c_3 = (\alpha_2 + \alpha_1/\sqrt{\delta} + |\alpha_2 - c\alpha_1/\sqrt{\delta}|)^2$;

$$\|A_2u\|_2^2 \leq \big\|-\beta_2\partial_x^2u - (\beta_2\delta + \beta_1)u + (\beta_2\delta^2 + \beta_1\delta + \gamma)(\delta I - \partial_x^2)^{-1}u\big\|_2^2 \leq$$

$$\leq \big(\beta_2\|u_{xx}\|_2 + (\beta_2\delta + \beta_1)\|u\|_2 + \big(\beta_2\delta + \beta_1 + \gamma/\delta\big)\|u\|_2\big)^2 \leq$$

$$\leq 2\big(\beta_2^2\|u_{xx}\|_2^2 + \big(2(\beta_2\delta + \beta_1) + \gamma/\delta\big)^2\|u\|_2^2\big) \leq 2\beta_2^2\|u_{xx}\|_2^2 + c_4y(t),$$

where $c_4 = 2(2(\beta_2\delta + \beta_1) + \gamma/\delta)^2$.

Taking them into account, we have

$$\|u_{tt}\|_2^2 \leq \big(\|A_1u_t\|_2 + \|A_2u\|_2 + \|(\delta I - \partial_x^2)^{-1}u_{xx}f'(u_x)\|_2\big)^2 \leq$$

$$\leq 3\left(\|A_1u_t\|_2^2 + \|A_2u\|_2^2 + \frac{1}{\delta^2}\|u_{xx}f'(u_x)\|_2^2\right) \leq 3\left(c_3z(t) + 2\beta_2^2\|u_{xx}\|_2^2 + c_4y(t) + \frac{\sigma_9(t)}{\delta^2}\|u_{xx}\|_2^2\right),$$

whence follows the inequality

$$\|u_{tt}\|_2^2 \leq 3c_3z(t) + 3c_4y(t) + c_5\|u_{xx}\|_2^2, \quad t \in [0, t_2], \tag{42}$$

where $c_5 = 6\beta_2^2 + 3c_6\delta^2$, $c_6 = \max_{t\in[0,t_1]}\sigma_9(t)$.

Let's return to the conservation law (37) and obtain the relation from it

$$z(t) + \beta_2\|u_{xx}\|_2^2 + \gamma\|u\|_2^2 + 2\alpha_2\int_0^t \|u_{sx}\|_2^2ds \leq E_0 + \beta_1\|u_x\|_2^2 + 2\left|\int_{-\infty}^{+\infty} F(u_x)dx\right|. \tag{43}$$

Earlier, when proving the existence of a global solution, we assumed the fulfillment of condition (38) — non-negativity of the potential $F(\eta)$ on the whole numerical axis $\eta \in \mathbb{R}$. Now, when considering the destruction of the solution, we require to fulfill the inequality for the nonlinearity f

$$\left|\int_{-\infty}^{+\infty} dx\int_0^{w(x)} f(s)ds\right| \leq \left|\int_{-\infty}^{+\infty} w(x)f(w(x))dx\right|, \tag{44}$$

where $w(x)$ is an arbitrary function from $C[\mathbb{R}]$, for which the functions $F(w(x))$ and $w(x)f(w(x))$ belong to the space $L_1(\mathbb{R})$.

Using inequality (44), we evaluate the integral in the right-hand side of (43). Integrating by parts, applying the limit equality (29) and the Cauchy-Bunyakovsky inequality, we have

$$2\left|\int_{-\infty}^{+\infty} F(u_x)\,dx\right| \le 2\left|\int_{-\infty}^{+\infty} f(u_x)\,du(x)\right| = \left|u(x)f(u_x)|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} uf'(u_x)u_{xx}\,dx\right| \le$$

$$\le 2|(uf'(u_x), u_{xx})| \le 2\|uf'(u_x)\|_2\|u_{xx}\|_2 \le \|uf'(u_x)\|_2^2 + \|u_{xx}\|_2^2 \le$$

$$\le \sup_{x\in\mathbb{R}}(f'(u_x))^2 \int_{-\infty}^{+\infty} u_2^2\,dx + \|u_{xx}\|_2^2 \le (f'(\sigma_7(t)))^2\|u\|_2^2 + \|u_{xx}\|_2^2 = \sigma_9(t)\|u\|_2^2 + \|u_{xx}\|_2^2,$$

whence follows the inequality

$$2\left|\int_{-\infty}^{+\infty} F(u_x)dx\right| \le c_6\|u\|_2^2 + \|u_{xx}\|_2^2, \quad t \in [0, t_2]. \tag{45}$$

Applying the estimation (45) to the relation (43) under the condition

$$\beta_2 > 1, \tag{46}$$

we obtain the inequality

$$\|u_{xx}\|_2^2 \le \frac{E_0}{\beta_2 - 1} + \frac{\beta_1 + c_6}{\beta_2 - 1}y(t) - \frac{1}{\beta_2 - 1}z(t), \quad t \in [0, t_2],$$

using which we increase the right part of the estimate (42):

$$\|u_{tt}\|_2^2 \le \frac{c_5}{\beta_2 - 1}E_0 + \left(3c_4 + c_5\frac{(\beta_1 + c_6)}{\beta_2 - 1}\right)y(t) + \left(3c_3 - \frac{c_5}{\beta_2 - 1}\right)z(t), \quad t \in [0, t_2].$$

Let us calculate the second order derivative of the functional (31) and express its value through the second integral of energy (33):

$$y''(t) + 2(u_{tt}, u_{xx}) - 2\delta(u_{tt}, u) = 2z(t).$$

Using the estimates

$$2(u_{tt}, u_{xx}) \le 2|(u_{tt}, u_{xx})| \le \|u_{tt}\|_2^2 + \|u_{xx}\|_2^2 \le 3c_3z(t) + 3c_4y(t) + (c_5 + 1)\|u_{xx}\|_2^2 \le$$

$$\le \frac{c_5 + 1}{\beta_2 - 1}E_0 + \left(3c_4 + (c_5 + 1)\frac{\beta_1 + c_6}{\beta_2 - 1}\right)y(t) + \left(3c_3 - \frac{c_5 + 1}{\beta_2 - 1}\right)z(t),$$

$$-2\delta(u_{tt}, u) \le 2\delta|(u_{tt}, u)| \le \delta\|u_{tt}\|_2^2 + \delta\|u\|_2^2 \le 3\delta c_3z(t) + \delta(3c_4 + 1)y(t) + \delta c_5\|u_{xx}\|_2^2 \le$$

$$\le \frac{\delta c_5}{\beta_2 - 1}E_0 + \delta\left(3c_4 + 1 + c_5\frac{\beta_1 + c_6}{\beta_2 - 1}\right)y(t) + \delta\left(3c_3 - \frac{c_5}{\beta_2 - 1}\right)z(t),$$

increase the left side of it:

$$y''(t) + c_7 + c_8y(t) \ge c_9z(t), \quad t \in [0, t_2], \tag{47}$$

where

$$c_7 = \frac{(\delta + 1)c_5 + 1}{\beta_2 - 1}E_0, \quad c_8 = 3(\delta + 1)c_4 + \delta + ((\delta + 1)c_5 + 1)\frac{(\beta_1 + c_6)}{\beta_2 - 1}, \quad c_9 = 2 + \frac{(\delta + 1)c_5 + 1}{\beta_2 - 1} - 3(\delta + 1)c_3.$$

Let us now reduce the right-hand side of inequality (47):

$$y(t)y''(t) - \frac{c_9}{4}(y'(t))^2 + c_7y(t) + c_8y^2(t) \ge 0, \quad t \in [0, t_2]. \tag{48}$$

We require that the coefficient at the square of the derivative in inequality (48) be greater than one, i.e., we require the inequality $c_9/4 > 1$ or (in the detailed notation)

$$6(\delta+1)\beta_2^2 - (2+3(\delta+1)c_3)\beta_2 + 3(\delta+1)(c_6/\delta^2 + c_3) + 3 > 0. \tag{49}$$

Two cases arise here: if the discriminant of the quadratic trinomial

$$D_1 = D_1(\delta, c_3, c_6) = (2+3(\delta+1)c_3)^2 - 72(\delta+1)((\delta+1)(c_6/\delta^2 + c_3) + 1) < 0, \tag{50}$$

then inequality (49) is valid for all values of $\beta_2 > 1$; If $D_1 \geq 0$, then inequality (49) holds at

$$1 < \beta_2 < \frac{2+3(\delta+1)c_3 - \sqrt{D_1(\delta, c_3, c_6)}}{12(\delta+1)} \quad \text{or} \quad \beta_2 > \frac{2+3(\delta+1)c_3 + \sqrt{D_1(\delta, c_3, c_6)}}{12(\delta+1)}. \tag{51}$$

From condition (50), follows the inequality

$$9c_3^2 - 12\left(6 - \frac{1}{(\delta+1)^2}\right)c_3 - 72\frac{c_6}{\delta^2} - \frac{72(\delta+1)-4}{(\delta+1)^2} < 0, \tag{52}$$

and the discriminant of the quadratic trinomial

$$D_2 = D_2(\delta, c_6) = 36\left(6 - (\delta+1)^{-2}\right)^2 + 648(\delta^{-2}c_6 + (\delta + 17/18)(\delta+1)^{-2}) \geq 0,$$

therefore inequality (52), and hence (50), is satisfied at

$$0 < c_3 = \left(\alpha_2 + \frac{\alpha_1}{\sqrt{\delta}} + \left|\alpha_2 - \frac{\alpha_1}{\sqrt{\delta}}\right|\right)^2 < \frac{6(6-(\delta+1)^{-2}) + \sqrt{D_2(\delta, c_6)}}{9}, \tag{53}$$

i.e., if condition (53) is satisfied, the inequality $c_9/4 > 1$ is valid for any value of the parameter $\beta_2 > 1$.

In the case of $D_1 \geq 0$, inequality (49) is satisfied for parameter values satisfying conditions (51), in which the values δ, c_3 and c_6 are related by the relation

$$(\alpha_2 + \alpha_1/\sqrt{\delta} + |\alpha_2 - \alpha_1/\sqrt{\delta}|)^2 \geq \frac{6(6-(\delta+1)^{-2}) + \sqrt{D_2(\delta, c_6)}}{9}.$$

Comparing inequality (48) with one of the basic ordinary differential inequalities for the energy integral [10, Appendix A, § 5], we conclude that if the initial conditions are fulfilled

$$(\delta(\varphi, \psi) + (\varphi', \psi'))^2 > \left(\frac{c_8}{c_9 - 4}(\delta\|\varphi\|_2^2 + \|\varphi'\|_2^2) + \frac{c_7}{c_9 - 2}\right)(\delta\|\varphi\|_2^2 + \|\varphi'\|_2^2), \tag{54}$$

then the time t_2 of existence of the solution of the Cauchy problem (1), (3) cannot be arbitrarily large, namely, there is an estimate from above

$$t_2 \leq T_\infty \leq \frac{1}{c_{10}(\delta\|\varphi\|_2^2 + \|\varphi'\|_2^2)^{(c_9-4)/4}}, \tag{55}$$

where

$$c_{10}^2 = \frac{(c_9-4)^2}{4(\delta\|\varphi\|_2^2 + \|\varphi'\|_2^2)^{c_9/2}}\left((\delta(\varphi,\psi) + (\varphi',\psi'))^2 - \left(\frac{c_8(\delta\|\varphi\|_2^2 + \|\varphi'\|_2^2)}{c_9 - 4} + \frac{c_7}{c_9 - 2}\right)(\delta\|\varphi\|_2^2 + \|\varphi'\|_2^2)\right) > 0,$$

and for the functionality of $y(t)$, it is fair to estimate from below

$$y(t) = \int_{-\infty}^{+\infty}(\delta u^2 + u_x^2)dx \geq \frac{1}{((\delta\|\varphi\|_2^2 + \|\varphi'\|_2^2)^{(c_9-4)/4} - c_{10}t)^{4/(c_9-4)}}, \tag{56}$$

and, hence, there is no time-global solution of the Cauchy problem (1), (3).

Thus, the following theorem is proven

Theorem 4. *Let the conditions of lemma and theorem* 2 *be satisfied and let the parameters* $\alpha_i, \beta_i, i = 1, 2, \gamma, \delta$ *of equation* (1), *the nonlinearity* f *and the initial functions* $\varphi(x), \psi(x)$ *satisfy conditions* (30), (44), (46), (49), (54), *respectively, then the time* t_2 *of existence of the solution* $u(t,x)$ *of the Cauchy problem* (1), (3) *cannot be arbitrarily large, namely it is bounded from above and the estimation* (55) *takes place, and for the energy integral* $y(t)$ *the estimation from below* (56) *is valid.*

CONFLICT OF INTERESTS

The author of this paper declares that he has no conflict of interests.

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INTEGRAL EQUATIONS

ON THE SOLVABILITY OF A SYSTEM OF MULTIDIMENSIONAL INTEGRAL EQUATIONS WITH CONCAVE NONLINEARITIES

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Abstract. The work is devoted to the study of questions of existence and uniqueness of a continuous bounded and positive solution to one system of nonlinear multidimensional integral equations. The scalar analogue of the indicated system of integral equations, with different representations of the corresponding matrix kernel and nonlinearities, has important applied significance in a number of areas of physics and biology. This article proposes a special iterative approach for constructing a positive continuous and bounded solution to the system under study. It is possible to prove that the corresponding iterations uniformly converge to a continuous solution of the specified system. Using some a priori estimates for strictly concave functions, we also prove the uniqueness of the solution in a fairly wide subclass of continuous bounded and coordinately nonnegative vector functions. In the case when the integral of the matrix kernel has a unit spectral radius, it is proved that in a certain subclass of continuous bounded and coordinate-wise non-negative vector functions, this system has only a trivial solution, that is an eigenvector of the kernel integral matrix.

Keywords: *nonlinear integral equation, system of integral equations, positive solution, continuous solution, limited solution, trivial solution, iterative process*

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1. INTRODUCTION. PROBLEM STATEMENT

Consider a system of nonlinear multivariate integral equations

$$f_i(x_1,\ldots,x_n)=\sum_{j=1}^{N}\int_{\mathbb{R}^n}K_{ij}(x_1,\ldots,x_n,t_1,\ldots,t_n)G_j(f_j(t_1,\ldots,t_n))dt_1\ldots dt_n,\quad i=\overline{1,N},\tag{1}$$

with respect to the vector-function $f(x_1,\ldots,x_n)=(f_1(x_1,\ldots,x_n),\ldots,f_N(x_1,\ldots,x_n))^{\mathrm{T}}$ with non-negative continuous and bounded on the set $\mathbb{R}^n$ coordinates $f\,(x_1,\ldots,x_n)$, $i=\overline{1,N}$, where $(x_1,\ldots,x_n)\in\mathbb{R}^n$, $\mathbb{R}=(-\infty,+\infty)$, T is the transpose sign. In system (1) the matrix kernel

$$K(x,t):=(K_{ij}(x_1,\ldots,x_n,t_1,\ldots,t_n))_{i,j=\overline{1,N}}$$

satisfies the following conditions:

1) $K_{ij}(x_1,\ldots,x_n,t_1,\ldots,t_n)>0$, $(x_1,\ldots,x_n,t_1,\ldots,t_n)\in\mathbb{R}^{2n}$, $K_{ij}\in C(\mathbb{R}^{2n})$, $i,j=\overline{1,N}$;

2) there exist $a_{ij} := \sup_{(x_1,\dots,x_n)\in\mathbb{R}^n} \int_{\mathbb{R}^n} K_{ij}(x_1,\dots,x_n,t_1,\dots,t_n)\,dt_1\dots dt_n < +\infty$, $i,j=\overline{1,N}$, with $r(A)=1$, $A=(a_{ij})_{i,j=\overline{1,N}}$, where $r(A)$ is the spectral radius of the matrix A, i.e., the modulus of the largest modulo eigenvalue.

According to Perron's theorem (see [1, p. 260]), there exists a vector $\eta=(\eta_1,\dots,\eta_N)^{\mathrm{T}}$ with positive coordinates η_i such that

$$\sum_{j=1}^{N} a_{ij}\eta_j = \eta_i, \quad i=\overline{1,N}. \tag{2}$$

Let us fix the vector η and impose the following conditions on the nonlinearities of $\{G_j(u)\}_{j=\overline{1,N}}$ (Fig. 1):

a) $G_j \in C(\mathbb{R}^+), \mathbb{R}^+=[0,+\infty), G_j(u)$ are monotonically increasing on the set $\mathbb{R}^+, j=\overline{1,N}$;

b) $G_j(0)=0, G_j(\eta_j)=\eta_j, j=\overline{1,N}$;

c) $G_j(u), j=\overline{1,N}$, are strictly concave (convex upwards) on $\mathbb{R}^+$ and there exists a continuous mapping $\varphi : [0,1]\to[0,1]$ with properties

$$\varphi(0)=0, \varphi(1)=1, \varphi \text{monotonically increases on the interval } [0,1], \tag{3}$$

$$\varphi \text{ strictly concave on the segment } [0,1], \tag{4}$$

such that the following inequalities hold:

$$G_j(\sigma u) \ge \varphi(\sigma)G_j(u), \quad u\in[0,\eta_j], \sigma\in[0,1], j=\overline{1,N};$$

d) there exists a number $r>0$ such that the functional equations $G_i(u)=u/\varepsilon_i(r), i=\overline{1,N}$, have positive solutions d_i, where

$$\varepsilon_i(r) := \min_{j=\overline{1,N}} \left\{ \inf_{(x_1,\dots,x_n)\in\mathbb{R}^n\setminus B_r} \int_{\mathbb{R}^n\setminus B_r} K_{ij}(x_1,\dots,x_n,t_1,\dots,t_n)dt_1\dots dt_n \right\} \in (0,1), \quad i=\overline{1,N},$$

$$B_r := \{x:=(x_1,\dots,x_n) : |x| = \sqrt{x_1^2+\dots+x_n^2} \le r\}.$$

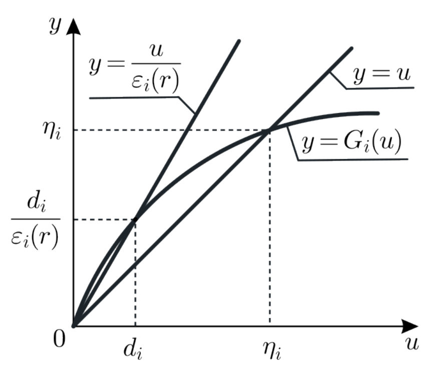

Fig. 1. Graph of the function $y=G_i(u)$

The main purpose of this paper is to investigate the existence and uniqueness of a continuous bounded and positive solution of system (1), as well as the uniform convergence to the solution of the corresponding iterative process with the rate of decreasing geometric progression.

The scalar analog of the system of nonlinear integral equations (1), besides purely theoretical interest, has a number of important applications to the study of various applied problems from physics and biology. In particular, under specific representations of the matrix kernel K and nonlinearities $\{G_j(u)\}_{j=\overline{1,N}}$, the scalar system (1) is encountered in problems from the dynamical theory of p-adic open, closed, and open-closed strings (see [2–5]) and in the mathematical theory of spatial and temporal pandemic propagation in the framework of the modified Atkinson–Roiter and Dickman–Kaper models (see [6, p. 318] and [7, p. 121], respectively).

Mathematical investigations of the system of the form (1) were mainly carried out in the one-dimensional case at $n = 1$. Thus, for example, in the case when $n = 1$ and the kernel K depends on the difference of its arguments, the system (1) is studied in [8–10]. The corresponding scalar analog of system (1) ($N = 1$) in the multidimensional case is studied in [5, 11–13], when the kernel K either depends on the difference of its arguments or is majorized by such a kernel. It should also be noted that the corresponding scalar one-dimensional equations under different restrictions on the kernel and on the nonlinearity have been studied (by different methods) in [2, 3, 14–17].

In this paper, under conditions 1), 2) and a)–d), we will first prove the constructive theorem of existence of a positive continuous and bounded solution of system (1). In the course of the proof of this theorem, we obtain a uniform estimate of the difference between the constructed solution and the corresponding successive approximations, with the right-hand side of the obtained inequality tending to zero as an infinitely decreasing geometric progression when the number of m-th approximation tends to infinity. Further, using some estimates for strictly concave and monotone functions, we prove the uniqueness of the solution of the system (1) in a sufficiently wide subclass of continuous bounded and coordinately nonnegative vector-functions. In the case when

$$C_{ij}(x_1,\dots,x_n) := \int_{\mathbb{R}^n} K_{ij}(x_1,\dots,x_n,t_1,\dots,t_n)dt_1\dots dt_n = a_{ij}$$

for all $(x_1,\dots,x_n) \in \mathbb{R}^n$ and $i,j = \overline{1,N}$, we show that in the above mentioned subclass of vector-functions, the only solution of the system (1) is only the vector $\eta = (\eta_1,\dots,\eta_N)_T$. In this paper, we give specific examples of the matrix kernel K and nonlinearities $\{G_j(u)\}_{j=\overline{1,N}}$, satisfying all conditions of the proved statements. Some of these examples have applications in the above-mentioned areas of physics and biology.

2. KEY NOTATIONS AND SUPPORTING RESULTS

The following lemma plays an important role in our further reasoning.

Lemma 1. *Let conditions* a), b), 1), 2) *be satisfied, and the graphs of the functions* $\{G_j(u)\}_{j=\overline{1,N}}$ *are strictly concave at* $\mathbb{R}^+$. *Then the inequality is true for any ordinally non-negative and bounded on* $\mathbb{R}^n$ *solution* $f^*(x_1,\dots,x_n) = (f_1^*(x_1,\dots,x_n),\dots,f_N^*(x_1,\dots,x_n))^{\mathrm{T}}$ *of the system* (1):

$$f_i^*(x_1,\dots,x_n) \le \eta_i, \quad (x_1,\dots,x_n) \in \mathbb{R}^n, \quad i = \overline{1,N},$$

where $\eta = (\eta_1,\dots,\eta_N)^{\mathrm{T}}$ *is the fixed vector of the matrix* A (see (2)).

Proof. Let us denote $\gamma_i := \sup_{(x_1,\dots,x_n)\in\mathbb{R}^n} f_i^*(x_1,\dots,x_n), i = \overline{1,N}$. Then from system (1) by virtue of conditions 1), 2), a) and relation (2) we will have

$$f_i^*(x_1,\dots,x_n) \le \sum_{j=1}^{N} a_{ij}G_j(\gamma_j) \le \max_{j=\overline{1,N}}\left\{\frac{G_j(\gamma_j)}{\eta_j}\right\}\sum_{j=1}^{N} a_{ij}\eta_j = \eta_i \max_{j=\overline{1,N}}\left\{\frac{G_j(\gamma_j)}{\eta_j}\right\},$$

$$(x_1,\dots,x_n) \in \mathbb{R}^n,\ i = \overline{1,N}.$$

It follows that

$$\gamma_i \le \eta_i \max_{j=\overline{1,N}}\left\{\frac{G_j(\gamma_j)}{\eta_j}\right\}, \quad i = \overline{1,N}. \tag{5}$$

Obviously, there exists an index $j^* \in \{1,2,\dots,N\}$ such that

$$\max_{j=\overline{1,N}}\left\{\frac{G_j(\gamma_j)}{\eta_j}\right\} = \frac{G_{j^*}(\gamma_{j^*})}{\eta_{j^*}}. \tag{6}$$

Replacing in inequality (5) the index i by the index j^*, we obtain $\gamma_{j^*} \le G_{j^*}(\gamma_{j^*})$. Let us see that the last inequality implies the evaluation of $\gamma_{j^*} \le \eta_{j^*}$. Assume the opposite: $\gamma_{j^*} > \eta_{j^*}$. By virtue of conditions a), b) and the strict concavity of the graph of $G_{j^*}(u)$, it follows that the function $\frac{G_{j^*}(u)}{u}$ is monotonically decreasing at $(0,+\infty)$. So $\frac{G_{j^*}(\gamma_{j^*})}{\gamma_{j^*}} < \frac{G_{j^*}(\eta_{j^*})}{\eta_{j^*}} = 1$. The latter inequality contradicts the inequality $\gamma_{j^*} \le G_{j^*}(\gamma_{j^*})$ obtained above. Thus, $\gamma_{j^*} \le \eta_{j^*}$. By virtue of this evaluation, relation (6) and conditions a), b), we arrive from (5) at the inequality $\gamma_i \le \eta_i, i = \overline{1,N}$. The lemma is proved.

The following is also useful

Lemma 2. *Let conditions* a), b), d), 1), *and* 2) *be satisfied and* $f(x_1,\dots,x_n)$ *be an arbitrary generically non-negative and continuous on* $\mathbb{R}^n$ *solution of system* (1). *Then if there exists an index* $j_0 \in \{1,2,\dots,N\}$ *such that* $\delta_{j_0} := \inf_{(x_1,\dots,x_n)\in\mathbb{R}^n\setminus B_r} f_{j_0}(x_1,\dots,x_n) > 0$*, then* $\inf_{(x_1,\dots,x_n)\in\mathbb{R}^n} f_i(x_1,\dots,x_n) > 0, i=\overline{1,N}$*, where the number* r *is defined under condition* d).

Proof. First of all, note that it follows from conditions a), b), d), 1) and, 2) that

$$f_i(x_1,\dots,x_n) \ge \sum_{j=1}^{N} \int_{\mathbb{R}^n\setminus B_r} K_{ij}(x_1,\dots,x_n,t_1,\dots,t_n) G_j(f_j(t_1,\dots,t_n))dt_1\dots dt_n \ge$$

$$\ge \int_{\mathbb{R}^n\setminus B_r} K_{ij_0}(x_1,\dots,x_n,t_1,\dots,t_n) G_{j_0}(f_{j_0}(t_1,\dots,t_n))dt_1\dots dt_n \ge$$

$$\ge G_{j_0}(\delta_{j_0}) \int_{\mathbb{R}^n\setminus B_r} K_{ij_0}(x_1,\dots,x_n,t_1,\dots,t_n)dt_1\dots dt_n, \quad (x_1,\dots,x_n)\in\mathbb{R}^n. \tag{7}$$

Next, let us consider the functions

$$\widetilde{C}_{ij_0}(x_1,\dots,x_n) := \int_{\mathbb{R}^n\setminus B_r} K_{ij_0}(x_1,\dots,x_n,t_1,\dots,t_n)dt_1\dots dt_n, \quad (x_1,\dots,x_n)\in\mathbb{R}^n, i=\overline{1,N},$$

and the following possible cases: A) $(x_1,\dots,x_n)\in\mathbb{R}^n\setminus B_r$, B) $(x_1,\dots,x_n)\in B_r$.

In case A), considering the definition of numbers $\varepsilon_i(r)$ in condition d) and inequality (7), we obtain

$$f_i(x_1,\dots,x_n) \ge G_{j_0}(\delta_{j_0})\varepsilon_i(r), \quad (x_1,\dots,x_n)\in\mathbb{R}^n\setminus B_r, i=\overline{1,N}. \tag{8}$$

Let us now consider the case B). It immediately follows from conditions 1), 2), that $\tilde{C}_{ij_0}\in C(\mathbb{R}^n)$, $\tilde{C}_{ij_0}(x_1,\dots,x_n) > 0$, $(x_1,\dots,x_n)\in\mathbb{R}^n$, $i=\overline{1,N}$. Given the compactness of the ball B_r, according to the Weierstrass theorem, we can assert that for each $i\in\{1,2,\dots,N\}$ there exists a point $x^i = (x_1^i,\dots,x_n^i)\in B_r$ such that

$$\min_{(x_1,\dots,x_n)\in B_r} \{\widetilde{C}_{ij_0}(x_1,\dots,x_n)\} = \widetilde{C}_{ij_0}(x_1^i,\dots,x_n^i) > 0. \tag{9}$$

From (7)–(9) we conclude that

$$\inf_{(x_1,\dots,x_n)\in\mathbb{R}^n} f_i(x_1,\dots,x_n) \ge \min\{\varepsilon_i(r), \widetilde{C}_{ij_0}(x_1^i,\dots,x_n^i)\}G_{j_0}(\delta_{j_0}), \quad (x_1,\dots,x_n)\in\mathbb{R}^n, i=\overline{1,N}.$$

The lemma is proved.

Now consider the functions $C_{ij}(x_1,\dots,x_n), i,j=\overline{1,N}$ and suppose that

e) there exist a point $(x_1,\dots,x_n)\in\mathbb{R}^n$ and indices $i_1,j_1\in\{1,2,\dots,N\}$ such that

$$C_{i_1,j_1}(x_1,\dots,x_n) < a_{i_1j_1}.$$

Lemma 3. *Let the conditions of Lemma* 1 *and* e) *be satisfied. Then, any continuous bounded and coordinate non-negative solution* $f(x_1,\dots,x_n)$ *of system* (1) *satisfies the inequalities* $f_i(x_1,\dots,x_n) < \eta_i, (x_1,\dots,x_n)\in\mathbb{R}^n$, $i=\overline{1,N}$.

Proof. According to lemma 1, the solution is $f_i(x_1,\dots,x_n)\le\eta_i, i=\overline{1,N}$. Let us verify that $f_i(x_1,\dots,x_n)\not\equiv\eta_i$, $i=\overline{1,N}$. Indeed, otherwise, from (1) with condition b) we obtain

$$\sum_{j=1}^{N} C_{ij}(x_1,\dots,x_n)\eta_j \equiv \eta_i, \quad i=\overline{1,N}.$$

Taking into account (2), we come to the equality

$$\sum_{j=1}^{N} \eta_j (C_{ij}(x_1, \ldots, x_n) - a_{ij}) \equiv 0, \quad i = \overline{1, N}. \tag{10}$$

Since $C_{ij}(x_1, \ldots, x_n) \le a_{ij}, \eta_j > 0, i, j = \overline{1, N}$, we arrive at a contradiction in (10) by virtue of condition e). Hence, there exists a point $(x_1^*, \ldots, x_n^*) \in \mathbb{R}^n$ and an index $j^* \in \{1, 2, \ldots, N\}$ such that $f_{j^*}(x_1^*, \ldots, x_n^*) < \eta_{j^*}$. Hence, by continuity of the function f_{j^*} it follows. That there exists a neighborhood $O_\delta(x_1^*, \ldots, x_n^*)$ of the point $(x_1^*, \ldots, x_n^*)$ such that

$$f_{j^*}(x_1, \ldots, x_n) < \eta_{j^*}, \quad (x_1, \ldots, x_n) \in O_\delta(x_1^*, \ldots, x_n^*). \tag{11}$$

By virtue of (11), relation (2) and inequality $C_{ij}(x_1, \ldots, x_n) \le a_{ij}$ from (1), taking into account conditions a), b) we will have

$$f_i(x_1, \ldots, x_n) = \sum_{j \ne j^*} \int\limits_{\mathbb{R}^n} K_{ij}(x_1, \ldots, x_n, t_1, \ldots, t_n) G_j(f_j(t_1, \ldots, t_n))\, dt_1 \ldots dt_n +$$

$$+ \int\limits_{\mathbb{R}^n} K_{ij^*}(x_1, \ldots, x_n, t_1, \ldots, t_n) G_{j^*}(f_{j^*}(t_1, \ldots, t_n))\, dt_1 \ldots dt_n \le$$

$$\le \sum_{j \ne j^*} C_{ij}(x_1, \ldots, x_n)\eta_j + \int\limits_{\mathbb{R}^n \setminus O_\delta(x_1^*, \ldots, x_n^*)} K_{ij^*}(x_1, \ldots, x_n, t_1, \ldots, t_n) G_{j^*}(f_{j^*}(t_1, \ldots, t_n))\, dt_1 \ldots dt_n +$$

$$+ \int\limits_{O_\delta(x_1^*, \ldots, x_n^*)} K_{ij^*}(x_1, \ldots, x_n, t_1, \ldots, t_n) G_{j^*}(f_{j^*}(t_1, \ldots, t_n))\, dt_1 \ldots dt_n \le$$

$$\le \sum_{j \ne j^*} C_{ij}(x_1, \ldots, x_n)\eta_j + \eta_{j^*} \int\limits_{\mathbb{R}^n \setminus O_\delta(x_1^*, \ldots, x_n^*)} K_{ij^*}(x_1, \ldots, x_n, t_1, \ldots, t_n)\, dt_1 \ldots dt_n +$$

$$+ \int\limits_{O_\delta(x_1^*, \ldots, x_n^*)} K_{ij^*}(x_1, \ldots, x_n, t_1, \ldots, t_n) G_{j^*}(f_{j^*}(t_1, \ldots, t_n))\, dt_1 \ldots dt_n <$$

$$< \sum_{j \ne j^*} C_{ij}(x_1, \ldots, x_n)\eta_j + \eta_{j^*} \int\limits_{\mathbb{R}^n \setminus O_\delta(x_1^*, \ldots, x_n^*)} K_{ij^*}(x_1, \ldots, x_n, t_1, \ldots, t_n)\, dt_1 \ldots dt_n +$$

$$+ \eta_{j^*} \int\limits_{O_\delta(x_1^*, \ldots, x_n^*)} K_{ij^*}(x_1, \ldots, x_n, t_1, \ldots, t_n) dt_1 \ldots dt_n =$$

$$= \sum_{j \ne j^*} C_{ij}(x_1, \ldots, x_n)\eta_j + C_{ij^*}(x_1, \ldots, x_n)\eta_{j^*} \le \sum_{j=1}^{N} a_{ij}\eta_j = \eta_i, \quad i, j = \overline{1, N}.$$

The lemma is proved.

3. THEOREM OF EXISTENCE OF BOUNDED SOLUTION

Let us now consider the following successive approximations for system (1):

$$f_i^{(m+1)}(x_1, \ldots, x_n) = \sum_{j=1}^{N} \int_{\mathbb{R}^n} K_{ij}(x_1, \ldots, x_n, t_1, \ldots, t_n) G_j(f_j^{(m)}(t_1, \ldots, t_n)) dt_1 \ldots dt_n,$$

$$f_i^{(0)}(x_1, \ldots, x_n) \equiv \eta_i, \quad (x_1, \ldots, x_n) \in \mathbb{R}^n, i = \overline{1, N}, m = 0, 1, 2, \ldots \tag{12}$$

Suppose that conditions a)–d), 1), and 2) are satisfied. By induction on m, it is not difficult to check the validity of the following statements:

$$f_i^{(m)}(x_1,\dots,x_n) \text{ monotonically decreasing on } m,\ m=0,1,2,\dots,\ i=\overline{1,N}, \tag{13}$$

$$f_i^{(m)} \in C(\mathbb{R}^n),\ \ i=\overline{1,N}, \tag{14}$$

$$f_i^{(m)}(x_1,\dots,x_n) > 0,\ \ m=0,1,2,\dots,\ i=\overline{1,N}. \tag{15}$$

Let us prove that for all $(x_1,\dots,x_n)\in\mathbb{R}^n\backslash B_r$ the following lower bound estimates hold:

$$f_i^{(m)}(x_1,\dots,x_n) \geq d_i, \quad m=0,1,2,\dots,i=\overline{1,N}, \tag{16}$$

where the numbers d_i are defined under condition d).

Let us check inequality (16) at $m=0$. Indeed, since the functions $G_i(u)/u$ are monotonically decreasing at $(0,+\infty)$, $i=\overline{1,N}$, then from the estimation of

$$1=\frac{G_i(\eta_i)}{\eta_i}<\frac{1}{\varepsilon_i(r)}=\frac{G_i(d_i)}{d_i}$$

we get that $d_i<\eta_i=f_i^{(0)}(x_1,\dots,x_n), i=\overline{1,N}$.

Suppose now that for $(x_1,\dots,x_n)\in\mathbb{R}^n\backslash B_r$, inequality (16) holds for some natural m. Then, using the conditions a), b), d), 1), and 2), from (12) and (15) we will have

$$f_i^{(m+1)}(x_1,\dots,x_n) \geq \sum_{j=1}^N \int_{\mathbb{R}^n\backslash B_r} K_{ij}(x_1,\dots,x_n,t_1,\dots,t_n)G_j(f_j^{(m)}(t_1,\dots,t_n))dt_1\dots dt_n \geq$$

$$\geq \sum_{j=1}^N G_j(d_j)\int_{\mathbb{R}^n\backslash B_r} K_{ij}(x_1,\dots,x_n,t_1,\dots,t_n)dt_1\dots dt_n \geq G_i(d_i)\varepsilon_i(r)=d_i, \quad i=\overline{1,N}.$$

If condition e) is satisfied, by analogy with the proof of Lemma 3, we can also verify that the inequalities hold

$$f_i^{(m)}(x_1,\dots,x_n)<\eta_i, \quad m=1,2,\dots,i=\overline{1,N},(x_1,\dots,x_n)\in\mathbb{R}^n. \tag{17}$$

Taking into account (14), (15) and the compactness of the ball B_r, we can say that for every $i\in\{1,2,\dots,N\}$ and $m\in\{0,1,2,\dots\}$, there exists a point $(x_1^{(m,i)},\dots,x_n^{(m,i)})\in B_r$ such that

$$\min_{(x_1,\dots,x_n)\in B_r} f_i^{(m)}(x_1,\dots,x_n)=f_i^{(m)}(x_1^{(m,i)},\dots,x_n^{(m,i)})>0, \quad (x_1,\dots,x_n)\in B_r. \tag{18}$$

Thus, it follows from (16) and (18) for $(x_1,\dots,x_n)\in\mathbb{R}^n$, that

$$f_i^{(m)}(x_1,\dots,x_n)\geq \min\{f_i^{(m)}(x_1^{(m,i)},\dots,x_n^{(m,i)}),d_i\}>0, \quad m=0,1,2,\dots,i=\overline{1,N}. \tag{19}$$

Let us now consider the functions $\chi_i(x_1,\dots,x_n)=\frac{f_i^{(2)}(x_1,\dots,x_n)}{f_i^{(1)}(x_1,\dots,x_n)}, i=\overline{1,N}$, on the set $\mathbb{R}^n$. From (13), (14), and (19) we have

$$\chi_i\in C(\mathbb{R}^n), \quad i=\overline{1,N},$$

$$\frac{\alpha_i}{\eta_i}\leq\chi_i(x_1,\dots,x_n)\leq 1, \quad (x_1,\dots,x_n)\in\mathbb{R}^n, i=\overline{1,N}, \tag{20}$$

where by virtue of (17), (19),

$$0<\alpha_i:=\min\{f_i^{(2)}(x_1^{(2,i)},\dots,x_n^{(2,i)}),d_i\}<\eta_i, \quad i=\overline{1,N}.$$

Let us denote by $\sigma_0=\min_{i=\overline{1,N}}(\alpha_i\eta_i)$. Obviously, $\sigma_0\in(0,1)$.

Consequently, considering (20) and (12) and conditions 1), a), we will have

$$\sum_{j=1}^{N}\int_{\mathbb{R}^n} K_{ij}(x_1,\ldots,x_n,t_1,\ldots,t_n)G_j(\sigma_0 f_j^{(1)}(t_1,\ldots,t_n))dt_1\ldots dt_n \le$$

$$\le f_i^{(3)}(x_1,\ldots,x_n) \le \sum_{j=1}^{N}\int_{\mathbb{R}^n} K_{ij}(x_1,\ldots,x_n,t_1,\ldots,t_n)G_j(f_j^{(1)}(t_1,\ldots,t_n))dt_1\ldots dt_n =$$

$$= f_i^{(2)}(x_1,\ldots,x_n), \quad (x_1,\ldots,x_n)\in\mathbb{R}^n, i=\overline{1,N}.$$

Hence, by virtue of condition c), we arrive at the inequalities

$$\varphi(\sigma_0)f_i^{(2)}(x_1,\ldots,x_n) \le f_i^{(3)}(x_1,\ldots,x_n) \le f_i^{(2)}(x_1,\ldots,x_n), \quad i=\overline{1,N}. \tag{21}$$

Now, using (21), (12), conditions 1), a), and c), let us write down

$$\varphi(\varphi(\sigma_0))f_i^{(3)}(x_1,\ldots,x_n) \le f_i^{(4)}(x_1,\ldots,x_n) \le f_i^{(3)}(x_1,\ldots,x_n), \quad i=\overline{1,N}.$$

Continuing this reasoning, at m-step we obtain the following estimate:

$$F_m(\sigma_0)f_i^{(m+1)}(x_1,\ldots,x_n) \le f_i^{(m+2)}(x_1,\ldots,x_n) \le f_i^{(m+1)}(x_1,\ldots,x_n),$$

$$(x_1,\ldots,x_n)\in\mathbb{R}^n, m=1,2,\ldots, i=\overline{1,N}, F_m(\sigma):=\underbrace{\varphi(\varphi\ldots\varphi(\sigma))}_{m\text{ times}}, \sigma\in[0,1]. \tag{22}$$

Then, using properties (3) and (4) of the function φ, we prove the validity of the inequality

$$F_m(\sigma_0) \ge k^m\sigma_0 + 1 - k^m, \quad m=1,2,\ldots, \tag{23}$$

where

$$k := \frac{1-\varphi(\frac{\sigma_0}{2})}{1-\frac{\sigma_0}{2}} \in (0,1), \quad \sigma_0 = \min_{i=\overline{1,N}}\left\{\frac{\alpha_i}{\eta_i}\right\} \in (0,1). \tag{24}$$

For this purpose, consider the line $y = ku + 1 - k$, passing through the points $(1,1)$ and $(\frac{\sigma_0}{2}, \varphi(\frac{\sigma_0}{2}))$, where the number k is given according to formula (24). From properties (3) and (4), it immediately follows that (Fig. 2)

$$\varphi(\sigma_0) \ge k\sigma_0 + 1 - k. \tag{25}$$

Since $k\sigma_0 + 1 - k \in (0,1)$, then taking into account the properties of concavity of the graph and monotonicity of the function φ from (25) we will have

$$F_2(\sigma_0) = \varphi(\varphi(\sigma_0)) \ge \varphi(k\sigma_0+1-k) \ge k(k\sigma_0+1-k)+1-k = k^2\sigma_0+1-k^2.$$

Continuing this process, at m-th step we obtain inequality (23).

Thus, in view of (22), (23), (17) and (13) we arrive at the following uniform estimate for successive approximations of (12):

$$0 \le f_i^{(m+1)}(x_1,\ldots,x_n) - f_i^{(m+2)}(x_1,\ldots,x_n) < \eta_i(1-\sigma_0)k^m,$$

$$(x_1,\ldots,x_n)\in\mathbb{R}^n, m=1,2,\ldots, i=\overline{1,N}. \tag{26}$$

From (26), we obtain uniform convergence of the sequence of continuous vector functions $f^{(m)}(x_1,\ldots,x_n) = (f_1^{(m)}(x_1,\ldots,x_n),\ldots,f_N^{(m)}(x_1,\ldots,x_n))^T, m=0,1,2,\ldots$, on the set $\mathbb{R}^n$:

$$\lim_{m\to\infty} f_i^{(m)}(x_1,\ldots,x_n) = f_i(x_1,\ldots,x_n), \quad (x_1,\ldots,x_n)\in\mathbb{R}^n, i=\overline{1,N},$$

and $f_i \in C(\mathbb{R}^n)$, $i=\overline{1,N}$.

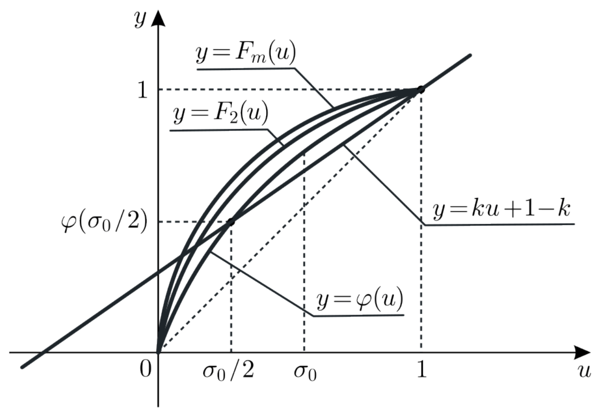

Fig. 2. Graph of the function $y = \varphi(u)$

By virtue of (13), conditions 1), 2), a), (14), (16), (26), and B. Levi's theorem (see [18, p. 303]), the limit vector function $f(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_N(x_1, \ldots, x_n))^{\mathrm{T}}$ satisfies the system (1) and the evaluation from below

$$f_i(x_1, \ldots, x_n) \geq d_i, \quad (x_1, \ldots, x_n) \in \mathbb{R}^n \backslash B_r, i = \overline{1, N}. \tag{27}$$

Given the estimate (27) and lemma 2, we conclude that

$$\inf_{(x_1,\ldots,x_n)\in\mathbb{R}^n} f_i(x_1, \ldots, x_n) > 0, \quad i = \overline{1, N}. \tag{28}$$

Then, taking into account condition e), the statement of lemma 3, and the monotonicity property (13), we arrive at the strict inequality

$$f_i(x_1, \ldots, x_n) < \eta_i, \quad (x_1, \ldots, x_n) \in \mathbb{R}^n, i = \overline{1, N}. \tag{29}$$

Now in evaluation (26), instead of m, we take $m+1, m+2, \ldots, m+p$. As a result, we obtain the following inequalities:

$$0 \leq f_i^{(m+2)}(x_1, \ldots, x_n) - f_i^{(m+3)}(x_1, \ldots, x_n) < \eta_i(1-\sigma_0)k^{m+1},$$
$$(x_1, \ldots, x_n) \in \mathbb{R}^n, m = 1, 2, \ldots, i = \overline{1, N},$$
$$0 \leq f_i^{(m+3)}(x_1, \ldots, x_n) - f_i^{(m+4)}(x_1, \ldots, x_n) < \eta_i(1-\sigma_0)k^{m+2},$$
$$(x_1, \ldots, x_n) \in \mathbb{R}^n, m = 1, 2, \ldots, i = \overline{1, N},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$0 \leq f_i^{(m+p+1)}(x_1, \ldots, x_n) - f_i^{(m+p+2)}(x_1, \ldots, x_n) < \eta_i(1-\sigma_0)k^{m+p},$$
$$(x_1, \ldots, x_n) \in \mathbb{R}^n, p, m = 1, 2, \ldots, i = \overline{1, N}.$$

Summarizing them with inequality (26), we arrive at a two-sided estimator

$$0 \leq f_i^{(m+1)}(x_1, \ldots, x_n) - f_i^{(m+p+2)}(x_1, \ldots, x_n) < \eta_i(1-\sigma_0)(k^m + k^{m+1} + \cdots + k^{m+p}),$$
$$(x_1, \ldots, x_n) \in \mathbb{R}^n, p, m = 1, 2, \ldots, i = \overline{1, N}. \tag{30}$$

From (30), in particular, it follows that

$$0 < f_i^{(m+1)}(x_1, \ldots, x_n) - f_i^{(m+p+2)}(x_1, \ldots, x_n) < \eta_i(1-\sigma_0)\frac{k^m}{1-k}. \tag{31}$$

Fixing the index m and decreasing $p \to \infty$ in (31), we obtain

$$0 < f_i^{(m+1)}(x_1, \dots, x_n) - f_i(x_1, \dots, x_n) < \eta_i(1 - \sigma_0)\frac{k^m}{1-k}. \tag{32}$$

Note also that if the functions $\{C_{ij}(x_1, \dots, x_n)\}_{i,j=\overline{1,N}}$ satisfy the additional condition

$$a_{ij} - C_{ij}(x_1, \dots, x_n) \in L_1(\mathbb{R}^n), \quad i, j = \overline{1, N}, \tag{33}$$

then, reasoning similarly to the proof of the main theorem (on the integral asymptotics of the solution) from [13], we can assert that there exist positive constants $D_1, D_2, \dots, D_N$ such that

$$0 \le \int_{\mathbb{R}^n} (\eta_i - f_i^{(m)}(x_1, \dots, x_n))dx_1 \dots dx_n \le D_i, \quad m = 0, 1, 2, \dots, i = \overline{1, N}.$$

Hence, according to the theorem of B. Levi, we conclude that $\eta_i - f_i \in L_1(\mathbb{R}^n), i = \overline{1, N}$, and

$$\int_{\mathbb{R}^n} (\eta_i - f_i(x_1, \dots, x_n))dx_1 \dots dx_n \le D_i, \quad i = \overline{1, N}.$$

Based on the above, the following is true

Theorem 1. *If conditions* a)–e), 1), 2) *are satisfied, the system of nonlinear multivariate integral equations* (1) *has an ordinally positive continuous and bounded on* $\mathbb{R}^n$ *solution* $f(x_1, \dots, x_n) = (f_1(x_1, \dots, x_n), \dots, f_N(x_1, \dots, x_n))^T$, *that is a uniform limit of successive approximations* (12). *Moreover, the estimates* (27)–(29) *and* (32) *hold. If in addition condition* (33) *is satisfied, then* $\eta_i - f_i \in L_1(\mathbb{R}^n), i = \overline{1, N}$.

4. SINGULARITY OF THE SOLUTION OF THE SYSTEM (1)

Let us consider the following subclass of continuous nonnegative and bounded vector functions on $\mathbb{R}^n$:

$$\mathbb{H} := \Big\{ f(x_1, \dots, x_n) = (f_1(x_1, \dots, x_n), \dots, f_N(x_1, \dots, x_n))^{\mathrm{T}} \colon f_i \in C_M(\mathbb{R}^n),$$

$$f_i(x_1, \dots, x_n) \ge 0,\ (x_1, \dots, x_n) \in \mathbb{R}^n,\ i = \overline{1, N},$$

$$\text{there is such } j_0 \in \{1, 2, \dots, N\} \text{ that } \inf_{(x_1, \dots, x_n) \in \mathbb{R}^n \setminus B_r} f_{j_0}(x_1, \dots, x_n) > 0 \Big\}, \tag{34}$$

where the number $r > 0$ is defined in condition d), through $C_M(\mathbb{R}^n)$, the space of continuous and bounded functions on the set $\mathbb{R}^n$ is denoted. The following holds

Theorem 2. *If conditions* a)–e), 1), 2) *are satisfied, the system of nonlinear multivariate integral equations* (1) *has no other solutions in the class* $\mathbb{H}$ *except for the solution* f, *constructed by means of successive approximations* (13).

Proof. Suppose the converse: the system (1) besides the solution $f \in \mathbb{H}$, constructed by means of successive approximations (12), also possesses another solution $f^* \in \mathbb{H}$. Then, using lemmas 2 and 3, we conclude that

$$f_i^*(x_1, \dots, x_n) < \eta_i, \quad (x_1, \dots, x_n) \in \mathbb{R}^n, i = \overline{1, N}, \tag{35}$$

$$\alpha_i^* := \inf_{(x_1, \dots, x_n) \in \mathbb{R}^n} f_i^*(x_1, \dots, x_n) > 0, \quad i = \overline{1, N}. \tag{36}$$

Applying the method of induction by m, it is easy to verify the validity of the following inequalities:

$$f_i^*(x_1, \dots, x_n) < f_i^{(m)}(x_1, \dots, x_n), \quad (x_1, \dots, x_n) \in \mathbb{R}^n, m = 0, 1, 2, \dots, i = \overline{1, N}. \tag{37}$$

In (37) by decreasing $m \to \infty$, we arrive at the inequality

$$f_i^*(x_1, \dots, x_n) \le f_i(x_1, \dots, x_n), \quad (x_1, \dots, x_n) \in \mathbb{R}^n, i = \overline{1, N}. \tag{38}$$

Consider the functions $B_i(x_1,\ldots,x_n) = f_i^*(x_1,\ldots,x_n)/f_i(x_1,\ldots,x_n)$, $(x_1,\ldots,x_n) \in \mathbb{R}^n$, $i = \overline{1,N}$. . Since $f, f^* \in \mathbb{H}$, then by virtue of (28), (29), (35), (36), (38), we have that $B_i \in C(\mathbb{R}^n), i = \overline{1,N}$, and

$$\frac{\alpha_i^*}{\eta_i} \le B_i(x_1,\ldots,x_n) \le 1, \quad (x_1,\ldots,x_n) \in \mathbb{R}^n, i = \overline{1,N}.$$

Let us denote $\sigma^* = \min_{i\in\overline{1,N}}\{\alpha_i^*/\eta_i\}$. By virtue of (35) and (36), the number $\sigma^* \in (0,1)$. Thus, we obtain the inequality

$$\sigma^* f_i(x_1,\ldots,x_n) \le f_i^*(x_1,\ldots,x_n) \le f_i(x_1,\ldots,x_n), \quad (x_1,\ldots,x_n) \in \mathbb{R}^n, i = \overline{1,N}. \tag{39}$$

Then, reasoning as in the proof of Theorem 1, we obtain the following estimates from (39):

$$0 \le f_i(x_1,\ldots,x_n) - f_i^*(x_1,\ldots,x_n) \le \eta_i(1-\sigma^*)k_*^m, \quad (x_1,\ldots,x_n) \in \mathbb{R}^n, m = 1,2,\ldots, i = \overline{1,N}, \tag{40}$$

where $k_* = \frac{1-\varphi\left(\frac{\sigma^*}{2}\right)}{1-\frac{\sigma^*}{2}} \in (0,1)$.

In (40), by decreasing the number $m \to \infty$, we arrive at the equality $f_i(x_1,\ldots,x_n) = f_i^*(x_1,\ldots,x_n)$, $(x_1,\ldots,x_n) \in \mathbb{R}^n, i = \overline{1,N}$. The theorem is proved.

Similarly, the following is proved

Theorem 3. *Let the conditions* a)–d), 1), 2) *be satisfied and the following relations hold*

$$C_{ij}(x_1,\ldots,x_n) = a_{ij}, \quad (x_1,\ldots,x_n) \in \mathbb{R}^n, i,j = \overline{1,N}.$$

Then the system (1) *in the class* $\mathbb{H}$ *possesses only a trivial solution* $\eta = (\eta_1,\ldots,\eta_N)^{\mathrm{T}}$.

5. EXAMPLES

To illustrate the theoretical results obtained, we give examples of the matrix kernel K and nonlinearities $\{G_j(u)\}_{j=\overline{1,N}}$.

Core K examples:

p$_1$) $K_{ij}(x_1,\ldots,x_n,t_1,\ldots,t_n) = \mathring{K}_{ij}(x_1-t_1,x_2-t_2,\ldots,x_n-t_n), (x_1,\ldots,x_n),(t_1,\ldots,t_n) \in \mathbb{R}^n, 2i,j = \overline{1,N}$, where $\mathring{K}_{ij}(\tau_1,\tau_2,\ldots,\tau_n) > 0$, $\mathring{K}_{ij} \in C(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \mathring{K}_{ij}(\tau_1,\ldots,\tau_n)\,d\tau_1\ldots d\tau_n = a_{ij} < 1$, $i,j = \overline{1,N}$, $r(A) = 1$, $A = (a_{ij})_{i,j=\overline{1,N}}$, $(\tau_1,\ldots,\tau_n) \in \mathbb{R}^n$.

p$_2$) $K_{ij}(x_1,\ldots,x_n,t_1,\ldots,t_n) = \lambda_{ij}(|x|)\mathring{K}_{ij}(x_1-t_1,x_2-t_2,\ldots,x_n-t_n)$, $(x_1,\ldots,x_n)$, $(t_1,\ldots,t_n) \in \mathbb{R}^n$, $|x| = \sqrt{x_1^2+\ldots+x_n^2}$, $0 < \inf_{v\ge 0}\lambda_{ij}(v) \le \lambda_{ij}(v) < 1$, $v \ge 0$, $1-\lambda_{ij} \in L_1(0,+\infty)$, $i,j = \overline{1,N}$.

p$_3$) $K_{ij}(x_1,\ldots,x_n,t_1,\ldots,t_n) = C_{ij}^*(x_1,\ldots,x_n)\mathring{K}_{ij}(x_1-t_1,\ldots,x_n-t_n)$, $(x_1,\ldots,x_n),(t_1,\ldots,t_n) \in \mathbb{R}^n$, $\inf_{(x_1,\ldots,x_n)\in\mathbb{R}^n} C_{ij}^*(x_1,\ldots,x_n) > 0$, $C_{ij}^* \in C(\mathbb{R}^n)$, $\sup_{(x_1,\ldots,x_n)\in\mathbb{R}^n} C_{ij}^*(x_1,\ldots,x_n) = 1$, $i,j = \overline{1,N}$.

Here are also *examples of functions* $\mathring{K}_{ij}$, λ_{ij}, C_{ij}^*, $i,j = \overline{1,N}$:

q$_1$) $\mathring{K}_{ij}(\tau_1,\ldots,\tau_n) = \pi^{-n/2}a_{ij}e^{-(\tau_1^2+\ldots+\tau_n^2)}$, $r(A) = 1$, $A = (a_{ij})_{i,j=\overline{1,N}}$, $\tau_j \in \mathbb{R}$, $i,j = \overline{1,N}$,

q$_2$) $\mathring{K}_{ij}(\tau_1,\ldots,\tau_n) = \int_a^b e^{-(|\tau_1|+\ldots+|\tau_n|)s}\,dQ_{ij}(s)$, $\tau_j \in \mathbb{R}$, $i,j = \overline{1,N}$, where $Q_{ij}(s)$ — are monotonically increasing functions on $[a,b)$, $0 < a < b \le +\infty$, with

$$2^n \int_a^b \frac{1}{s^n}\,dQ_{ij}(s) = a_{ij}, \quad i,j = \overline{1,N};$$

q$_3$) $\lambda_{ij}(|x|) = 1 - \varepsilon_{ij}e^{-(x_1^2+\ldots+x_n^2)}$, $0 < \varepsilon_{ij} < 1$ — are parameters, $(x_1,\ldots,x_n) \in \mathbb{R}^n$, $i,j = \overline{1,N}$,

q$_4$) $C_{ij}^*(x_1,\ldots,x_n) = 1 - \varepsilon_{ij}e^{-(|x_1|+\ldots|x_n|)}$, $(x_1,\ldots,x_n) \in \mathbb{R}^n$, $i,j = \overline{1,N}$.

Let us now turn to *examples of nonlinearities* $\{G_j(u)\}_{j=\overline{1,N}}$:

r$_1$) $G_j(u) = u^{\beta_j}\eta_j^{1-\beta_j}$, $u \in [0, +\infty)$, $\beta_j \in (0,1)$, $j = \overline{1,N}$;

r$_2$) $G_j(u) = \eta_j(u^{\beta_j} + u^{\delta_j})/(\eta_j^{\beta_j} + \eta_j^{\delta_j})$, $u \in [0, +\infty)$, $\beta_j, \delta_j \in (0,1)$, $j = \overline{1,N}$;

r$_3$) $G_j(u) = l_j(1 - e^{-u^{\beta_j}})$, $u \in [0, +\infty)$, $\beta_j \in (0,1)$, $l_j = \eta_j/(1 - \exp\{-\eta_j^{\beta_j}\})$, $j = \overline{1,N}$.

Let us elaborate on examples p$_3$), q$_1$), r$_3$) and verify that conditions 2) and d) are satisfied. First of all, note that in this case

$$\sup_{(x_1,\dots,x_n)\in\mathbb{R}^n} \int_{\mathbb{R}^n} K_{ij}(x_1,\dots,x_n,t_1,\dots,t_n)dt_1\dots dt_n =$$

$$= \sup_{(x_1,\dots,x_n)\in\mathbb{R}^n} \left(C^*_{ij}(x_1,\dots,x_n) \int_{\mathbb{R}^n} \mathring{K}_{ij}(x_1-t_1,\dots,x_n-t_n)dt_1\dots dt_n \right) =$$

$$= \sup_{(x_1,\dots,x_n)\in\mathbb{R}^n} \left(C^*_{ij}(x_1,\dots,x_n) \int_{\mathbb{R}^n} \mathring{K}_{ij}(\tau_1,\dots,\tau_n)d\tau_1\dots d\tau_n \right) =$$

$$= a_{ij} \sup_{(x_1,\dots,x_n)\in\mathbb{R}^n} C^*_{ij}(x_1,\dots,x_n) = a_{ij}, \quad i,j = \overline{1,N}.$$

Since $r(A) = 1$ (see Example q$_1$)), condition 2) is satisfied. For completeness, let us give an example of the matrix $A = (a_{ij})_{i,j=\overline{1,N}}$ with unit spectral radius and with elements $a_{ij} \in (0,1)$, $i,j = \overline{1,N}$ (in the case when $N = 2$):

$$A = \begin{pmatrix} 7/9 & 1/3 \\ 1/3 & 1/2 \end{pmatrix}.$$

Let's check condition d). First evaluate the integral of the function $\mathring{K}_{ij}(x_1-t_1,\dots,x_n-t_n)$ over the set $\mathbb{R}^n \backslash B_r$:

$$\int\limits_{\mathbb{R}^n\backslash B_r} \mathring{K}_{ij}(x_1-t_1,\dots,x_n-t_n)\,dt_1\dots dt_n =$$

$$= \int\limits_{\mathbb{R}^n} \mathring{K}_{ij}(x_1-t_1,\dots,x_n-t_n)\,dt_1\dots dt_n - \int\limits_{B_r} \mathring{K}_{ij}(x_1-t_1,\dots,x_n-t_n)\,dt_1\dots dt_n =$$

$$= a_{ij} - \int\limits_{B_r} \mathring{K}_{ij}(x_1-t_1,\dots,x_n-t_n)\,dt_1\dots dt_n \ge a_{ij} - \int\limits_{-r}^{r} \int\limits_{\mathbb{R}^{n-1}} \mathring{K}_{ij}(x_1-t_1,\dots,x_n-t_n)\,dt_1\dots dt_n =$$

$$= a_{ij} - \int\limits_{-r}^{r} \Phi_{ij}(x_n-t_n)\,dt_n = a_{ij} - \int\limits_{x_n-r}^{x_n+r} \Phi_{ij}(\tau_n)\,d\tau_n,$$

where $\Phi_{ij}(\tau) := \int_{\mathbb{R}^{n-1}} \mathring{K}_{ij}(t_1,\dots,t_{n-1},\tau)dt_1\dots dt_{n-1}$.

Consider the functions $F_{ij}(x_n) := \int_{x_n-r}^{x_n+r} \Phi_{ij}(\tau_n)d\tau_n$, $i,j = \overline{1,N}$, $x_n \in \mathbb{R}$. Since $F_{ij}(x_n) \to 0$ at $|x_n| \to \infty$, for every fixed $i,j \in \{1,2,\dots,N\}$, there exists a number $r_0 > 0$ such that at $|x_n| > r_0$

$$F_{ij}(x_n) \le \frac{a_{ij}}{2}.$$

But since $F_{ij} \in C(\mathbb{R})$ and $\mathring{K}_{ij}(t_1,\dots,t_n) > 0$, $(t_1,\dots,t_n) \in \mathbb{R}^n$, then for $x_n \in [-r_0, r_0]$

$$F_{ij}(x_n) \le \max_{x_n\in[-r_0,r_0]} \left\{ \int_{x_n-r}^{x_n+r} \Phi_{ij}(\tau_n)d\tau_n \right\} =: \delta_{ij} < a_{ij}.$$

Hence, $F_{ij}(x_n) \le \max\{a_{ij}/2, \delta_{ij}\} < a_{ij}$, $x_n \in \mathbb{R}$, $i,j = \overline{1,N}$.

Thus, we have

$$\inf_{(x_1,\dots,x_n)\in\mathbb{R}^n\setminus B_r}\int_{\mathbb{R}^n\setminus B_r}\mathring{K}_{ij}(x_1-t_1,\dots,x_n-t_n)dt_1\dots dt_n\geq$$

$$\geq\inf_{(x_1,\dots,x_n)\in\mathbb{R}^n}\int_{\mathbb{R}^n\setminus B_r}\mathring{K}_{ij}(x_1-t_1,\dots,x_n-t_n)dt_1\dots dt_n\geq a_{ij}-\max\{a_{ij}/2,\delta_{ij}\}>0,\quad i,j=\overline{1,N},$$

whence it follows that

$$\varepsilon_i(r)\geq\min_{j=\overline{1,N}}\{C^0_{ij}(a_{ij}-\max\{\frac{a_{ij}}{2},\delta_{ij}\})\}>0,$$

where $C^0_{ij}:=\inf_{(x_1,\dots,x_n)\in\mathbb{R}^n}C^*_{ij}(x_1,\dots,x_n)$.

On the other hand, it is obvious that $\varepsilon_i(r)\leq a_{ij}<1$, $i,j=\overline{1,N}$.

We now verify that, for Example p$_3$), the equations $G_i(u)=u/\varepsilon_i(r)$ have positive solutions d_i. Indeed, since $G_i\in C(\mathbb{R}^+)$, $G_i(\eta_i)=\eta_i$, $\lim_{u\to+0}G_i(u)/u=+\infty$, $\lim_{u\to+\infty}G_i(u)/u=0$, $i=\overline{1,N}$, and $\varepsilon_i(r)\in(0,1)$; and $G_i(u)/u$ decreases monotonically at $(0,+\infty)$, then for every $i\in\{1,2,\dots,N\}$, there exists a single $d_i>0$ such that $G_i(d_i)/d_i=1/\varepsilon_i(r)$.

The verification of conditions 2) and d) for the rest of the examples is done in the same way.

Now let us give a specific example of a nonlinear multidimensional integral equation having an application in the theory of p-adic string (see [5]):

$$\varphi^p(x_1,\dots,x_n)=\pi^{-n/2}\int_{\mathbb{R}^n}e^{-((x_1-t_1)^2+\dots+(x_n-t_n)^2)}\varphi(t_1,\dots,t_n)dt_1\dots dt_n,\quad(x_1,\dots,x_n)\in\mathbb{R}^n,$$

where $p>2$ is an odd number. Using the notation $f(x_1,\dots,x_n)=\varphi^p(x_1,\dots,x_n)$, this equation is reduced to a multivariate equation of the form (1) with concave nonlinearity with respect to the sought non-negative function $f(x_1,\dots,x_n)$.

We also give an example of a one-dimensional convolutional integral equation with exponential nonlinearity arising in the mathematical theory of the geographical spread of an epidemic:

$$f(x)=a\int_{-\infty}^{\infty}K(x-t)(1-e^{-f(t)})dt,\quad x\in\mathbb{R},$$

where $a>1$ is a numerical parameter, the kernel $K(x)>0$, $x\in\mathbb{R}$, $\int_{-\infty}^{\infty}K(x)dx=1$ (see [6, p. 318] in the formulation of Theorem 1 ($f(x)=-\chi(x)$)).

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CONFLICT OF INTERESTS

The author of this paper declares that he has no conflict of interests.

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CONTROL THEORY

STABLE SOLUTION OF PROBLEMS OF TRACKING AND DYNAMICAL RECONSTRUCTION UNDER MEASURING PHASE COORDINATES AT DISCRETE TIME MOMENTS

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Abstract. The problem of dynamic reconstruction of input actions in a system of ordinary differential equations and the problem of tracking a trajectory of a system by some trajectory of another one influenced by an unknown disturbance are under consideration. An input action is assumed to be an unbounded function, namely, an element of the space of square integrable functions. Two solving algorithms, which are stable with respect to informational noises and computational errors and oriented to program realization, are designed. Upper estimates of their convergence rates are established. The algorithms are based on constructions from feedback control theory. They operate under conditions of (inaccurate) measuring the phase states of the given systems at discrete times.

Keywords: *problem of tracking, reconstruction*

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1. INTRODUCTION. PROBLEM STATEMENT

We consider a system of ordinary differential equations

$$\dot{y}(t) = f(t, y(t)) + Bu(t), \quad t \in T = [0, \vartheta], \tag{1}$$

with the initial condition

$$y(0) = y_0. \tag{2}$$

Here $0 < \vartheta < +\infty$, $y \in \mathbb{R}^N$, $u \in \mathbb{R}^r$ is the input influence, $f(t, y)$ is a Lipschitz (with Lipschitz constant L) vector function over a set of variables, B – a stationary matrix of dimension $N \times r$, $n, r \in \mathbb{N}$.

It is assumed that the system (1) is subjected to an unknown input influence $u(\cdot) \in L_2(T; \mathbb{R}^r)$. At discrete, sufficiently frequent, moments of time $\tau_i \in \Delta = \{\tau_i\}_{i=\overline{0,m}}$ ($\tau_0 = 0$, $\tau_m = \vartheta$, $\tau_{i+1} = \tau_i + \delta$) the phase states $y(\tau_i) = y(\tau_i; y_0, u(\cdot))$ of system (1) are measured. The states $y(\tau_i)$, $i = \overline{0, m-1}$, are measured with error. The measurement results are vectors $\xi_i^h \in \mathbb{R}^N$, satisfying the inequalities

$$|y(\tau_i) - \xi_i^h|_N \le h, \tag{3}$$

where $h \in (0, 1)$ is the level of measurement error, $|\cdot|_N$ denotes the Euclidean norm in the space $\mathbb{R}^N$.

It is required to specify an algorithm for approximate restoration of the input impact based on the results of inaccurate measurements $y(\tau_i)$. For this purpose, we consider the problem consisting in the construction of an algorithm that, based on the current measurements of values $y(\tau_i)$ in "real time", forms (according to the feedback principle) the function $u = u^h(\cdot)$, that is an approximation (in the space metric $L_2(T;\mathbb{R}^r)$) of some input influence generating the solution $y(\cdot)$ of equation (1).

The formulated problem is a problem of dynamic recovery (reconstruction). One of the approaches to its solution was developed in [1, pp. 7–87; 2, pp. 400–415; 3, pp. 13–93; 4–12]. In [1–10], the case of instantaneous constraints on perturbations was considered; the case of absence of such constraints is described in [3, pp. 41–64; 6; 11; 12]. The approach is based on a combination of methods of the theory of positional control [13], according to which for dynamic, realized at the rate of "real time", restoration of the perturbation acting on the system (1), one proceeds as follows: some controlled system, quite often called a model, is introduced; after that, the restoration task is replaced by the task of forming the control of this model according to the feedback principle in such a way, that at a suitable matching of the measurement error h, the value of the measurement interval δ (as well as, perhaps, some other parameters, e.g., regulation parameter), the control u^h (.) — in one or some other metrics — approximates some input influence that induces a measured solution y (.) of system (1). Usually, when speaking of approximation, one means uniform (space metric C) or mean-square (space metric L_2) metrics. When implementing this approach, in many cases the right-hand side of the model has the same structure as the real system (system (1)). However, instead of the phase vector of the model in its right part there are the values ξ_i^h, i.e., the results of measurements of phase states of the real system instead of the states of the model. Quite often (see, for example, [1, p. 23; 4; 5]) the model has the following form:

$$\dot{y}^h(t) = f(\tau_i, \xi_i^h) + Bu_i^h \quad \text{at a.e.} \quad t \in \delta_i = [\tau_i, \tau_{i+1}), \quad i = \overline{0, m-1}. \tag{4}$$

In this case, the control $u^h(\cdot)$ in the model is formed according to some rule U in the form of feedback:

$$u^h(t) = u_i^h = U(\tau_i, \xi_i^h, y^h(\tau_i)) \quad \text{at a.e.} \quad t \in \delta_i, \quad i = \overline{0, m-1}. \tag{5}$$

In mathematical control theory, one of the "classical" problems is the so-called tracking problem, the study of which began in the fifties of the XX century and was caused by practical problems arising in aviation and astronautics. This problem has not lost its relevance nowadays, in particular, due to the needs of flight dynamics development. The tracking problem is also in demand when analyzing processes arising in control problems of mechanical systems [14, 15], as well as systems functioning under uncertainty [16]. It also plays an important role in the framework of positional differential games [13].

The essence of the tracking problem in the simplest case is as follows. There is a system (1) with an unknown input influence $u(\cdot)$, satisfying usually the instantaneous constraint $u(t) \in P$ at a.e. $t \in T$, where $P \subset \mathbb{R}^r$ is a compact set. Along with the system (1) there is another system of the same type

$$\dot{x}(t) = f(t, x(t)) + Bv(t), \quad t \in T, \tag{6}$$

$$x(0) = x_0$$

and control $v(\cdot)$, that obeys the same constraints as the function $u(\cdot)$. At moments τ_i, the phase states of systems (1) and (6), $y(\tau_i)$ and $x(\tau_i)$, respectively, are measured (with error). The measurement results are vectors $\xi_i^h \in \mathbb{R}^N$ and $\psi_i^h \in \mathbb{R}^N$, satisfying the inequalities

$$|\xi_i^h - y(\tau_i)|_N \le h, \quad |\psi_i^h - x(\tau_i)|_N \le h.$$

The essence of the tracking task consists in designing such an algorithm for forming the control of $v = v^h(\cdot)$ system (6) according to the feedback principle

$$v^h(t) = v_i^h = V(\tau_i, \xi_i^h, \psi_i^h) \quad \text{at a.e.} \quad t \in \delta_i, \quad i = \overline{0, m-1}, \tag{7}$$

that, at appropriate coordination of values h and δ the solutions of systems (1) and (6), will be close, as a rule in uniform metric (in case of proximity of initial states of these systems), whatever the admissible realization of input influence $v(\cdot)$ is. Thus, when solving the tracking problem, it is necessary to construct such a law V of control formation (7), that whatever the number $\varepsilon > 0$, the numbers h_* and δ_* are specified, such that for all $h \in (0, h_*)$ and $\delta \in (0, \delta_*)$, the inequality is true

$$\sup_{t \in T} |x(t; x_0, v^h(\cdot)) - y(t; y_0, u(\cdot))|_N \le \varepsilon,$$

if the value $|x_0 - y_0|_N$ is small enough. Here $x(\cdot; x_0, v^h(\cdot))$ is the solution of the system (6) generated by the control $v^h(\cdot)$ of the form (7). Note that both in the reconstruction problem and in the tracking problem, the input influence of the given system is unknown.

If the algorithms for solving the reconstruction problem described in the papers cited above allowed us to obtain for an arbitrary measurable input influence $u(\cdot)$ (possibly constrained by some specified instantaneous constraints) estimates of the convergence rate (to $u(\cdot)$) of $u^h(\cdot)$ (in model (4) formed according to rule (5)) in a uniform or mean-square metric, then, while solving the reconstruction problem, we would simultaneously solve the tracking problem. Unfortunately, however, such estimates can be obtained only for special classes $u(\cdot)$, for example, for functions with bounded variation. In the case when $u(\cdot)$ is not such a function, the algorithms from these works guarantee only convergence of the controls $u^h(\cdot)$ to $u(\cdot)$.

A question naturally arises: can we choose not the system of the form (4), but the system of the form (6), i.e., a complete copy of the system (1), as a model in reconstruction algorithms? Then, while solving the reconstruction problem in accordance with the described approach, we would simultaneously solve the tracking problem. Unfortunately, for arbitrary f and B, even if smooth enough, it is not possible to give a positive answer to it. The purpose of this paper is to specify two classes of systems of the form (1), for which the answer to the question is positive. For each of these two classes, a different rule of control formation will be specified. The first class is a system being linear both in phase variables and perturbation; the second is a system with a monotonic function in phase variable f. It should be noted that the approach to solving problems of dynamic reconstruction developed, in this paper, was applied when solving problems of reconstruction of unknown structural characteristics of a bioreactor with recharge [3], the problem of formation of flight telemetry using indirect data [3], and problems of modeling of pollution spreading processes [17].

Thereafter, for each $h \in (0,1)$, we fix a family Δ_h of partitions of the segment T by control time instants $\tau_{h,i}$:

$$\Delta_h = \{\tau_{h,i}\}_{i=\overline{0,m_h}}, \quad \tau_{h,0} = 0, \quad \tau_{h,m_h} = \vartheta, \quad \tau_{h,i+1} = \tau_{h,i} + \delta(h), \quad \delta(h) \in (0,1). \tag{8}$$

It should be noted that the same solution of the system (1) can be conditioned by more than one influence. Let $\mathcal{U}(y(\cdot))$ be the set of all input influences from $L_2(T;\mathbb{R}^r)$, generating the solution $y(\cdot)$ of system (1), i.e.,

$$\mathcal{U}(y(\cdot)) = \{\tilde{u}(\cdot) \in L_2(T;\mathbb{R}^r) : \dot{y}(t) - f(t, y(t)) = B\tilde{u}(t) \text{ at a.e. } t \in T\}.$$

By the symbol $u_*(\cdot)$ we denote the minimal element of the set $\mathcal{U}(y(\cdot))$, i.e., by $L_2(T;\mathbb{R}^r)$-norm.

$$u_*(\cdot) = \arg \min_{u(\cdot) \in U(y(\cdot))} |u(\cdot)|_{L_2(T;\mathbb{R}^r)}.$$

Such an element exists and is unique. Following the approach adopted in the theory of incorrect problems, we will recover $u_*(\cdot)$. Hereinafter $c^{(0)}, c^{(1)}, \ldots, c_0, c_1, \ldots, k^{(1)}, k^{(2)}, \ldots, k_1, k_2, \ldots$ denote positive constants that can be written out explicitly, $(\cdot, \cdot)$ is the scalar product in the corresponding finite-dimensional Euclidean space, and $|\cdot|$ is the modulus of a number.

2. SOLUTION ALGORITHM IN CASE OF A LINEAR SYSTEM

Let us consider the case when the system (1) is linear, i.e., has the form

$$\dot{y}(t) = Ay(t) + Bu(t) + f_1(t). \tag{9}$$

Here, A and B are constant matrices of corresponding dimensions, $f_1(\cdot) \in L_2(T;\mathbb{R}^N)$ is a given function. The model is a copy of the system (9):

$$\dot{y}^h(t) = Ay^h(t) + Bu^h(t) + f_1(t) \tag{10}$$

initialized

$$y^h(0) = \xi_0^h.$$

Let's fix the function $\alpha(h) : (0,1) \to (0.1)$. In the future we will need the following

Condition A. With $h \to 0$, we have $\alpha(h) \to 0$, $\delta(h)\alpha^{-2}(h) \to 0$, $h^2(\alpha(h)\delta(h))^{-1} \to 0$.

Let us denote by $\mathcal{Y}(t)$ the fundamental matrix of the system of equations $\dot{y}(t) = Ay(t)$. The inequality

$$\|\mathcal{Y}(t)\| \le \exp\{\chi t\}, \quad t \ge 0,$$

where $\chi = \|A\|$, $\|A\|$ is the Euclidean norm of the matrix A, is true.

Before the algorithm starts, we fix the value $h \in (0,1)$, the partition $\Delta_h = \{\tau_{h,i}\}_{i=\overline{0,m_h}}$ of the form (8) and the number $\alpha = \alpha(h)$. The algorithm operation is divided into a finite number of steps of the same type. At the i-th step, carried out at the time interval $\delta_i = [\tau_i, \tau_{i+1})$, $\tau_i = \tau_{h,i}$, the following operations are performed: at the moment τ_i, the vector u_i^h is calculated according to formula (5), in which

$$U(\tau_i, \xi_i^h, y^h(\tau_i)) = \alpha^{-1} \exp\{-2\chi\tau_{i+1}\} B'(\xi_i^h - y^h(\tau_i)) \tag{11}$$

(here dash means transpose); then the input of system (10) at all $t \in \delta_i$ is given control $u^h(t)$ of the form (5), (11), under the action of which the system (10) passes from the state $y^h(\tau_i)$ to the state $y^h(\tau_{i+1})$. The work of the algorithm ends at the moment ϑ.

Let's introduce the functional

$$\lambda(t) = \exp\{-2\chi t\}|y^h(t) - y(t)|_N^2.$$

In the future, we'll need the following

Lemma 1 (Gronwall's discrete inequality [18, p. 311]). *Let $\phi_j \geq 0$, $f_j \geq 0$ at $j = \overline{0,m}$ and $f_j \leq f_{j+1}$ at $j = \overline{0, m-1}$. Then from the inequalities*

$$\phi_{j+1} \leq c_0\delta \sum_{i=1}^{j} \phi_i + f_j, \quad j = \overline{1, m-1},$$

inequalities follow

$$\phi_{j+1} \leq f_j \exp\{c_0 j\delta\}, \quad j = \overline{0, m-1},$$

if $c_0 > 0$, $\phi_1 \leq f_0$.

Lemma 2. *Let condition* A *be satisfied. Then it is possible to specify such a number $h_* \in (0,1)$, that for all $h \in (0, h_*)$ the inequalities are true.*

$$\max_{i \in \overline{0, m_h - 1}} \lambda(\tau_{i+1}) \leq d_1\{\alpha + \delta + h^2\delta^{-1}\}, \tag{12}$$

$$\int_0^\vartheta |u^h(s)|_r^2\, ds \leq (1 + d_2\delta\alpha^{-2}) \int_0^\vartheta |u_*(s)|_r^2\, ds + d_3 h^2(\alpha\delta)^{-1}, \tag{13}$$

where d_j, $j = 1,2,3$ are positive constants independent of h, δ and α.

Proof. Let's estimate the change in the value of

$$\varepsilon(t) = \lambda(t) + \alpha \int_0^t (|u^h(\tau)|_r^2 - |u_*(\tau)|_r^2) d\tau.$$

Here $\alpha = \alpha(h)$, $\delta = \delta(h)$. It is easy to see that the inequality is true

$$\varepsilon(\tau_{i+1}) \leq \varepsilon(\tau_i) + \lambda_{1i} + \mu_{1i} + \alpha \int_{\tau_i}^{\tau_{i+1}} (|u^h(\tau)|_r^2 - |u_*(\tau)|_r^2) d\tau, \tag{14}$$

where

$$\lambda_{1i} = 2\left(S_i^h, \int_{\tau_i}^{\tau_{i+1}} Y(\tau_{i+1} - \tau) B(u^h(\tau) - u_*(\tau)) d\tau\right),$$

$$\mu_{1i} = \delta \exp\{-2\chi\tau_{i+1}\} \int_{\tau_i}^{\tau_{i+1}} |Y(\tau_{i+1} - \tau) B(u^h(\tau) - u_*(\tau))|_N^2 d\tau,$$

$$S_i^h = \exp\{-2\chi\tau_{i+1}\} Y(\delta) s_i^h, \quad s_i^h = y^h(\tau_i) - y(\tau_i).$$

Note that at $t \in [0, \delta_*]$, $\delta_* \in (0,1)$,

$$\|\mathcal{Y}(t) - I\| \leq c_* t, \quad c_* = c_*(\delta_*),$$

where I is a unit matrix of dimension $N \times N$. Therefore

$$|S_i^h - \exp\{-2\chi\tau_{i+1}\}s_i^h|_N \le \delta c_* \exp\{-2\chi\tau_{i+1}\}|s_i^h|_N \le \delta c_*|s_i^h|_N. \tag{15}$$

In this case, taking into account (15) and the inequality $|S_i^h|_N \le |s_i^h|_N$, we have

$$|(S_i^h, \mathcal{Y}(\delta)Bu) - \exp\{-2\chi\tau_{i+1}\}(s_i^h, Bu)| \le$$

$$\le |S_i^h|_N|\mathcal{Y}(\delta) - I|_N|Bu|_N + |(S_i^h, Bu) - \exp\{-2\chi\tau_{i+1}\}(s_i^h, Bu)| \le 2\delta c^{(0)}|s_i^h|_N|Bu|_N. \tag{16}$$

Further, by virtue of (16), the inequality is true

$$\lambda_{1i} \le 2\exp\{-2\chi\tau_{i+1}\}\left(y^h(\tau_i) - y(\tau_i), \int_{\tau_i}^{\tau_{i+1}} B\{u_i^h - u_*(\tau)\}d\tau\right) + I_{1i},$$

where

$$I_{1i} = \delta c^{(1)}|s_i^h|_N \int_{\tau_i}^{\tau_{i+1}} |u_i^h - u_*(\tau)|_r d\tau.$$

It is not difficult to see that there is an estimation

$$I_{1i} \le \delta^2\lambda(\tau_i) + c^{(2)}\delta \int_{\tau_i}^{\tau_{i+1}} (|u_i^h|_r^2 + |u_*(\tau)|_r^2)d\tau. \tag{17}$$

Considering (17) and the rule for choosing the control $u^h(\cdot)$ (see (5), (11)), we obtain

$$\lambda_{1i} + \alpha\int_{\tau_i}^{\tau_{i+1}} (|u^h(s)|_r^2 - |u_*(s)|_r^2)ds \le$$

$$\le \delta^2\lambda(\tau_i) + c^{(3)}h\int_{\tau_i}^{\tau_{i+1}} (|u_i^h|_r + |u_*(s)|_r)ds + c^{(2)}\delta\int_{\tau_i}^{\tau_{i+1}} (|u_i^h|_r^2 + |u_*(s)|_r^2)ds. \tag{18}$$

In addition, the estimates are correct

$$\mu_{1i} \le \delta c^{(4)}\int_{\tau_i}^{\tau_{i+1}} (|u_i^h|_r^2 + |u_*(\tau)|_r^2)d\tau,$$

$$c^{(3)}h\int_{\tau_i}^{\tau_{i+1}} (|u_i^h|_r + |u_*(s)|_r)ds \le \delta c^{(5)}\int_{\tau_i}^{\tau_{i+1}} (|u_i^h|_r^2 + |u_*(s)|_r^2)ds + c^{(6)}h^2. \tag{19}$$

From (14), using (18), (19), we establish the validity of the inequality

$$\gamma(\tau_{i+1}) = \lambda(\tau_{i+1}) + \alpha\int_{\tau_i}^{\tau_{i+1}} |u^h(s)|_r^2 ds \le$$

$$\le (1+\delta^2)\lambda(\tau_i) + \alpha\int_{\tau_i}^{\tau_{i+1}} |u_*(\tau)|_r^2 d\tau + \delta c^{(7)}\int_{\tau_i}^{\tau_{i+1}} (|u_*(\tau)|_r^2 + |u_i^h|_r^2)d\tau + c^{(6)}h^2. \tag{20}$$

In turn, by virtue of (3), (11) we have

$$|u_i^h|_r^2 \le \alpha^{-2}c^{(8)}(h^2 + |y^h(\tau_i) - y(\tau_i)|_N^2)^2 \le \alpha^{-2}c^{(9)}(\lambda(\tau_i) + h^2) \le \alpha^{-2}c^{(9)}(\gamma(\tau_i) + h^2). \tag{21}$$

From (20), (21) follows the estimation of

$$\gamma(\tau_{i+1}) \le (1+\delta^2)\gamma(\tau_i) + (\alpha + c^{(7)}\delta)\int_{\tau_i}^{\tau_{i+1}} |u_*(s)|_r^2 ds + c^{(6)}h^2 + c^{(9)}\delta^2\alpha^{-2}(\gamma(\tau_i) + h^2). \tag{22}$$

Taking into account condition A, we conclude that it is possible to specify the number $h_1 \in (0,1)$ such that the inequality holds

$$\sup_{h\in(0,h_1)} \delta(h)\alpha^{-2}(h) \le 1.$$

From (22), we derive in the standard way (see, e.g., [13, p. 59–64]) the relation

$$\gamma(\tau_{i+1}) \le \left((\alpha + c^{(7)}\delta) \int_{\tau_i}^{\tau_{i+1}} |u_*(s)|_r^2 ds + c^{(6)} h^2 \delta^{-1} + c^{(9)} h^2 \right) \exp\{\delta(1 + c^{(9)}\alpha^{-2})\tau_{i+1}\}. \tag{23}$$

Note that $\delta(h)\alpha^{-2}(h) \to 0$ at $h \to 0$. Therefore, we can specify a number $c^{(10)} > 0$, such that for all $h \in (0, h_1)$ the inequality is true

$$\exp\{\delta(1 + c^{(9)}\alpha^{-2})\vartheta\} \le 1 + \delta c^{(10)}(1 + \alpha^{-2}).$$

Then from (23) follows the relation

$$\int_0^\vartheta |u^h(s)|_r^2 ds \le (1 + c^{(7)}\delta\alpha^{-1})(1 + c^{(10)}\delta\alpha(1 + \alpha^{-2})) \int_0^\vartheta |u_*(s)|_r^2 ds + c^{(11)} h^2 (\delta\alpha)^{-1}. \tag{24}$$

By virtue of condition A, there is such a number $h_* \in (0, h_1)$ such that for all $h \in (0, h_*)$

$$(1 + c^{(7)}\delta\alpha^{-1})(1 + c^{(10)}\delta(1 + \alpha^{-2})) \le 1 + d_2\delta\alpha^{-2}. \tag{25}$$

Inequality (13) follows from (24) and (25). In turn, inequality (12) follows from (23). The lemma is proved.

Remark. If $\delta(h) = d_4 h$, $\alpha(h) = d_5 h^{1/2-\varepsilon}$, where d_4 and d_5 are positive constants, $\varepsilon \in (0, 1/2)$, then the inequalities hold

$$\max_{i=0,m_h-1} \lambda(\tau_{i+1}) \le d_6 h^{1/2-\varepsilon},$$

$$\int_0^\vartheta |u^h(s)|_r^2 ds \le (1 + d_7 h^{2\varepsilon}) \int_0^\vartheta |u_*(s)|_r^2 ds + d_8 h^{1/2+\varepsilon}.$$

It follows from lemma 2

Theorem 1. *Let the conditions of lemma 2 be satisfied. Then there is convergence $u^h(\cdot) \to u_*(\cdot)$ at $h \to 0$.*

The proof of this theorem follows the standard scheme (see, for example, the proof of Theorem 1.2.3 in [3, pp. 21-27]).

Under some additional conditions, an estimate of the convergence rate of the algorithm can be obtained. To justify it, we need the following

Lemma 3 [3, p. 29]. *Let $x_1(\cdot) \in L_\infty(T_*; \mathbb{R}^n)$, $y_1(\cdot) \in W(T_*; \mathbb{R}^n)$, $T_* = [a, b]$, $-\infty < a < b < +\infty$,*

$$\left| \int_a^t x_1(\tau) d\tau \right|_n \le \varepsilon, \quad |y_1(t)|_n \le K, \quad t \in T_*.$$

Then the inequality is true for all $t \in T_*$:

$$\left| \int_a^t (x_1(\tau), y_1(\tau)) d\tau \right| \le \varepsilon(K + \mathrm{var}(T_*; y_1(\cdot))).$$

Here, $\mathrm{var}(T_*; y_1(\cdot))$ denotes the variation of the function $y_1(\cdot)$ on the segment T_*, and $W(T_*; \mathbb{R}^n)$ denotes the set of functions $y(\cdot) : T_* \to \mathbb{R}^n$ with bounded variation.

Lemma 4. *Let $u_*(\cdot)$ be a function of bounded variation, B be a matrix independent of t and y (stationary) matrix, $N \ge r$,* rank $B = r$*. Let the conditions of Lemma 2 also be satisfied. Then we can specify a number $d_9 > 0$ such that for all $h \in (0, h_*)$ the inequality is true.*

$$\int_0^\vartheta |u^h(\tau) - u_*(\tau)|_r^2 d\tau \le d_9(\alpha^{1/2} + h^2(\alpha\delta)^{-1} + \delta\alpha^{-2} + h^{1/2} + h\delta^{-1/2}). \tag{26}$$

Proof. Note that for any $t_1, t_2 \in T$, $t_1 < t_2$, the following relation is true

$$\left| \int_{t_1}^{t_2} B\{u^h(t) - u_*(t)\} dt \right|_N = \left| \int_{t_1}^{t_2} [\dot{y}^h(\tau) - \dot{y}(\tau) - A(y^h(\tau) - y(\tau))] d\tau \right|_N \le$$

$$\le |\mu_h(t_2) - \mu_h(t_1)|_N + k^{(1)} \int_{t_1}^{t_2} |\mu_h(\tau)|_N d\tau,$$

where $\mu_h(t) = y^h(t) - y(t)$. It is not difficult to see that the inequalities are true at $t \in \delta_i$

$$|\mu_h(t)|_N^2 \le k^{(2)}\lambda(\tau_i) + k^{(3)} \left| \int_{\tau_i}^{t} Y(t-s)B(u^h(s) - u_*(s))ds \right|_N \le$$

$$\le k^{(2)}\lambda(\tau_i) + k^{(4)} \int_{\tau_i}^{t} (|u^h(s)|_r + |u_*(s)|_r)ds. \tag{27}$$

In turn, by virtue of (12) and (21) at $t \in \delta_i$, we have

$$\int_{\tau_i}^{t} |u^h(s)|_r ds \le k^{(5)}\delta\alpha^{-1}(\lambda^{1/2}(\tau_i) + h) \le k^{(6)}\delta\alpha^{-1}(\alpha^{1/2} + \delta^{1/2} + h\delta^{-1/2}). \tag{28}$$

Given the convergence of $\delta(h)\alpha^{-2}(h) \to 0$ at $h \to 0$, we conclude that at $h \in (0, h_*)$, the following estimates are valid

$$\delta\alpha^{-1/2} \le k^{(7)}\alpha^{3/2}, \quad \delta^{3/2}\alpha^{-1} \le k^{(8)}\alpha^2, \quad h\delta^{1/2}\alpha^{-1} \le k^{(9)}h. \tag{29}$$

Moreover, in view of (28) and (29) at $t \in \delta_i$, the following estimates are true

$$\int_{\tau_i}^{t} |u^h(s)|_r ds \le k^{(10)}(h + \alpha^{3/2}),$$
$$\int_{\tau_i}^{t} |u_*(s)|_r ds \le k^{(11)}\delta^{1/2} \le k^{(12)}\alpha. \tag{30}$$

From (27), taking into account (30), we derive the following relation, which is valid at $t \in \delta_i$

$$|\mu_h(t)|_N^2 \le k^{(2)}\lambda(\tau_i) + k^{(13)}(h + \alpha). \tag{31}$$

In this case, by virtue of (12), from (31) we obtain

$$\sup_{t \in T} |\mu_h(t)|_N \le k^{(14)}(\alpha + h + h^2\delta^{-1})^{1/2}.$$

Hence we deduce

$$\left| \int_{t_1}^{t_2} (u^h(t) - u_*(t))dt \right|_r \le k^{(15)} \left| \int_{t_1}^{t_2} B(u^h(t) - u_*(t))dt \right|_N \le k^{(16)}(\alpha^{1/2} + h^{1/2} + h\delta^{-1/2}). \tag{32}$$

Again using lemma 2 (see (13)), we set

$$\int_0^{\vartheta} |u^h(\tau) - u_*(\tau)|_r^2 d\tau = \int_0^{\vartheta} |u^h(\tau)|_r^2 d\tau - 2\int_0^{\vartheta} (u^h(\tau), u_*(\tau))d\tau + \int_0^{\vartheta} |u_*(\tau)|_r^2 d\tau \le$$

$$\le (2 + d_2\alpha^{-2}\delta) \int_0^{\vartheta} |u_*(\tau)|_r^2 d\tau - \int_0^{\vartheta} (u^h(\tau), u_*(\tau))d\tau + d_3h^2(\alpha\delta)^{-1} =$$

$$= 2\int_0^{\vartheta} (u_*(\tau) - u^h(\tau), u_*(\tau))d\tau + d_2\alpha^{-2}\delta \int_0^{\vartheta} |u_*(\tau)|_r^2 d\tau + d_3h^2(\alpha\delta)^{-1}. \tag{33}$$

Considering lemma 3 and also (32), we obtain

$$\sup_{t \in T} \left| \int_0^{t} (u_*(\tau) - u^h(\tau), u_*(\tau))d\tau \right| \le k^{(17)}(\alpha^{1/2} + h^{1/2} + h\delta^{-1/2}). \tag{34}$$

Thus, inequality (26) is true for all $h \in (0, h_*)$, $t \in T$, by virtue of (33), (34). The lemma is proved.

3. SOLUTION ALGORITHM IN CASE OF NONLINEAR SYSTEM

Let us specify the algorithm for solving the problem under consideration in the case when the system is nonlinear in phase variable. Let the system (1) have the following form:

$$\dot{y}(t) = f(t, y(t)) + Bu(t), \tag{35}$$

where B is a constant matrix of dimension $N \times r$. Let us assume that the function f is continuous on t, monotone on x, i.e., at some $\omega \geq 0$ the inequality is satisfied

$$(f(t,x) - f(t,y), x - y) \leq -\omega |x - y|_N^2, \quad t \in T, x, y \in \mathbb{R}^N,$$

and satisfies the growth condition

$$|f(t,x)|_N \leq c(1 + |x|_N), \quad t \in T, x \in \mathbb{R}^N,$$

where $c => 0$. If these conditions are satisfied, it is known that at any $u(\cdot) \in L_2(T; \mathbb{R}^r)$, there exists a single solution of the system (35), understood in the sense of Carathéodory. As a model, we take a copy of (35), namely the system

$$\dot{y}^h(t) = f(t, y^h(t)) + Bu^h(t) \tag{36}$$

with initial state of

$$y^h(0) = \xi_0^h.$$

The algorithm for solving the problem, in this case, is similar to the algorithm described above for the linear system. First of all, we select some family Δ_h (8) of partitions of the segment T, as well as the function $\alpha(h) : (0,1) \to (0,1)$.

The values $h \in (0,1)$, $\alpha = \alpha(h)$ and the partition $\Delta_h = \{\tau_{h,i}\}_{i=\overline{0,m_h}}$ of the form (8) are fixed before the algorithm starts. The work of the algorithm is divided into $m - 1$, $m = m_h$ steps of the same type. At i-th step, carried out at the time interval $\delta_i = [\tau_i, \tau_{i+1})$, $\tau_i = \tau_{h,i}$, the following operations are performed. First (at the moment τ_i), the vector u_i^h is calculated according to formula (5), in which

$$U(\tau_i, \xi_i^h, y^h(\tau_i)) = \alpha^{-1} B'(\xi_i^h - y^h(\tau_i)). \tag{37}$$

Then, the control $u^h(t)$ of the form (5), (37) is applied to the input of the system (36). Under the action of this control, the system (36) changes from the state $y^h(\tau_i)$ to the state $y^h(\tau_{i+1})$. The operation of the algorithm ends at the moment ϑ.

As in the linear case, it turns out that at a certain agreement of the values h, $\delta(h)$ and $\alpha(h)$ the function $u^h(\cdot)$ is an approximation of $u_*(\cdot)$. Before proceeding to the proof of this fact, we give a lemma that will be needed later.

Lemma 5. *It is possible to specify such a number $d_{10} > 0$, such that the inequality is satisfied uniformly over all $t \in T$, $y_0 \in \mathbb{R}^N$, $u(\cdot) \in L_2(T; \mathbb{R}^r)$.*

$$\int_0^t |\dot{y}(s; y_0, u(\cdot))|_N^2 ds \leq d_{10} \left(|y_0|_N^2 + \int_0^t |u(s)|_r^2 ds \right).$$

Here $y(\cdot; y_0, u(\cdot))$ is the solution of system (1) with initial state (2) generated by $u(\cdot) \in L_2(T; \mathbb{R}^r)$.

Lemma 6. *Let $\alpha(h) \to 0$, $\delta(h)\alpha^{-2}(h) \to 0$ at $h \to 0$. Then we can specify such a number $h_1 \in (0,1)$, such that for all $h \in (0, h_1)$, $t \in T$ for some positive d_{11}, d_{12}, d_{13}, the inequalities are true.*

$$\max_{i=\overline{0,m_h-1}} \varepsilon_1(\tau_i) \leq d_{11}(\alpha + \delta + h^2\delta^{-1}), \tag{38}$$

$$\int_0^\vartheta |u^h(\tau)|_r^2 d\tau \leq (1 + d_{12}\delta\alpha^{-2}) \int_0^\vartheta |u_*(\tau)|_r^2 d\tau + d_{13}(h^2(\alpha\delta)^{-1} + \delta\alpha^{-1}), \tag{39}$$

where $\varepsilon_1(t) = |y^h(t) - y(t)|_N^2$, $\alpha = \alpha(h)$, $\delta = \delta(h)$.

Proof. Consider the change in the value of $\varepsilon_1(t)$ at $t \in T$. For $t \in \delta_i = [\tau_i, \tau_{i+1})$, $i = \overline{0, m-1}$, we have

$$\frac{d\varepsilon_1(t)}{dt} = 2(y^h(t) - y(t), f(t, y^h(t)) - f(t, y(t)) + B(u_i^h - u_*(t))) \leq$$

$$\le -2\omega\varepsilon_1(t) + 2(y^h(t) - y(t), B(u_i^h - u_*(t))) \le -2\omega\varepsilon_1(t) + \sum_{j=1}^{3} I_{ji}(t), \tag{40}$$

where

$$I_{1i}(t) = 2(y^h(\tau_i) - \xi_i^h, B(u_i^h - u_*(t))),$$
$$I_{2i}(t) = 2\|B\|h(|u_i^h|_r + |u_*(t)|_r),$$
$$I_{3i}(t) = 2\|B\|(|u_i^h|_N + |u_*(t)|_N)\int_{\tau_i}^{\tau_{i+1}} |\dot y^h(s) - \dot y(s)|_N ds.$$

From (40) follows the inequality

$$\varepsilon_1(\tau_{i+1}) \le \varepsilon_1(\tau_i) - 2\omega\int_{\tau_i}^{\tau_{i+1}} \varepsilon_1(s)ds + \int_{\tau_i}^{\tau_{i+1}} \sum_{j=1}^{3} I_{ji}(s)ds. \tag{41}$$

Further, at $t \in \delta_i$ we have

$$\varepsilon_1(\tau_i) = \left| y^h(t) - y(t) - \int_{\tau_i}^{t} (\dot y^h(s) - \dot y(s))ds \right|_N^2 \le 2\varepsilon_1(t) + 2\delta\int_{\tau_i}^{t} |\dot y^h(s) - \dot y(s)|_N^2 ds,$$

therefore

$$-\omega\varepsilon_1(\tau_i) \ge -2\omega\varepsilon_1(t) - 2\omega\delta\int_{\tau_i}^{t} |\dot y^h(s) - \dot y(s)|_N^2 ds.$$

Thus, at $t \in \delta_i$ the inequality is true

$$-2\omega\varepsilon_1(t) \le -\omega\varepsilon(\tau_i) + 2\omega\delta\int_{\tau_i}^{t} |\dot y^h(s) - \dot y(t)|_N^2 ds.$$

Hence, after integration at $t \in [\tau_i, \tau_{i+1}]$, we obtain

$$-2\omega\int_{\tau_i}^{t} \varepsilon_1(s)ds \le -\omega\delta\varepsilon_1(\tau_i) + 2\omega\delta^2\int_{\tau_i}^{t} |\dot y^h(s) - \dot y(s)|_N^2 ds. \tag{42}$$

From (41), (42), considering in (42) $t = \tau_{i+1}$, we deduce

$$\varepsilon_1(\tau_{i+1}) \le (1 - \omega\delta)\varepsilon_1(\tau_i) + \tilde I_{1i} + \sum_{j=1}^{3}\int_{\tau_i}^{\tau_{i+1}} I_{ji}(s)ds, \tag{43}$$

where

$$\tilde I_{1i} = 4\omega\delta^2\int_{\tau_i}^{\tau_{i+1}} (|\dot y^h(s)|_N^2 + |\dot y(s)|_N^2)ds.$$

Further, taking into account the definition of u_i^h (see (5), (37)), we conclude that the following inequality holds

$$\int_{\tau_i}^{\tau_{i+1}} (I_{1i}(t) + \alpha(|u_i^h|_r^2 - |u_*(t)|_r^2))dt \le 0. \tag{44}$$

It's not hard to see that

$$\int_{\tau_i}^{\tau_{i+1}} I_{2i}(t)dt \le c_0h^2 + \tilde I_{2i}, \tag{45}$$

where

$$\tilde I_{2i} = \delta\int_{\tau_i}^{\tau_{i+1}} (|u_i^h|_r^2 + |u_*(t)|_r^2)dt.$$

In turn, by virtue of (5), (37) and (3), the inequality is true

$$|u_i^h|_r \le \alpha^{-1}c_1(h + \varepsilon_1(\tau_i)),$$

therefore

$$\delta \int_{\tau_i}^{\tau_{i+1}} |u^h(s)|_r^2 ds \le 2\delta^2\alpha^{-2}c_1^2(h^2 + \varepsilon_1(\tau_i)), \tag{46}$$

hence,

$$\delta \int_0^{\tau_{i+1}} |u^h(s)|_r^2 ds \le 2\delta^2\alpha^{-2}c_1^2 \left(\sum_{j=0}^{i} \varepsilon_1(\tau_j) + \vartheta h^2 \delta^{-1} \right). \tag{47}$$

Considering (47), we obtain

$$\sum_{j=0}^{i} \tilde{I}_{2j} \le \delta \int_0^{\tau_{i+1}} |u_*(s)|_r^2 ds + 2\vartheta c_1^2 \delta h^2 \alpha^{-2} + 2c_1^2\delta^2\alpha^{-2} \sum_{j=0}^{i} \varepsilon_1(\tau_j). \tag{48}$$

Then we have

$$\int_{\tau_i}^{\tau_{i+1}} I_{3i}(t)dt \le \tilde{I}_{3i} + \tilde{I}_{2i}, \tag{49}$$

where

$$\tilde{I}_{3i} = \|B\|^2 \delta \int_{\tau_i}^{\tau_{i+1}} (|\dot{y}^h(s)|_N^2 + |\dot{y}(s)|_N^2) ds.$$

By virtue of lemma 5, for all $i = \overline{1, m}$, the following relation is correct

$$\int_0^{\tau_i} (|\dot{y}^h(s)|_N^2 + |\dot{y}(s)|_N^2) ds \le c_2 \left(1 + \int_0^{\tau_i} (|u^h(s)|_r^2 + |u_*(s)|_r^2) ds \right). \tag{50}$$

Then,

$$\sum_{j=0}^{i} \tilde{I}_{1j} \le c_3 \delta \left(1 + \sum_{j=0}^{i} \tilde{I}_{2j} \right), \quad \sum_{j=0}^{i} \tilde{I}_{3j} \le c_4 \left(\delta + \sum_{j=0}^{i} \tilde{I}_{2j} \right).$$

In this case, taking into account (49), we conclude that the inequality holds

$$\sum_{j=0}^{i} \int_{\tau_j}^{\tau_{j+1}} I_{3j}(s) ds \le c_5 \delta + c_6 \sum_{j=0}^{i} \tilde{I}_{2j}. \tag{51}$$

Then from (45), (47), (48), and (51), we obtain

$$\sum_{j=0}^{i} \left(\tilde{I}_{1j} + \int_{\tau_j}^{\tau_{j+1}} (I_{2j}(t) + I_{3j}(t)) dt \right) \le c_7 h^2 \delta^{-1} + c_8 \delta + c_9 \left(\delta^2 \alpha^{-2} \sum_{j=0}^{i} \varepsilon_1(\tau_j) + \delta h^2 \alpha^{-2} \right). \tag{52}$$

In turn, from (43), taking advantage of (44) and (52), we derive the estimation

$$\varepsilon_1(\tau_{i+1}) + \alpha \int_0^{\tau_{i+1}} (|u^h(s)|_r^2 - |u_*(s)|_r^2) ds \le$$

$$\le \varepsilon_1(0) + c_7 h^2 \delta^{-1} + c_8 \delta + c_9 \delta h^2 \alpha^{-2} + c_9 \delta^2 \alpha^{-2} \sum_{j=0}^{i} \varepsilon_1(\tau_j). \tag{53}$$

By virtue of the discrete Gronwall inequality (see lemma 1) from (53), we have

$$\varepsilon_1(\tau_{i+1}) + \alpha \int_0^{\tau_{i+1}} |u^h(s)|_r^2 ds \le$$

$$\le \left(\varepsilon_0(0) + c_7 h^2 \delta^{-1} + c_8 \delta + c_9 \delta h^2 \alpha^{-2} + \alpha \int_0^{\tau_{i+1}} |u_*(s)|_r^2 ds \right) \exp\{c_9 (i+1) \delta^2 \alpha^{-2}\}. \tag{54}$$

Note that

$$\varepsilon_1(0) \le h^2, \quad \exp\{c_9(i+1)\delta^2\alpha^{-2}\} \le \exp\{c_9\vartheta\delta\alpha^{-2}\}.$$

Furthermore, if $\delta(h)\alpha^{-2}(h) \to 0$ at $h \to 0$, then the inequalities are satisfied at $h \in (0, h_1)$, $h_1 \in (0, 1)$

$$\exp\{c_9\vartheta\delta\alpha^{-2}\} \le 1 + c_{10}\delta\alpha^{-2}, \quad \delta\alpha^{-2} \le c_{11},$$

where $c_{10} = c_{10}(h_1) > 0$, $c_{11} = c_{11}(h_1) > 0$.

Thus, in view of (54) at $h \in (0, h_1)$, $i = \overline{0, m-1}$, the inequality is true

$$\varepsilon_1(\tau_{i+1}) + \alpha \int_0^{\tau_{i+1}} |u^h(s)|_r^2 ds \le \alpha(1 + c_{12}\delta\alpha^{-2}) \int_0^{\tau_{i+1}} |u_*(s)|_r^2 ds + c_{13}(h^2\delta^{-1} + \delta),$$

from which inequalities (38) and (39) follow. The lemma is proved.

By means of lemma 6, it can be proved that

Theorem 2. *Let the conditions of lemma* 6 *be satisfied. Suppose also* $h^2(\alpha(h)\delta(h))^{-1} \to 0$ *at* $h \to 0$. *Then, there is convergence of* $u^h(\cdot) \to u_*(\cdot)$ *at* $h \to 0$.

As in the case of a linear system, we can write out an estimate of the convergence rate of the algorithm.

Lemma 7. *Let the conditions of Theorem* 2 *be satisfied. Let also the function* $y \to f(t, y)$ *be a Lipschitz function,* $r \le N$, rank $B = r$. *Then at* $h \in (0, h_1)$*, the following estimate of the convergence rate of the algorithm takes place:*

$$\int_0^{\vartheta} |u^h(s) - u_*(s)|_r^2 ds \le d_{14}(\alpha^{1/2} + \delta^{1/2} + h\delta^{-1/2} + h\alpha^{-1/2} + \delta\alpha^{-2} + h^2(\alpha\delta)^{-1}). \tag{55}$$

Proof. The proof of the lemma is similar to the proof of Lemma 4. Indeed, let L be the Lipschitz constant of the function f. It is easy to see that at a.e. $t \in \delta_i$, the following relation is true

$$\dot{\varepsilon}_1(t) \le -2\omega\varepsilon_1(t) + I_{4i}(t) + I_{3i}(t) \le I_{4i}(t) + I_{3i}(t), \tag{56}$$

where

$$I_{4i}(t) = 2(y^h(\tau_i) - y(\tau_i), B(u_i^h - u_*(t))).$$

Note that the inequality is true at $t \in \delta_i$

$$\left| \int_{\tau_i}^t I_{4i}(s) ds \right|_N \le \varepsilon_1(\tau_i) + 2\|B\|^2 \tilde{I}_{2i},$$

therefore (see (49)) for all $t \in \delta_i$, the estimate is true:

$$\left| \int_{\tau_i}^t (I_{4i}(s) + I_{3i}(s)) ds \right|_N \le \varepsilon_1(\tau_i) + \tilde{I}_{3i} + (1 + 2\|B\|^2)\tilde{I}_{2i}. \tag{57}$$

Under the conditions of Theorem 2, we can consider that at $h \in (0, h_1)$, the following relations take place:

$$\max_{i=\overline{0, m_h}} \varepsilon_1(\tau_i) \le k_1, \quad \delta\alpha^{-2} \le k_2. \tag{58}$$

Using (39), we obtain

$$\int_0^{\vartheta} |u^h(s)|_N^2 ds \le k_3(1 + \delta\alpha^{-2} + h^2\delta^{-1}\alpha^{-1}). \tag{59}$$

In turn, by virtue of (46), (50), (58), (59) and lemma 6, the inequalities are true at $h \in (0, h_1)$

$$\tilde{I}_{2i} \le k_4\delta + k_5\delta^2\alpha^{-2}(h^2 + \varepsilon_1(\tau_i)) \le k_6\delta, \tag{60}$$

$$\tilde{I}_{3i} \le k_7\delta + k_8\delta \int_0^{\tau_i} |u^h(s)|_r^2 ds \le k_9\delta + k_{10}(h^2\alpha^{-1} + \delta^2\alpha^{-2}) \le k_{11}(\delta + h^2\alpha^{-1}). \tag{61}$$

In view of (58)

$$\alpha^{-1} \le k_{12}\delta^{-1/2} \le k_{13}\delta^{-1}.$$

In this case, taking into account (57), (60), (61), from (56) we obtain the relations valid at $t \in \delta_i$

$$\varepsilon_1(t) \le 2\varepsilon_1(\tau_i) + k_{11}(\delta + h^2\alpha^{-1}) \le 2\varepsilon(\tau_i) + k_{14}(\delta + h^2\delta^{-1}). \tag{62}$$

Hence, by virtue of (38) and (62) at $t \in \delta_i$ there is a chain of inequalities

$$\left| \int_0^t (u^h(s) - u_*(s))ds \right|_r \le k_{15} \left| \int_0^t (\dot{y}^h(s) - \dot{y}(s) - f(s, y^h(s)) + f(s, y(s)))ds \right|_N \le$$

$$\le k_{15} \left(\varepsilon_1^{1/2}(t) + \varepsilon_1^{1/2}(0) + L \int_0^t \varepsilon_1^{1/2}(s)ds \right) \le k_{16}(\alpha + \delta + h^2\delta^{-1} + h^2\alpha^{-1})^{1/2}.$$

In addition, similarly to (33), (34), the estimates are established

$$\int_0^\vartheta |u^h(s) - u_*(s)|_r^2 ds \le$$

$$\le 2 \int_0^\vartheta (u_*(s) - u^h(s), u_*(s))ds + d_{12}\delta\alpha^{-2} \int_0^\vartheta |u_*(s)|_r^2 ds + d_{13}(h^2(\alpha\delta)^{-1} + \delta\alpha^{-1}), \tag{63}$$

$$\sup_{t \in T} \left| \int_0^t (u^h(s) - u_*(s), u_*(s))ds \right| \le k_{18}(\alpha + \delta + h^2\delta^{-1} + h^2\alpha^{-1})^{1/2}. \tag{64}$$

Lemma 3 is used to derive inequality (64). Inequality (55) follows from inequalities (63) and (64). The lemma is proved.

CONFLICT OF INTERESTS

The author of this paper declares that he has no conflict of interests.

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CONTROL THEORY

ON THE PROBLEM OF PURSUING A GROUP OF COORDINATED EVADERS IN A GAME WITH FRACTIONAL DERIVATIVES

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Abstract. In a finite-dimensional Euclidean space, the problem of pursuing of a group of evaders by a group of pursuers is considered, described by a linear non-stationary system of differential equations with fractional Caputo derivatives. Sets of admissible players' controls — compacts, terminal sets — origin of coordinates. Sufficient conditions have been obtained for the capture of at least one evader and all evaders under the condition that the evaders use the same control. In the study, the method of matrix and scalar resolving functions is used as a basic one. It is shown that differential games described by equations with fractional derivatives have properties that are different from those of differential games described by ordinary differential equations.

Keywords: *differential game, group pursuit, pursuer, evader, fractional derivative*

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1. INTRODUCTION

One of the directions of development of the modern theory of differential games is the study of pursuit-evasion problems with participation of a group of participants [1–4], and besides deepening of classical methods of investigation the search of game problems to which previously developed methods are applicable is actively conducted.

Differential games with fractional derivatives were first considered in [5], where the method of scalar resolving functions was used for the study. Differential games with fractional derivatives based on the Hamilton-Jacobi equation were studied in [6]. In [7], the problem of pursuit by a group of pursuers of a single evader in differential games described by equations with fractional derivatives was considered. The problem of conflict interaction between a group of pursuers and a group of evaders in games with fractional dynamics was considered in [8], scalar resolving functions were used for analysis. A. A. Chikrii, in his paper [9], notes that scalar resolving functions attract a terminal set with images of some multivalued mappings that occur in a cone stretched over this set, which limits the possibilities for the pursuer's maneuver, and also proposes to use matrix resolving functions to analyze two-person pursuit games. In [10], matrix resolution functions were applied to study the problem of pursuit by a group of pursuers of a single evader described by a stationary linear system with fractional Caputo derivatives.

In [11], the problem of pursuit by a group of pursuers of a group of evaders in linear stationary differential games with simple matrices under the condition that all evaders use the same control was considered. Sufficient conditions for catching at least one evader were obtained. The pursuit problem in which all evaders use the same control will be referred to as the *coordinated evaders pursuit problem*.

In this paper we consider the problem of conflict interaction between a group of pursuers and a group of evaders in a differential game described by a nonstationary linear system of differential equations with fractional Caputo derivatives. Under the condition that the evaders use the same control, sufficient conditions for catching at least one evader are obtained, using matrix or scalar resolving functions. The study of the nonstationary case is supplemented by some results for games described by linear stationary systems with a simple matrix.

1. PROBLEM STATEMENT

In the space $\mathbb{R}^k$ ($k \geqslant 2$) we consider a differential game of $n+m$ persons: n pursuers $P_1, \ldots, P_n$ and m evaders $E_1, \ldots, E_m$, described by a system of the form

$$(D^{(\alpha)})z_{ij} = A_{ij}(t)z_{ij} + u_i - v, \quad z_{ij}(t_0) = z_{ij}^0, \quad u_i \in U_i, v \in V. \tag{1}$$

Here, $i \in I = \{1, \ldots, n\}$, $j \in J = \{1, \ldots, m\}$, $z_{ij}, u_i, v \in \mathbb{R}^k$, U_i, V are compact sets $\mathbb{R}^k$, $\alpha \in (0,1)$, $D^{(\alpha)}x$ is Caputo derivative of the function x of order α [12], $A_{ij}(t)$ are continuous matrix functions of order $k \times k$. Terminal sets M_{ij}^* of the form

$$M_{ij}^* = M_{ij} + M_{ij}^0,$$

where M_{ij} is a linear subspace of $\mathbb{R}^k$, M_{ij}^0 are convex compact sets from L_{ij} – the orthogonal complement of M_{ij} to $\mathbb{R}^k$. We consider $z_{ij}^0 \notin M_{ij}^*$ for all $i \in I$, $j \in J$.

The actions of the evaders can be interpreted as follows: there is a center that, for all evaders $E_1, \ldots, E_m$, chooses the same control $v(\cdot)$.

Let $v : [t_0, +\infty) \to V$ be a measurable function, which we will call *admissible*. The *prehistory* of $v_t(\cdot)$, at the moment t of the function $v(\cdot)$, will be called the contraction of the function v at $[t_0, t]$.

2. SUFFICIENT CATCHING CONDITIONS

Definition 1. We will say that a quasi-strategy $\mathcal{U}_i$ of the pursuer P_i is defined, if a mapping U_i^0, that puts the measurable function $u_i(t)$ with values in U_i in accordance with the initial positions of $z^0 = (z_{ij}^0, i \in I, j \in J)$, the moment t, and an arbitrary control prehistory $v_t(\cdot)$ of the evader $E_j, j \in J$, is defined.

Let's denote this game by $G(n, m, z^0)$.

Definition 2. A capture of at least one evader occurs in the game $G(n, m, z^0)$, if there exist moment $T > 0$, quasi-strategies $\mathcal{U}_1, \ldots, \mathcal{U}_n$ of pursuers $P_1, \ldots, P_n$ such that for any measurable function $v(\cdot), v(t) \in V, t \in [t_0, T]$, there exist moment $\tau \in [t_0, T]$ and numbers $p \in I, q \in J$, for which $z_{pq}(\tau) \in M_{pq}$.

Let us introduce the following notations: E^0 is a identity matrix of order $k \times k$, $\pi_{ij} : \mathbb{R}^k \to L_{ij}$ is an orthogonal projection operator,

$$\Gamma(\beta) = \int_0^{+\infty} s^{\beta-1} e^{-s} ds, \quad {}_\tau J_t f(t) = \frac{1}{\Gamma(\alpha)} \int_\tau^t (t-s)^{\alpha-1} f(s) ds,$$

$$G_{ij}^0(t,\tau) = \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} E^0,$$

$$G_{ij}^{l+1}(t,\tau) = {}_\tau J_t (A_{ij}(t) G_{ij}^l(t,\tau)), \quad l = 0, 1, \ldots, \quad \Phi_{ij}(t,\tau) = \sum_{l=0}^{+\infty} G_{ij}^l(t,\tau),$$

$$\widetilde{G}_{ij}^0(t,\tau) = E^0, \quad \widetilde{G}_{ij}^{l+1}(t,\tau) = {}_\tau J_t (A_{ij}(t) \widetilde{G}_{ij}^l(t,\tau)), \quad l = 0, 1, \ldots, \quad \Psi_{ij}(t,\tau) = \sum_{l=0}^{+\infty} \widetilde{G}_{ij}^l(t,\tau),$$

$$W_{ij}(t,\tau,v) = \pi_{ij} \Phi_{ij}(t,\tau)(U_i - v), \quad W_{ij}(t,\tau) = \bigcap_{v \in V} W_{ij}(t,\tau,v),$$

Int A, co A are the interior and the convex hull of the set A, respectively.

Assumption 1. *There exists a mapping* $q : I \to J$*, such that for all* $i \in I, t \geqslant t_0, \tau \in [t_0, t]$ *the following condition is satisfied*

$$W_{iq(i)}(t,\tau) \neq \emptyset.$$

Remark 1. Fulfillment of assumption 1 will allow further organizing the pursuit of evaders, so that each pursuer will carry out the capture of "its" evader.

It follows from the measurable choice theorem [13, Theorem 8.1.3], that for every $i \in I$ for any $\geqslant t_0$, there exists at least one measurable selector $\gamma_{iq(i)}(t,\tau) \in W_{iq(i)}(t,\tau)$ for all $t \geqslant t_0, \tau \in [t_0, t]$. Let us choose arbitrary measurable selectors $\gamma_{iq(i)}(t,\tau)$, fix them and denote

$$\xi_{iq(i)}(t) = \pi_{iq(i)}\Psi_{iq(i)}(t,t_0)z^0_{iq(i)} + \int_{t_0}^{t} \gamma_{iq(i)}(t,\tau)d\tau.$$

Theorem 1. *Let Assumption* 1 *be satisfied, and there exist* $T > t_0, l \in I$ *such that* $\xi_{lq(l)}(T) \in M^0_{lq(l)}$. *Then a capture occurs in the game* $G(n, m, z^0)$.

Proof. Let's consider the multivalued mapping ($\tau \in [t_0, T], v \in V$):

$$U_l(T,\tau,v) = \{u_l \in U_l : \pi_{lq(l)}\Phi_{lq(l)}(T,\tau)(u - v) - \gamma_{lq(l)}(T,\tau) = 0\}.$$

By assumption 1, $U_l(T,\tau,v) \neq \emptyset$ for all $\tau \in [t_0, T], v \in V$. It follows from the measurable choice theorem [13, Theorem 8.1.3], that there exists a measurable selector $u^*_l(\tau, v) \in U_l(T,\tau,v)$. We assume the control of the pursuer P_l is equal to

$$u_l(\tau) = u^*_l(\tau, v(\tau)), \quad \tau \in [t_0, T].$$

The controls of the other pursuers are set arbitrarily. The solution of the Cauchy problem for the system (1) is represented as [14]

$$z_{lq(l)}(T) = \Psi_{lq(l)}(T,t_0)z^0_{lq(l)} + \int_{t_0}^{T} \Phi_{lq(l)}(T,s)(u_l(s) - v(s))ds,$$

therefore

$$\pi_{lq(l)}z_{lq(l)}(T) = \xi_{lq(l)}(T) + \int_{t_0}^{T} (\pi_{lq(l)}\Phi_{lq(l)}(T,s)(u_l(s) - v(s)) - \gamma_{lq(l)}(T,s))ds = \xi_{lq(l)}(T) \in M^0_{lq(l)}.$$

This means that a capture of at least one evader occurs in the game $G(n, m, z^0)$. The theorem is proven.

In the following, we will assume that $\xi_{iq(i)}(t) \notin M^0_{iq(i)}$ is for all $i \in I, t \geqslant t_0$.

Consider an arbitrary diagonal square matrix Λ_i of order $k_i \times k_i$, where k_i is the dimension of $L_{iq(i)}$, of the form

$$\Lambda_i = \begin{pmatrix} \lambda_{i1} & 0 & \dots & 0 \\ 0 & \lambda_{i2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_{ik_i} \end{pmatrix} = \text{diag}(\lambda_{i1}, \lambda_{i2}, \dots, \lambda_{ik_i}).$$

We will identify the matrix Λ_i with the vector $(\lambda_{i1}, \dots, \lambda_{ik_i})$. We will understand the inequality $\Lambda_i \geqslant 0$ coordinatewise. Let us introduce multivalued mappings

$$M_i(t,\tau,v) = \{\Lambda_i : \Lambda_i \geqslant 0, \Lambda_i(M^0_{iq(i)} - \xi_{iq(i)}(t)) \cap (W_{iq(i)}(t,\tau,v) - \gamma_{iq(i)}(t,\tau)) \neq \emptyset\}.$$

Due to the properties of the parameters of the conflict-controlled process, the mappings $M_i(t,\tau,v)$ are (τ, v) measurable mappings [15]. Let us define the scalar functions

$$\lambda^0_i(t,\tau,v) = \sup_{\Lambda_i \in M_i(t,\tau,v)} \min_{l \in J_i} \lambda_{il}(t,\tau,v), \quad J_i = \{1, \dots, k_i\}. \tag{2}$$

Assumption 2. *For all* $t \geqslant t_0, \tau \in [t_0, t], v \in V$, *an exact upper bound is achieved in* (2).

We consider this assumption to be satisfied. Let us define the set

$$M^*_i(t,\tau,v) = \{\Lambda_i(t,\tau,v) \in M_i(t,\tau,v) : \lambda^0_i(t,\tau,v) = \min_{l \in J_i} \lambda_{il}(t,\tau,v)\}.$$

It follows from [15], that under the assumptions made, $M^*_i(t,\tau,v)$ is measurable by (τ, v) and closed-valued at any $t \geqslant 0$. By the measurable choice theorem [13, Theorem 8.1.3], for each $i \in I$ in $M^*_i(t,\tau,v)$, there exists at least one selector measurable by (τ, v) at any $t \geqslant 0$. Let us fix these selectors and denote them by $\Lambda^*_i(t,\tau,v) = \text{diag}(\lambda^*_{i1}(t,\tau,v), \dots, \lambda^*_{ik_i}(t,\tau,v))$. Let further

$$\delta(t,\tau) = \inf_{v \in V} \max_{i \in I} \min_{l \in J_i} \lambda^*_{il}(t,\tau,v).$$

Lemma 1. *Let assumptions* 1, 2 *be satisfied,*

$$\lim_{t\to+\infty}\int_{t_0}^{t}\delta(t,s)ds=+\infty. \tag{3}$$

Then, there exists a moment $T>t_0$ such that for each measurable function $v(\cdot)$, $v(t)\in V$, $t\in[t_0,T]$, there exists a number $l\in I$, such that for all $p\in J_l$ the inequalities are true:

$$\int_{t_0}^{T}\lambda_{lp}^*(T,s,v(s))ds\geqslant 1.$$

Proof. Let $v(\cdot)$ be an arbitrary admissible function. Then for all $t\geqslant t_0$, $s\in[t_0,t]$, $l\in I$, $p\in J_l$, the inequalities are true:

$$\lambda_{lp}^*(t,s,v(s))\geqslant\lambda_l^*(t,s,v(s)). \tag{4}$$

In addition, relations are true,

$$\max_{l\in I}\int_{t_0}^{t}\lambda_l^*(t,s,v(s))ds\geqslant\frac{1}{n}\int_{t_0}^{t}\sum_{l\in I}\lambda_l^*(t,s,v(s))ds\geqslant\frac{1}{n}\int_{t_0}^{t}\max_{l\in I}\lambda_l^*(t,s,v(s))ds\geqslant\frac{1}{n}\int_{t_0}^{t}\delta(t,s)ds.$$

It follows from condition (3) that there exists a number $T>t_0$, for which

$$\frac{1}{n}\int_{t_0}^{T}\delta(T,s)ds\geqslant 1.$$

Hence,

$$\max_{l\in I}\int_{t_0}^{T}\lambda_l^*(T,s,v(s))ds\geqslant 1,$$

so there is a number $l\in I$, for which

$$\int_{t_0}^{T}\lambda_l^*(T,s,v(s))ds\geqslant 1.$$

From the last inequality and inequality (4), the validity of the statement of the lemma follows.

Let's find the number

$$T_0=\inf\left\{t\geqslant t_0:\inf_{v(\cdot)}\max_{l\in I}\min_{p\in J_l}\int_{t_0}^{t}\lambda_{lp}^*(t,s,v(s))ds\geqslant 1\right\}.$$

Consider the sets ($i\in I$, $p\in J_l$)

$$T_{ip}(v(\cdot))=\left\{t\geqslant t_0:\int_{t_0}^{t}\lambda_{ip}^*(T_0,s,v(s))ds\geqslant 1\right\}.$$

Let's determine the values

$$t_{ip}^*(v(\cdot))=\begin{cases}\inf\{t:t\in T_{ip}(v(\cdot))\}, & \text{if } T_{ip}(v(\cdot))\neq\emptyset,\\ +\infty, & \text{if } T_{ip}(v(\cdot))=\emptyset.\end{cases}$$

Assumption 3. 1) *For all $\tau\in[t_0,T_0]$, $v\in V$, $l\in I$, $J_l^0\subset J_l$, selectors $B_l(T_0,\tau,v)=diag(\beta_{l1}(T_0,\tau,v),\ldots$ $\ldots,\beta_{lk_l}(T_0,\tau,v))$ where*

$$\beta_{lp}(T_0,\tau,v)=\begin{cases}\lambda_{lp}^*(T_0,\tau,v), & \text{if } p\in J_l^0,\\ 0, & \text{if } p\notin J_l^0,\end{cases}$$

satisfy the condition $B_l(T_0,\tau,v)\subset M_l(T_0,\tau,v)$.

2) $\int_{t_0}^{T_0}B_l(T_0,s,v(s))M_{lq(l)}^0ds\subset M_{lq(l)}^0$.

Theorem 2. *Let assumptions* 1–3 *and condition* (19) *be satisfied. Then, at least one evader is captured in the game $G(n,m,z^0)$.*

Proof. It follows from lemma 1, that $T_0 < +\infty$. Let $v : [t_0, T_0] \to V$ be an arbitrary admissible function. Let us introduce the functions $B_l^*(T_0, t, v) = \text{diag}(\beta_{l1}^*(T_0, t, v), \ldots, \beta_{lk_l}^*(T_0, t, v))$, where

$$\beta_{lp}^*(T_0, t, v) = \begin{cases} \lambda_{lp}^*(T_0, t, v), & \text{if } t \in [t_0, t_{lp}^*(v(\cdot))), \\ 0, & \text{if } t \in [t_{lp}^*(v(\cdot)), T_0]. \end{cases}$$

By assumption 3, $B_i^*(T_0, t, v)$ is a measurable selector of $M_i(T_0, t, v)$. Consider multivalued mappings

$$U_i(T_0, t, v) = \{u_i \in U_i : \pi_{iq(i)}\Phi_{iq(i)}(T_0, t)(u_i - v) - \gamma_{iq(i)}(T_0, t) \in B_i^*(T_0, t, v)(M_{iq(i)}^0 - \xi_{iq(i)}(T_0))\}.$$

Then $U_i(T_0, t, v) \neq \emptyset$ for all $i \in I$, $t \in [t_0, T_0]$, $v \in V$, and hence by the measurable choice theorem [13, Theorem 8.1.3], $U_i(T_0, t, v)$ has at least one measurable selector $u_i^*(T_0, t, v)$. We define the pursuers' controls by assuming $u_i(t) = u_i^*(T_0, t, v(t))$. We'll show that this evaders' control guarantees the capture of at least one evader.

The solution of the Cauchy problem of the system (1) has the form [14]

$$z_{iq(i)}(t) = \Psi_{iq(i)}(t, t_0)z_{iq(i)}^0 + \int_{t_0}^{t} \Phi_{iq(i)}(t, s)(u_i(s) - v(s))ds,$$

therefore

$$\begin{aligned} \pi_{iq(i)}z_{iq(i)}(T_0) &= \pi_{iq(i)}\Psi_{iq(i)}(T_0, t_0)z_{iq(i)}^0 + \int_{t_0}^{T_0} \gamma_{iq(i)}(T_0, s)ds + \\ &+ \int_{t_0}^{T_0} (\pi_{iq(i)}\Phi_{iq(i)}(T_0, s)(u_i(s) - v(s)) - \gamma_{iq(i)}(T_0, s))ds = \\ &= \xi_{iq(i)}(T_0) + \int_{t_0}^{T_0} (\pi_{iq(i)}\Phi_{iq(i)}(T_0, s)(u_i(s) - v(s)) - \gamma_{iq(i)}(T_0, s))ds \in \\ &\in \xi_{iq(i)}(T_0) + \int_{t_0}^{T_0} B_i^*(T_0, s, v(s))(M_{iq(i)}^0 - \xi_{iq(i)}(T_0))ds = \\ &= \xi_{iq(i)}(T_0)\left(E^0 - \int_{t_0}^{T_0} B_i^*(T_0, s, v(s))ds\right) + \int_{t_0}^{T_0} B_i^*(T_0, s, v(s))M_{iq(i)}^0 ds. \end{aligned}$$

From the definition of $B_i^*(T_0, s, v)$ and lemma 1, it follows that there exists a number $l \in I$, for which

$$\int_{t_0}^{T_0} B_l^*(T_0, s, v(s))ds = E^0.$$

Then,

$$\pi_{lq(l)}z_{lq(l)}(T_0) = \int_{t_0}^{T_0} B_l^*(T_0, s, v(s))M_{lq(l)}^0 ds \subset M_{lq(l)}^0.$$

The theorem is proved.

Remark 2. Scalar resolving functions are a special case of matrix resolving functions, since they are represented in the form λE^0, where λ is a non-negative real number.

Example 1. Let the system (1) $k = 2$, $n = m = 1$, $t_0 = 0$, $A_{11}(t) = 0$ for all t, $V = \{0\}$, $z_{11}^0 = (2, 1)$, $M_{11}^* = \{0\}$, $U_1 = \{(u_1, u_2) : u_1 = 0, u_2 \in [-1, 1]\} \cup \{(u_1, u_2) : u_2 = 0,\ u_1 \in [-1, 1]\} \cup \{(u_1, u_2) : u_1 = = u_2 \in [-1, 1]\}$. Then

$$\Psi_{11}(t, t_0) = E^0, \quad \Phi_{11}(t, s) = \frac{(t - s)^{\alpha - 1}}{\Gamma(\alpha)}, \quad W_{11}(t, s, v) = W_{11}(t, s) = \frac{(t - s)^{\alpha - 1}}{\Gamma(\alpha)}U_1.$$

Let's take $\gamma_{11}(t, s) = 0$ for all (t, s), then $\xi_{11}(t) = z_{11}^0$,

$$M_1(t, s, v) = \left\{\begin{pmatrix} 0 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \lambda_2 = \frac{\lambda(t - s)^{\alpha - 1}}{\Gamma(\alpha)}, \lambda \in [0, 1]\right\} \cup$$
$$\cup \left\{\begin{pmatrix} \lambda_2/2 & 0 \\ 0 & 0 \end{pmatrix}, \lambda_2 = \frac{\lambda(t - s)^{\alpha - 1}}{\Gamma(\alpha)}, \lambda \in [0, 1]\right\} \cup \left\{\begin{pmatrix} \lambda_2/2 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \lambda_2 = \frac{\lambda(t - s)^{\alpha - 1}}{\Gamma(\alpha)}, \lambda \in [0, 1]\right\},$$

$$\lambda_1^*(t,s,v) = \sup_{\Lambda \in M_1(t,s,v)} \min_{l \in J_1} \lambda_{1l}(t,s,v) = \frac{(t-s)^{\alpha-1}}{2\Gamma(\alpha)}.$$

Hence,

$$M_1^*(t,s,v) = \operatorname{diag}\left(\frac{(t-s)^{\alpha-1}}{2\Gamma(\alpha)}, \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\right), \quad \delta(t,s) = \frac{(t-s)^{\alpha-1}}{2\Gamma(\alpha)}.$$

We have $\lim_{t\to+\infty}\int_0^t \delta(t,s)ds = +\infty$, so $T_0 = (2\Gamma(\alpha+1))^{1/\alpha}$. Let $T_1 = T_0 - (\Gamma(\alpha+1))^{1/\alpha}$. The control of pursuer P_1 has the form

$$u_1(t) = \begin{cases} (-1,-1), & t \in [0, T_1), \\ (-1, 0), & t \in [T_1, T_0], \end{cases}$$

then [14]

$$z_{11}(T_0) = z_{11}^0 + \frac{1}{\Gamma(\alpha)}\int_0^{T_0} (T_0 - s)^{\alpha-1} u_1(s)ds = 0.$$

Note that the use of scalar resolving functions, i.e., functions of the form

$$\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix},$$

does not allow us to prove the solvability of the pursuit problem, since in this case the condition $-\Lambda z_{11}^0 \in U_1 - v$ is satisfied only for the zero matrix Λ.

Example 2. Consider the game $G(n,1,z^0)$, in which the system (1) has the form

$$\begin{cases} (D^{(\alpha)})z_{i1} = tz_{i2}, \\ (D^{(\alpha)})z_{i2} = u_i - v, \end{cases} \qquad z_i(0) = z_i^0. \tag{5}$$

Here $z_i = (z_{i1}, z_{i2}) \in \mathbb{R}^{2k}$, $U_i = V = \{v \in \mathbb{R}^k : \|v\| \leqslant 1\}$, $M_{i1}^* = \{(z_{i1}, z_{i2}) \in \mathbb{R}^{2k} : z_{i1} = 0\}$, so ($i \in I$)

$$M_{i1}^0 = \{(z_{i1}, z_{i2}) \in \mathbb{R}^{2k} : z_{i1} = z_{i2} = 0\}, \quad M_{i1} = \{(z_{i1}, z_{i2}) \in \mathbb{R}^{2k} : z_{i1} = 0\},$$

$$L_{i1} = \{(z_{i1}, z_{i2}) \in \mathbb{R}^{2k} : z_{i2} = 0\}, \quad \pi_{i1} = \begin{pmatrix} E^0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Let's denote

$$p(t,\tau) = \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}, \quad q(t,\tau) = \frac{\alpha(t-\tau)^{2\alpha-1}(t+\tau)}{\Gamma(2\alpha+1)}, \quad r(t,\tau) = \frac{(t-\tau)^{\alpha}(t+\alpha\tau)}{\Gamma(\alpha+2)}.$$

Then [14]

$$\Psi_i(t,\tau) = \begin{pmatrix} E^0 & r(t,\tau)E^0 \\ 0 & E^0 \end{pmatrix}, \quad \Phi_i(t,\tau) = \begin{pmatrix} p(t,\tau)E^0 & q(t,\tau)E^0 \\ 0 & p(t,\tau)E^0 \end{pmatrix}.$$

Hence,

$$W_i(t,\tau,v) = q(t,\tau)(V - v), \quad W_i(t,\tau) = \{0\}, \quad \gamma_i(t,\tau) = 0, \quad \xi_i(t) = \pi_i \Psi_i(t,0) z_i^0 = z_{i1}^0 + r(t,0) z_{i2}^0,$$

$$\lambda_i(t,\tau,v) = q(t,\tau)\frac{(\xi_i(t), v) + \sqrt{(\xi_i(t), v)^2 + \|\xi_i(t)\|^2(1 - \|v\|^2)}}{\|\xi_i(t)\|^2}.$$

Assertion. *Let $z_{i2}^0 = 0$ for all $i \in I$ and $0 \in \operatorname{Int} \operatorname{co}\{z_{i1}^0, i \in I\}$. Then a capture occurs in the game $G(n,1,z^0)$.*

Proof. In this case, $\xi_{i1}(t) = z_{i1}^0$ for all $t > 0$. It follows from [16], that

$$\delta(t,\tau) = \min_v \max_i \lambda_i(t,\tau,v) \geqslant q(t,\tau)\delta_0$$

for all t, τ with some $\delta_0 > 0$. Therefore, all conditions of Theorem 2 are satisfied and, hence there is a capture in the game $G(n,1,z^0)$. The assertion is proved.

Note that in [14], the problem of pursuit by one pursuer of one evader described by system (5), in which the pursuer has an advantage over the evader, was considered in the space $\mathbb{R}^2$.

3. SUFFICIENT CAPTURE CONDITIONS IN THE LINEAR STATIONARY CASE WITH SIMPLE MATRICES

Theorem 3. *Let in the system (1) for all* i, j $A_{ij}(t) = a_{ij}E^0$ *for any* t, $M_{ij}^* = \{0\}$, $t_0 = 0$, $U_i = V = \{v : \|v\| \leqslant 1\}$, *there exists a mapping* $q : I \to J$ *such that* $a_{iq(i)} < 0$ *for all* $i \in I$ *and*

$$0 \in \operatorname{Int} \operatorname{co}\{z_{iq(i)}^0, i \in I\}. \tag{6}$$

Then a capture of at least one evader occurs in the game $G(n, m, z^0)$.

Proof. In this case

$$\Psi_{iq(i)}(t, t_0) = E_{1/\alpha}(a_{iq(i)}t^\alpha, 1), \quad \Phi_{iq(i)}(t, \tau) = (t - \tau)^{\alpha-1}E_{1/\alpha}(a_{iq(i)}(t - \tau)^\alpha, \alpha),$$

where $E_\rho(z, \mu) = \sum_{l=0}^\infty z^l/\Gamma(l\rho^{-1} + \mu)$ is the Mittag-Leffler function. Assumption 1 is fulfilled.

Let's take $\gamma_{iq(i)}(t, \tau) = 0$ as selectors for all $i \in I$, $t \geqslant 0$, $\tau \in [0, t]$. Then $\xi_{iq(i)}(t) = \pi_{iq(i)}E_{1/\alpha}(a_{iq(i)}t^\alpha, 1)z_{iq(i)}^0$. Let

$$\lambda(z, v) = \sup\{\lambda \geqslant 0 : -\lambda z \in V - v\}, \quad \delta = \min_{v\in V} \max_{i\in I} \lambda(z_{iq(i)}^0, v), \quad a = \min_{i\in I} a_{iq(i)}.$$

It follows from condition (6) and from [16], that $\delta > 0$. Let us show that there exists $T > 0$ such that for any admissible function $v(\cdot)$ there exists $l \in I$, for which

$$E_{1/\alpha}(a_{lq(l)}T^\alpha, 1) - \int_0^T (T - s)^{\alpha-1}E_{1/\alpha}(a_{lq(l)}(T - s)^\alpha, \alpha)\lambda(z_{lq(l)}^0, v(s))ds \leqslant 0. \tag{7}$$

Consider the functions

$$h_i(t, v(\cdot)) = E_{1/\alpha}(a_{iq(i)}t^\alpha, 1) - \int_0^t (t - s)^{\alpha-1}E_{1/\alpha}(a_{iq(i)}(t - s)^\alpha, \alpha)\lambda(z_{iq(i)}^0, v(s))ds.$$

It follows from [17], that for all $t \geqslant 0$, $\tau \in [0, t]$, $i \in I$ the inequalities hold

$$E_{1/\alpha}(a_{iq(i)}(t - \tau)^\alpha, \alpha) \geqslant E_{1/\alpha}(a(t - \tau)^\alpha, \alpha).$$

It follows from Theorem 4.1.1 of [18], that for all $t \geqslant 0$, $\tau \in [0, t]$, the inequality $E_{1,\alpha}(a(t-\tau)^\alpha, \alpha) \geqslant 0$ is true. From the last two inequalities we obtain

$$\begin{aligned}
&\sum_{i=1}^n \int_0^t (t - s)^{\alpha-1}E_{1/\alpha}(a_{iq(i)}(t - s)^\alpha, \alpha)\lambda(z_{iq(i)}, v(s))\, ds \geqslant \\
&\geqslant \int_0^t (t - s)^{\alpha-1}E_{1/\alpha}(a(t - s)^\alpha, \alpha) \max_{i\in I} \lambda(z_{iq(i)}, v(s))\, ds \geqslant \\
&\geqslant \delta \int_0^t (t - s)^{\alpha-1}E_{1/\alpha}(a(t - s)^\alpha, \alpha)\, ds = \delta t^\alpha E_{1/\alpha}(at^\alpha, \alpha + 1),
\end{aligned}$$

hence

$$F(t) = \sum_{i=1}^n h_i(t, v(\cdot)) \leqslant \sum_{i=1}^n E_{1/\alpha}(a_{iq(i)}t^\alpha, 1) - \delta t^\alpha E_{1/\alpha}(at^\alpha, \alpha + 1).$$

Since $a_{iq(i)} < 0$ for all $i \in I$, it follows from [18] that the asymptotic representation is valid at $t \to +\infty$

$$E_{1/\alpha}(a_{iq(i)}t^\alpha, 1) = -\frac{1}{a_{iq(i)}t^\alpha\Gamma(\alpha + 1)} + O\left(\frac{1}{t^{2\alpha}}\right), \quad E_{1/\alpha}(at^\alpha, \alpha + 1) = -\frac{1}{at^\alpha} + O\left(\frac{1}{t^{2\alpha}}\right),$$

$$F(t) = -\sum_{i=1}^n \frac{1}{a_{iq(i)}t^\alpha\Gamma(\alpha + 1)} + \frac{1}{a} + O\left(\frac{1}{t^\alpha}\right),$$

hence $\lim_{t\to+\infty} F(t) < 0$. So $\lim_{t\to+\infty}\sum_{i=1}^n h_i(t, v(\cdot)) < 0$. Since $\sum_{i=1}^n h_i(0, v(\cdot)) > 0$, there exists $T > 0$, for which for any admissible function $v(\cdot)$ the inequality $\sum_{i=1}^n h_i(T, v(\cdot)) < 0$ is true. Thus, inequality (9) is proved.

Let

$$T_0 = \min\left\{t : \inf_{v(\cdot)}\min_{i\in I}\left(E_{1/\alpha}(a_{iq(i)}t^\alpha, 1) - \int_0^t (t-s)^{\alpha-1}E_{1/\alpha}(a_{iq(i)}(t-s)^\alpha, \alpha)\lambda(z^0_{iq(i)}, v(s))ds\right) \leqslant 0\right\}.$$

It follows from inequality (7), that $T_0 < +\infty$. Let $v(\cdot)$ be an admissible fleeing control. Consider the sets

$$T_i(v(\cdot)) = \left\{t \geqslant 0 : E_{1/\alpha}(a_{iq(i)}T_0^\alpha, 1) - \int_0^t (T_0-s)^{\alpha-1}E_{1/\alpha}(a_{iq(i)}(T_0-s)^\alpha, \alpha)\lambda(z^0_{iq(i)}, v(s))ds \leqslant 0\right\}.$$

Let the following be

$$t_i(v(\cdot)) = \begin{cases}\inf\{t : t\in T_i(v(\cdot))\}, & \text{if } T_i(v(\cdot)) \neq \emptyset,\\ +\infty, & \text{if } T_i(v(\cdot)) = \emptyset,\end{cases}$$

$$\beta_i(t, v(\cdot)) = \begin{cases}\lambda(z_{iq(i)}, v(t)), & t\in[0, t_i(v(\cdot))],\\ 0, & t\in[t_i(v(\cdot)), T_0].\end{cases}$$

Let's set the controls of the pursuers P_i, $i \in I$, assuming

$$u_i(t) = v(t) - \beta_i(t, v(\cdot))z^0_{iq(i)}.$$

The solution of the Cauchy problem of the system (1) is represented in the form [19]

$$z_{iq(i)}(T_0) = E_{1/\alpha}(a_{iq(i)}T_0^\alpha, 1)z^0_{iq(i)} + \int_0^{T_0}(T_0-s)^{\alpha-1}E_{1/\alpha}(a_{iq(i)}(T_0-s)^\alpha, \alpha)(u_i(s)-v(s))ds =$$
$$= \left(E_{1/\alpha}(a_{iq(i)}T_0^\alpha, 1) - \int_0^{T_0}(T_0-s)^{\alpha-1}E_{1/\alpha}(a_{iq(i)}(T_0-s)^\alpha, \alpha)\beta_i(s, v(s))ds\right)z^0_{iq(i)} =$$
$$= \left(E_{1/\alpha}(a_{iq(i)}T_0^\alpha, 1) - \int_0^{t_i(v(\cdot))}(T_0-s)^{\alpha-1}E_{1/\alpha}(a_{iq(i)}(T_0-s)^\alpha, \alpha)\beta_i(s, v(s))ds\right)z^0_{iq(i)}.$$

It follows from the previous proof that there exists a number $l \in I$, for which $z_{lq(l)}(T_0) = 0$. The theorem is proved.

Example 3. Let $k = 2$, $I = \{1, 2, 3, 4\}$, $J = \{1, 2\}$, $A_{ij}(t) = a_{ij}E^0$, $a_{ij} < 0$, $U_i = V = \{v : \|v\| \leqslant 1\}$, $z^0_{11} = (1, 3)$, $z^0_{21} = (-1, 3)$, $z^0_{31} = (-1, 1)$, $z^0_{41} = (1, 1)$, $z^0_{12} = (0, -1)$, $z^0_{22} = (-2, -1)$, $z^0_{32} = (-2, -3)$, $z^0_{42} = (0, -3)$. Define a mapping $q : I \to J$ as follows: $q(1) = 2$, $q(2) = q(3) = q(4) = 1$. The conditions of Theorem 3 are satisfied, and so a capture of at least one evader occurs in the game $G(4, 2, z^0)$. Note that $0 \notin \operatorname{Int}\operatorname{co}\{z^0_{i1}, i \in I\}$ and $0 \notin \operatorname{Int}\operatorname{co}\{z^0_{i2}, i \in I\}$.

We show that if $a_{iq(i)} > 0$, then condition (6) in Theorem 3 does not guarantee capture.

Example 4. Let $k = 2$, $n = 3$, $m = 1$, $I = \{1, 2, 3\}$, $M^*_{i1} = \{0\}$, $t_0 = 0$, $z^0_{11} = (0, 1)$, $z^0_{21} = (1/2, -\sqrt{3}/2)$, $z^0_{31} = (-1/2, -\sqrt{3}/2)$, $U_i = V = \{v : \|v\| \leqslant 1\}$. System (1) has the form

$$(D^{(1/2)})z_{i1} = z_{i1} + u_i - v.$$

Let's take $v(t) = 0$ for all $t \geqslant 0$. Then we have

$$z_{i1}(t) = E_2(\sqrt{t}, 1)z^0_{i1} + \int_0^t (t-s)^{-1/2}E_2((t-s)^{1/2}, 1/2)u_i(s)ds.$$

Suppose that there exist $T > 0$, function $u_l(\cdot)$, $l \in \{1, 2, 3\}$, for which $z_{l1}(T) = 0$. Then [20, p. 120, formula (1.15)]

$$E_2(\sqrt{T}, 1) = \|E_2(\sqrt{T}, 1)z^0_{l1}\| = \left\|\int_0^T (T-s)^{-1/2}E_2\big((T-s)^{1/2}, 1/2\big)u_l(s)\,ds\right\| \leqslant$$
$$\leqslant \int_0^T (T-s)^{-1/2}E_2\big((T-s)^{1/2}, 1/2\big)\,ds = \sqrt{T}E_2(\sqrt{T}, 3/2).$$

By virtue of [20, p. 118, formula (1.4)],

$$E_2(\sqrt{T}, 3/2) = \frac{1}{\sqrt{T}}(E_2(\sqrt{T}, 1) - 1).$$

Relation (7) entails the inequality

$$E_2(\sqrt{T}, 1) \leqslant E_2(\sqrt{T}, 1) - 1,$$

which is impossible. Consequently, in this game $G(3, 1, z^0)$, capture does not occur.

4. CAPTURE OF ALL EVADERS

In the space $\mathbb{R}^k$ ($k \geqslant 2$), we consider a differential game $G(1, m, z^0)$ involving $1 + m$ persons: one pursuer P_1 and m evaders $E_1, \ldots, E_m$. The law of motion of the pursuer P_1 has the form

$$(D^{(\alpha)})x_1 = ax_1 + u, \quad x_1(0) = x_1^0, \quad u \in V;$$

the law of motion of each of the evaders E_j is of the form

$$(D^{(\alpha)})y_j = ay_j + v_j, \quad y_j(0) = y_j^0, \quad v_j \in V.$$

Here $V = \{v : \|v\| \leqslant 1\}$, $\alpha \in (0, 1)$, $a \in \mathbb{R}^1$, $D^{(\alpha)}f$ is the Caputo derivative of the function f of order α, $j \in J = \{1, \ldots, m\}$. We consider $x_1^0 \neq y_j^0$ for all $j \in J$.

Let's denote

$$f(t) = E_{1/\alpha}(at^\alpha, 1), \quad F(t) = t^\alpha E_{1/\alpha}(at^\alpha, \alpha + 1), \quad z_j^0 = y_j^0 - x_1^0.$$

Lemma 2. *Let* $a < 0$, $T_2 > T_1 \geqslant 0$,

$$h(t) = \int_{T_1}^{T_2} (t - s)^{\alpha - 1} E_{1/\alpha}(a(t - s)^\alpha, \alpha) ds.$$

Then $\lim_{t\to+\infty} t^\alpha h(t) = 0$.

Proof. By substituting $t - s = \tau$ we get

$$h(t) = \int_{t-T_2}^{t-T_1} \tau^{\alpha-1} E_{1/\alpha}(a\tau^\alpha, \alpha) d\tau.$$

By virtue of formula (2.32) from [20, p. 136], the inequality

$$|E_{1/\alpha}(a\tau^\alpha, \alpha)| \leqslant \frac{M}{\tau^\alpha}, \quad M > 0,$$

is true for all $t > T2$, therefore

$$|h(t)| = \left| \int_{t-T_2}^{t-T_1} \tau^{\alpha-1} E_{1/\alpha}(a\tau^\alpha, \alpha) d\tau \right| \leqslant \int_{t-T_2}^{t-T_1} \frac{M\tau^{\alpha-1}}{\tau^\alpha} d\tau = M(\ln(t - T_1) - \ln(t - T_2)).$$

Then

$$|t^\alpha h(t)| \leqslant Mt^\alpha(\ln(t - T_1) - \ln(t - T_2)) = Mt^\alpha \ln\left(1 + \frac{T_2 - T_1}{t - T_2}\right) \leqslant \frac{Mt^\alpha(T_2 - T_1)}{t - T_2}.$$

Since $\lim_{t\to+\infty} \frac{t^\alpha}{t-T_2} = 0$, then $\lim_{t\to+\infty} t^\alpha h(t) = 0$. The lemma is proved.

Theorem 4. *Let* $a < 0$, $M_{1j}^* = \{0\}$ *for all* $j \in J$*, there is* $v_0 \in V$, $\|v_0\| = 1$*, such that* $(y_j^0 - x_1^0, v_0) < 0$ *for all* $j \in J$*. All evaders use constant control* v_0*, the pursuer* P_1 *knows* v_0*. Then a capture of all evaders occurs in the game* $G(1, m, z^0)$.

Proof. 1. We show that there exist a moment T_m and a vector u_m, $\|u_m\| = 1$, for which the equality $x_1(T_m) = y_m(T_m)$ holds, where $x_1(t)$ is the trajectory of the pursuer P_1, using constant control u_m.

Let the pursuer P_1 uses the constant control u on the interval $[0, T_m]$. Then, by virtue of the Cauchy formula [19] and formula (1.15) from [20, p. 120], we have

$$x_1(t) = f(t)x_1^0 + \int_0^t (t-s)^{\alpha-1} E_{1/\alpha}(a(t-s)^\alpha, \alpha) ds \cdot u = f(t)x_1^0 + F(t)u,$$

$$y_m(t) = f(t)y_m^0 + F(t)v_0.$$

The $x_1(t) = y_m(t)$ can be represented as

$$F(t)u = f(t)z_m^0 + F(t)v_0.$$

Let us require that $\|u\| = 1$. For this purpose, consider the function

$$g_m(t) = \|f(t)z_m^0 + F(t)v_0\|^2 - F^2(t) = f^2(t)\|z_m^0\|^2 + 2f(t)F(t)(z_m^0, v_0),$$

where (a, b) is the scalar product of the vectors a and b. It follows from Theorem 4.1.1 [18], that $f(t) > 0$, $F(t) > 0$ for all $t > 0$. Therefore, the equation $g_m(t) = 0$ is equivalent to the equation

$$\frac{f(t)}{F(t)} = -\frac{2(z_m^0, v_0)}{\|z_m\|^2}. \tag{8}$$

Note that $\lim_{t\to+0} \frac{f(t)}{F(t)} = +\infty$. By virtue of Theorem 1.2.1 of [18], we have the asymptotic estimates

$$f(t) = -\frac{1}{at^\alpha \Gamma(1-\alpha)} + O(1/t^{2\alpha}), \quad F(t) = -\frac{1}{a} + O(1/t^\alpha), \tag{9}$$

therefore $\lim_{t\to+\infty} \frac{f(t)}{F(t)} = 0$. Hence, equation (8) has at least one positive root T_m. We now assume that the control of the pursuer P_1 on the interval $[0, T_m]$ is equal to

$$u_m = \frac{f(T_m)}{F(T_m)} z_m^0 + v_0.$$

We obtain that at time T_m, the pursuer P_1 will realize the capture of the evader E_m.

2. Let us further construct a control for the pursuer P_1, that guarantees the capture of E_{m-1}. Suppose that at $[T_m, T_{m-1}]$, the pursuer P_1 uses the constant control u (the moment T_{m-1} will be defined below). Then, by virtue of the Cauchy formula [19] ($t > T_m$),

$$x_1(t) = f(t)x_1^0 + \int_0^{T_m} (t-s)^{\alpha-1} E_{1/\alpha}(a(t-s)^\alpha, \alpha) ds \cdot u_m + \int_{T_m}^t (t-s)^{\alpha-1} E_{1/\alpha}(a(t-s)^\alpha, \alpha) ds \cdot u,$$

$$y_{m-1}(t) = f(t)y_{m-1}^0 + F(t)v_0.$$

Let's denote

$$H_m(t) = \int_{T_m}^t (t-s)^{\alpha-1} E_{1/\alpha}(a(t-s)^\alpha, \alpha) ds, \quad h_m(t) = \int_0^{T_m} (t-s)^{\alpha-1} E_{1/\alpha}(a(t-s)^\alpha, \alpha) ds.$$

Note that $H_m(t) + h_m(t) = F(t)$. Then the equality $x_1(t) = y_{m-1}(t)$ can be represented as

$$f(t)x_1^0 + h_m(t)u_m + H_m(t)u = f(t)y_{m-1}^0 + F(t)v_0$$

or

$$H_m(t)u = f(t)z_{m-1}^0 + F(t)v_0 - h_m(t)u_m.$$

Consider the function

$$g_{m-1}(t) = \|f(t)z_{m-1}^0 + F(t)v_0 - h_m(t)u_m\|^2 - H_m^2(t).$$

Then

$$g_{m-1}(T_m) = \|f(T_m)z^0_{m-1} + F(T_m)v_0 - h_m(T_m)u_m\|^2.$$

Since $F(T_m) = h_m(T_m)$ and $F(T_m)(v_0 - u_m) = -f(T_m)z^0_m$, then

$$g_{m-1}(T_m) = \|f(T_m)z^0_{m-1} - f(T_m)z^0_m\|^2 = f^2(T_m)\|z^0_{m-1} - z^0_m\|^2 > 0.$$

The function $t^\alpha g_{m-1}(t)$ can be written as

$$t^\alpha g_{m-1}(t) = t^\alpha f^2(t)\|z^0_{m-1}\|^2 + 2t^\alpha f(t)F(t)(z^0_{m-1}, v_0) - 2t^\alpha f(t)h_m(t)(z^0_{m-1}, u_m) - \\ - 2t^\alpha F(t)h_m(t)(v_0, u_m) + 2t^\alpha F(t)h_m(t).$$

By virtue of asymptotic estimates (9) and lemma 2, we obtain that the following relations are true

$$\lim_{t\to+\infty} t^\alpha f(t)F(t) = \frac{1}{a^2\Gamma(1-\alpha)}, \quad \lim_{t\to+\infty} t^\alpha f^2(t) = 0,$$

$$\lim_{t\to+\infty} t^\alpha f(t)h_m(t) = 0, \quad \lim_{t\to+\infty} t^\alpha F(t)h_m(t) = 0,$$

so it follows from the inequality $(z^0_{m-1}, v_0) < 0$, that $\lim_{t\to+\infty} t^\alpha g_{m-1}(t) = -\infty$, and hence there exists a moment $T_{m-1} > T_m$, for which $g_{m-1}(T_{m-1}) = 0$.

Choosing now on the interval $[T_m, T_{m-1}]$ control u_{m-1} of the form

$$u_{m-1} = f(T_{m-1})z^0_{m-1} + F(T_{m-1})v_0 - h_m(T_{m-1})u_m/H_m(T_{m-1}),$$

the pursuer P_1 at the moment T_{m-1} will catch the evader E_{m-1}.

3. Let's denote

$$h_l(t) = \int_{T_{l+1}}^{T_l} (t-s)^{\alpha-1}E_{1/\alpha}(a(t-s)^\alpha, \alpha)ds, \quad H_{k+1}(t) = \int_{T_{k+1}}^{t} (t-s)^{\alpha-1}E_{1/\alpha}(a(t-s)^\alpha, \alpha)ds,$$

$$s_l(t) = h_m(t)u_m + \dots + h_l(t)u_l, \quad \hat{s}_l(t) = h_m(t) + \dots + h_l(t), \quad l = m-1, \dots, k+1.$$

Suppose that the vectors $u_m, \dots, u_{k+1}$ and the moments of time $T_m < T_{m-1} < \dots < T_{k+1}$, guaranteeing the pursuer P_1 to catch the evaders $E_m, \dots, E_{k+1}$, are defined, and on the interval $[T_{k+2}, T_{k+1}]$ the vector u_{k+1} is equal to

$$u_{k+1} = f(T_{k+1})z^0_{k+1} + F(T_{k+1})v_0 - s_{k+2}(T_{k+1})/H_{k+2}(T_{k+1}). \tag{10}$$

Let us further construct a control of the pursuer P_1, which guarantees him to catch the evader E_k. Suppose that at $[T_{k+1}, T_k]$, the pursuer P_1 uses the constant control u (the moment T_k will be defined below). Then for $t > T_{k+1}$, by virtue of the Cauchy formula [19], we have

$$x_1(t) = f(t)x^0_1 + \int_0^{T_m} (t-s)^{\alpha-1}E_{1/\alpha}(a(t-s)^\alpha, \alpha)ds \cdot u_m + \int_{T_m}^{T_{m-1}} (t-s)^{\alpha-1}E_{1/\alpha}(a(t-s)^\alpha, \alpha)ds \cdot u_{m-1} + \dots$$

$$\dots + \int_{T_{k+2}}^{T_{k+1}} (t-s)^{\alpha-1}E_{1/\alpha}(a(t-s)^\alpha, \alpha)ds \cdot u_{k+1} + \int_{T_{k+1}}^{t} (t-s)^{\alpha-1}E_{1/\alpha}(a(t-s)^\alpha, \alpha)ds \cdot u,$$

$$y_k(t) = f(t)y^0_k + F(t)v_0.$$

The inequality $x_1(t) = y_k(t)$ can be represented as

$$f(t)x^0_1 + s_{k+1}(t) + H_{k+1}(t)u = f(t)y^0_k + F(t)v_0 \quad \text{или} \quad H_{k+1}(t)u = f(t)z^0_k - s_{k+1}(t) + F(t)v_0.$$

Consider the function

$$g_k(t) = \|f(t)z^0_k - s_{k+1}(t) + F(t)v_0\|^2 - H^2_{k+1}(t),$$

then

$$g_k(T_{k+1}) = \|f(T_{k+1})z^0_k - s_{k+1}(T_{k+1}) + F(T_{k+1})v_0\|^2.$$

It follows from the definition of the functions $H_{k+2}(\cdot)$ and $h_{k+2}(\cdot)$ that $H_{k+2}(T_{k+1}) = h_{k+1}(T_{k+1})$.

Since $s_{k+1}(T_{k+1}) = s_{k+2}(T_{k+1}) + h_{k+1}(T_{k+1})u_{k+1}$, then

$$s_{k+1}(T_{k+1}) = s_{k+2}(T_{k+1}) + H_{k+2}(T_{k+1})u_{k+1}. \tag{11}$$

Using formula (10), let us write equality (11) as

$$s_{k+1}(T_{k+1}) = f(T_{k+1})z^0_{k+1} + F(T_{k+1})v_0.$$

Then

$$g_k(T_{k+1}) = \|f(T_{k+1})z^0_k - f(T_{k+1})z^0_{k+1}\|^2 = f^2(T_{k+1})\|z^0_k - z^0_{k+1}\|^2 > 0.$$

Since $H_{k+1}(t) = F(t) - \hat{s}_{k+1}(t)$, the function $t^\alpha g_k(t)$ can be represented as

$$t^\alpha g_k(t) = t^\alpha f^2(t)\|z^0_k\|^2 + 2t^\alpha f(t)F(t)(z^0_k, v_0) + t^\alpha\|s_{k+1}(t)\|^2 -$$
$$-2t^\alpha F(t)(s_{k+1}(t), v_0) - 2t^\alpha f(t)(s_{k+1}(t), z^0_k) + 2t^\alpha F(t)\hat{s}_{k+1}(t) - t^\alpha \hat{s}^2_{k+1}(t).$$

It follows from lemma 2, that for any l and p

$$\lim_{t\to+\infty} t^\alpha h_l(t)h_p(t) = 0,$$

therefore

$$\lim_{t\to+\infty} t^\alpha\|s_{k+1}(t)\|^2 = \lim_{t\to+\infty} t^\alpha \hat{s}^2_{k+1}(t) = \lim_{t\to+\infty} t^\alpha f^2(t) = 0,$$

hence $\lim_{t\to+\infty} t^\alpha g_k(t) = -\infty$. Therefore, there is a moment $T_k > T_{k+1}$, for which $g_k(T_k) = 0$. Choosing its control u_k on the interval $[T_{k+1}, T_k]$ in the form of

$$u_k = f(T_k)z^0_k + F(T_k)v_0 - s_{k+1}(T_k)/H_{k+1}(T_k),$$

the pursuer P_1 at the moment T_k will catch the fleeing E_k. The theorem is proved.

Corollary. *Let $a < 0$, there exists a hyperplane H such that $y^0_j \in H$ for all $j \in J$, $x^0_1 \notin H$, v_0 the unit normal vector of the hyperplane H, directed into the half-space containing x^0_1. The evaders use constant control v_0. Then a capture of all evaders occurs in the game $G(1, m, z^0)$.*

The validity of this statement follows directly from Theorem 4, since $(y^0_j - x^0_1, v_0) < 0$ for all $j \in J$.

Remark 3. Let the corollary conditions be satisfied and the laws of motion of each participant have the form

$$\dot{x}_1 = ax_1 + u, \quad \dot{y}_j = ay_j + v_j, \quad u, v_j \in V, \quad j \in J. \tag{12}$$

In [2], the problem of evasion a group of evaders from a group of pursuers described by system (12) was considered, where it was shown that in the game $G(1, m, z^0)$, the pursuer P_1 will realize the capture of no more than one evader [2, Corollary 6.3.3, p. 333].

Thus, Theorem 4 shows that differential games described by equations with fractional derivatives have properties that differential games described by ordinary differential equations do not have.

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CONFLICT OF INTERESTS

The authors of this paper declare that they have no conflict of interests.

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BRIEF REPORTS

BASS-GURA FORMULA FOR LINEAR SYSTEM WITH DYNAMIC OUTPUT FEEDBACK

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Abstract. In this paper, we solve the problem of assigning the desired characteristic polynomial of a linear stationary dynamic system with one input and output dynamic feedback in the form of a first-order dynamic compensator. Necessary and sufficient conditions for the existence of the solution of the problem are considered. An explicit formula for the compensator parameters, analogous to the Bass–Gura formula for a state feedback system, is derived.

Keywords: *linear system, output feedback, Bass–Gura formula*

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1. INTRODUCTION. PROBLEM STATEMENT

The Ackerman and Bass-Gura formulas are known from mathematical control theory [1, p. 360], used to solve the problem of assigning the desired characteristic polynomial of a linear stationary system with one input and state feedback, whose behavior is described by equations

$$\dot{x} = Ax + bu, \quad u = -f^{\mathrm{T}}x, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}$ is the scalar control, $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$, $f \in \mathbb{R}^n$.

The characteristic polynomial of the system (1) is the characteristic polynomial of the matrix of the closed-loop system $A - bf^{\mathrm{T}}$. Let us denote by

$$a(\lambda) = \lambda^n + a_1\lambda^{n-1} + \dots + a_n, \quad d(\lambda) = \lambda^n + d_1\lambda^{n-1} + \dots + d_n$$

characteristic polynomial of the matrix A and the desired characteristic polynomial of the matrix $A - bf^{\mathrm{T}}$. Suppose that the matrix

$$X(A,b) = \begin{bmatrix} b & Ab & \dots & A^{n-1}b \end{bmatrix}$$

is nonsingular, which corresponds to the controllability condition of the system (1).

According to Ackerman's formula the required vector f is equal to

$$f^{\mathrm{T}} = \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix} X(A,b)^{-1} d(A).$$

According to the Bass-Gura formula

$$f^{\mathrm{T}} = (\bar{d} - \bar{a})^{\mathrm{T}} H^{-1} X(A,b)^{-1},$$

where

$$H = \begin{bmatrix} a_{n-1} & a_{n-2} & \dots & 1 \\ a_{n-2} & a_{n-3} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \dots & 0 \end{bmatrix}, \quad \bar{a} = \begin{bmatrix} a_n \\ a_{n-1} \\ \vdots \\ a_1 \end{bmatrix}, \quad \bar{d} = \begin{bmatrix} d_n \\ d_{n-1} \\ \vdots \\ d_1 \end{bmatrix}.$$

In papers [2, 3], the Ackerman formula and the Bass–Gura formula were generalized for systems with multiple inputs and state feedback. The purpose of this paper is to obtain a generalization of the Bass–Gura formula for a system with dynamic output feedback in the form of a first-order dynamic compensator.

It is known [4] that dynamic feedback significantly expands the possibilities of output feedback compared to static feedback. Dynamic output feedback can include state observers as well as dynamic compensators of the general kind. According to the seminal work [5], a dynamic compensator of order $\min\{p_c, p_o\}$, where p_c and p_o are the controllability and observability indices of the system, respectively, can be constructed for a fully controllable and fully observable system. In the case of a system with one input, the minimum order of the compensator is equal to the observability index p_o.

Let us consider a linear stationary system with one input

$$\dot{x} = Ax + bu, \quad y = Cx,$$

where $x \in \mathbb{R}^n$ is the state vector, $y \in \mathbb{R}^l$ is the measurement vector, $u \in \mathbb{R}$ is the scalar control, $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$, $C \in \mathbb{R}^{l\times n}$, $l < n$.

We will search for the control in the form of a dynamic compensator of the first order

$$u = -f^{\mathsf{T}}y - z, \quad \dot{z} + pz = q^{\mathsf{T}}y,$$

where $f \in \mathbb{R}^l$, $p \in \mathbb{R}$, $q \in \mathbb{R}^l$ are the compensator parameters. The system with compensator is described by the equations

$$\dot{x} = (A - bf^{\mathsf{T}}C)x - bz, \quad \dot{z} = q^{\mathsf{T}}Cx - pz. \tag{2}$$

The characteristic polynomial of the system (2) is the characteristic polynomial of the matrix of the closed-loop system

$$D = \begin{bmatrix} A - bf^{\top}C & -b \\ q^{\top}C & -p \end{bmatrix}.$$

We will search for the compensator parameters taking into account the properties of the given characteristic polynomial of the matrix D. For this purpose, it is necessary to obtain an explicit formula for the feedback parameters similar to the Bass–Gura formula.

2. KEY FINDINGS

Let's denote by

$$a(\lambda) = \det(\lambda E - A) = \lambda^n + a_1\lambda^{n-1} + \dots + a_n$$

the characteristic polynomial of the matrix A. Let us introduce a column vector $g(\lambda) = C(\lambda E - A)^*b$, where $(\lambda E - A)^*$ is the adjoint matrix to $\lambda E - A$.

Lemma. *The characteristic polynomial of the matrix D is*

$$\det(\lambda E - D) = (\lambda + p)a(\lambda) + (f^{\top}(\lambda + p) + q^{\top})g(\lambda). \tag{3}$$

Proof. Following simple transformations in the determinant of the matrix $\lambda E - D$, we obtain

$$\det(\lambda E - D) = \det\begin{bmatrix} \lambda E - A + bf^{\top}C & b \\ -q^{\top}C & \lambda + p \end{bmatrix} = \det\begin{bmatrix} \lambda E - A + b(f^{\top} + (\lambda + p)^{-1}q^{\top})C & b \\ 0 & \lambda + p \end{bmatrix} =$$

$$= (\lambda + p)a(\lambda) + (f^{\top}(\lambda + p) + q^{\top})C(\lambda E - A)^*b = (\lambda + p)a(\lambda) + (f^{\top}(\lambda + p) + q^{\top})g(\lambda).$$

Here the $\det(A + bc^{\top}) = \det A + c^{\top}A^*b$ is applied, where A is a square matrix, A^* is a adjoint matrix to A, b is a column vector, $c^{\top}$ is a row vector [6, p. 133]. The lemma is proved.

The following theorem formulates necessary and sufficient conditions for the existence of a solution to the problem and simultaneously describes the algorithm for calculating the compensator parameters.

Theorem. *The characteristic polynomial of the matrix D can be arbitrarily set by choosing the compensator parameters f, p, q, only when*

$$\operatorname{rank} X(A, b) = n, \quad \operatorname{rank} Y(A, C) = n,$$

where

$$X(A,b) = \begin{bmatrix} b & Ab & \dots & A^{n-1}b \end{bmatrix}, \quad Y(A,C) = \begin{bmatrix} C & CA \end{bmatrix}^{\mathrm{T}}.$$

Proof. Let's denote by

$$d(\lambda) = \lambda^{n+1} + d_1\lambda^n + \dots + d_{n+1} \tag{4}$$

the desired characteristic polynomial of the matrix D. We will search for the compensator parameters from the condition of coincidence of polynomials (3) and (4).

Let's denote

$$\pi_k(\lambda) = \begin{bmatrix} \lambda^k & \lambda^{k-1} & \dots & \lambda & 1 \end{bmatrix}^\top, \quad \bar a = \begin{bmatrix} a_2 & a_3 & \dots & a_n \end{bmatrix}, \quad \bar d = \begin{bmatrix} d_2 & d_3 & \dots & d_{n+1} \end{bmatrix}.$$

Then

$$a(\lambda) = \lambda^n + \begin{bmatrix} a_1 & \bar a \end{bmatrix} \pi_{n-1}(\lambda), \quad \lambda a(\lambda) = \lambda^{n+1} + a_1\lambda^n + \begin{bmatrix} \bar a & 0 \end{bmatrix} \pi_{n-1}(\lambda),$$
$$d(\lambda) = \lambda^{n+1} + d_1\lambda^n + \bar d \pi_{n-1}(\lambda).$$

Let us write the matrix $(\lambda E - A)^*$ as a matrix polynomial [7, p. 91]

$$(\lambda E - A)^* = E\lambda^{n-1} + A_1\lambda^{n-2} + \dots + A_{n-1},$$

where

$$A_1 = A + a_1E, \quad A_2 = A^2 + a_1A + a_2E = AA_1 + a_2E, \quad \dots$$
$$\dots, \quad A_{n-1} = A^{n-1} + a_1A^{n-2} + \dots + a_{n-1}E = AA_{n-2} + a_{n-1}E.$$

Note that by the Cayley-Hamilton theorem,

$$A_n = A^n + a_1A^{n-1} + \dots + a_nE = AA_{n-1} + a_nE = 0.$$

Let's introduce the matrix

$$G = \begin{bmatrix} 1 & a_1 & \dots & a_{n-1} \\ 0 & 1 & \dots & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

Vectors $g(\lambda)$ and $\lambda g(\lambda)$ can be written in the form

$$g(\lambda) = C(\lambda E - A)^* b = C \begin{bmatrix} b & A_1b & \dots & A_{n-1}b \end{bmatrix} \pi_{n-1}(\lambda) = CX(A,b)G\pi_{n-1}(\lambda),$$
$$\lambda g(\lambda) = Cb\lambda^n + C \begin{bmatrix} A_1b & A_2b & \dots & A_nb \end{bmatrix} \pi_{n-1}(\lambda) =$$
$$= Cb\lambda^n + CAX(A,b)G\pi_{n-1}(\lambda) + Cb \begin{bmatrix} a_1 & \bar a \end{bmatrix} \pi_{n-1}(\lambda).$$

The characteristic polynomial (3) of the matrix D is equal to

$$\det(\lambda E - D) = \lambda^{n+1} + a_1\lambda^n + \begin{bmatrix} \bar a & 0 \end{bmatrix} \pi_{n-1}(\lambda) + p\lambda^n + p \begin{bmatrix} a_1 & \bar a \end{bmatrix} \pi_{n-1}(\lambda) +$$
$$+ f^{\mathrm{T}}Cb\lambda^n + f^{\mathrm{T}}CAX(A,b)G\pi_{n-1}(\lambda) + f^{\mathrm{T}}Cb \begin{bmatrix} a_1 & \bar a \end{bmatrix} \pi_{n-1} + (fp+q)^{\mathrm{T}}CX(A,b)G\pi_{n-1}(\lambda). \tag{5}$$

The given polynomial (4) and the polynomial (5) coincide if and only if

$$a_1 + p + f^\top Cb = d_1, \tag{6}$$

$$\begin{bmatrix} \bar a & 0 \end{bmatrix} + p \begin{bmatrix} a_1 & \bar a \end{bmatrix} + f^\top CAX(A,b)G + f^\top Cb \begin{bmatrix} a_1 & \bar a \end{bmatrix} + (fp+q)^\top CX(A,b)G = \bar d. \tag{7}$$

Let us denote $r = fp + q$. From equation (6) express p and substitute it into (7). Then (7) takes the form

$$\begin{bmatrix} r^\top & f^{\mathrm{T}} \end{bmatrix} Y(A,C)X(A,b)G = \bar d - \begin{bmatrix} \bar a & 0 \end{bmatrix} - (d_1 - a_1) \begin{bmatrix} a_1 & \bar a \end{bmatrix}. \tag{8}$$

Let f and r be solutions of equation (8). Then from relation (6), we obtain $p = d_1 - a_1 - fCb$, and $q = r - fp$.

Equation (8) has a solution with respect to the unknowns f and r for any vector $\bar{d}$, if and only if $\operatorname{rank} Y(A,C)X(A,b)G = n$. The matrix G is nonsingular. The matrices $X(A,b)$ and $Y(A,C)$ have the dimensions $n \times n$ and $2l \times n$, respectively. Hence, $\operatorname{rank} Y(A,C)X(A,b)G = n$ if and only if $\operatorname{rank} X(A,b) = n$ and $\operatorname{rank} Y(A,C) = n$. The theorem is proved.

Remark. It follows from the theorem, that the necessary condition for the existence of a solution to the problem is the condition $2l \geq n$. Consequently, the problem has a solution at a sufficiently large number of output variables. For example, at $n = 5$ the number of output variables should be at least 3. This is a significant limitation of the considered output feedback.

If the conditions of the theorem are satisfied and $2l = n$, then the solution of equation (8) is unique. If $2l > n$, then equation (8) has infinitely many solutions.

In the case of a unique solution

$$\begin{bmatrix} r^{\mathrm{T}} & f^{\mathrm{T}} \end{bmatrix} = \left(\bar{d} - \begin{bmatrix} \bar{a} & 0 \end{bmatrix} - (d_1 - a_1)\begin{bmatrix} a_1 & \bar{a} \end{bmatrix}\right) G^{-1} X(A,b)^{-1} Y(A,C)^{-1}. \tag{9}$$

Formula (9) can be considered as an analog of the Bass-Gura formula for a system with state feedback.

Let the conditions of the theorem be satisfied and $2l > n$. Then we can find a partial solution of equation (8):

$$\begin{bmatrix} r^{\mathrm{T}} & f^{\mathrm{T}} \end{bmatrix} = \left(\bar{d} - \begin{bmatrix} \bar{a} & 0 \end{bmatrix} - (d_1 - a_1)\begin{bmatrix} a_1 & \bar{a} \end{bmatrix}\right) G^{-1} X(A,b)^{-1} (Y(A,C)^{\mathrm{T}} Y(A,C))^{-1} Y(A,C)^{\mathrm{T}}.$$

3. NUMERICAL EXAMPLE

Let $n = 6$, $l = 3$,

$$A = \begin{bmatrix} -1.68 & 0.64 & 1.53 & -1.5 & -1.45 & -0.22 \\ 0.89 & 1.48 & 2.35 & 0.78 & -2.21 & -0.08 \\ -0.74 & 0.96 & 1.28 & -2.04 & 1.61 & 1.6 \\ 0.35 & -1.78 & 0.74 & -1.54 & -0.16 & -0.06 \\ 0.15 & -1.05 & -1.19 & 0.65 & -0.22 & -0.54 \\ -0.53 & 0.37 & 0.7 & -0.09 & 0.15 & -0.41 \end{bmatrix},$$

$$b = \begin{bmatrix} -0.47 \\ -0.53 \\ 1.87 \\ 0.79 \\ -0.56 \\ 0.46 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let us set the desired characteristic polynomial of the system (2):

$$d(\lambda) = (\lambda + 0.3)(\lambda + 0.4)(\lambda + 0.5)(\lambda + 0.2 + 0.7i)(\lambda + 0.2 - 0.7i)(\lambda + 0.1 + 0.3i)(\lambda + 0.1 - 0.3i) =$$
$$= \lambda^7 + 1.8\lambda^6 + 1.9\lambda^5 + 1.34\lambda^4 + 0.5979\lambda^3 + 0.17482\lambda^2 + 0.03367\lambda + 0.00318.$$

The conditions of the theorem are fulfilled. The parameters of the compensator are determined uniquely:

$$f^{\mathrm{T}} = \begin{bmatrix} 0.0891861 & -1.5061263 & 14.434942 \end{bmatrix},$$

$$q^{\mathrm{T}} = \begin{bmatrix} 7.3744718 & -10.250088 & 53.52229 \end{bmatrix}, \quad p = -3.0716998.$$

The verification shows that the characteristic polynomial of the matrix D coincides with the given one.

CONFLICT OF INTERESTS

The author of this paper declares that he has no conflict of interests.

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BRIEF REPORTS

OPTIMAL TRAJECTORIES IN THE GRUSHIN α-PLANE

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Abstract. For the Grushin α-plane, optimal trajectories, cutting time, and cutting set are described.

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1. INTRODUCTION. PROBLEM STATEMENT

In sub-Riemannian geometry [1,§ 9.2], the Grushin plane is well known as the simplest example of an almost Riemannian manifold (such a manifold is Riemannian in complement to a special submanifold of dimension one). A natural generalization of this example is α-the Grushin plane when the degeneracy on a special set is of order $\alpha \geq 1$. Extremal trajectories for such a case were parameterized in [2], and their optimality was investigated based on this in [3]. In this paper, an independent study of the optimality of extremal trajectories is carried out using a qualitative approach that does not use the parameterization of these trajectories.

The optimal control problem for the classical Grushin plane is posed as follows [1, § 9.2]:

$$\dot{q} = u_1 X_1 + u_2 X_2, \quad q = (x, y) \in M = \mathbb{R}^2, \quad u = (u_1, u_2) \in \mathbb{R}^2, \tag{1}$$

$$q(0) = q_0, \quad q(t_1) = q_1, \quad l = \int_0^{t_1} (u_1^2 + u_2^2)^{1/2} dt \to \min, \tag{2}$$

where $X_1 = \frac{\partial}{\partial x}$, $X_2 = x\frac{\partial}{\partial y}$.

A natural generalization of this problem (α-Grushin plane) [2, 3] is posed similarly, but for vector fields:

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = |x|^\alpha \frac{\partial}{\partial y}, \quad \alpha \in \mathbb{R}, \alpha \geq 1. \tag{3}$$

Problem (1)–(3) is called an *almost Riemannian problem on the α-plane of Grushin*.

Let us denote the cost function in this problem – the almost Riemannian distance – as $d(q_0, q_1) = \inf\{l(q(\cdot)) : q(\cdot)$ the trajectory of system (1), (2)$\}$. A special set is the set of points $q \in M$, where the set of admissible velocities $\{\dot{q} = u_1 X_1 + u_2 X_2\}$ is not full-dimensional: $Z = \{q = (x, y) \in M : x = 0\}$. If $q_0 \in M \setminus Z$, then the problem locally becomes Riemannian, so the case $q_0 \in Z$, that is considered in this paper, is of special interest.

2. BASIC CONCEPTS AND PROPERTIES

2.1. Symmetries

The problem (1)–(3) has obvious symmetries – reflections

$$(x,y)\mapsto(-x,y),\quad (x,y)\mapsto(x,-y),\quad (x,y)\mapsto(-x,-y). \tag{4}$$

Vector fields X_1, X_2 are independent of y, so parallel translations are also symmetries

$$(x,y)\mapsto(x,y+a),\quad a\in\mathbb{R}. \tag{5}$$

Another one-parameter group of symmetries is given by the flow of the vector field

$$V=x\frac{\partial}{\partial x}+(1+\alpha)y\frac{\partial}{\partial y},\quad (x,y)\mapsto e^{tV}(x,y)=(e^t x, e^{(1+\alpha)t}y),\quad t\in\mathbb{R}, \tag{6}$$

since $[V,X_1]=-X_1$, $[V,X_2]=-X_2$. So the optimal synthesis and, in particular, the distance d are invariant with respect to symmetries (4), (5) and homogeneous of order 1 with respect to symmetry (6): $d(e^{tV}(q_0),e^{tV}(q_1))=e^t d(q_0,q_1)$, $q_i\in M$, $t\in\mathbb{R}$. Given the symmetry (5), we will further assume $q_0=(0,0)$.

2.2. Existence of solutions

System (1) is completely controllable in each of the Riemannian half-planes $\{q\in M : \operatorname{sign} x=\pm1\}$, since in them, the set of admissible velocities is full-dimensional. It is possible to move between these half-planes along the fields $\pm X_1$, so system (1) is quite controllable. Note that at the points $q\in Z$, the condition of the Rashevskii-Chow theorem [4, § 5.3; 5, § 2.2.4] is satisfied only at $\alpha\in 2\mathbb{N}$. All conditions of Filippov's theorem [4,§ 10.3; 5,§ 3.1.2] are satisfied, so optimal trajectories exist.

2.3. Extreme trajectories

As usual in sub-Riemannian geometry, we pass from length minimization (2) to energy minimization $J=0.5\int_0^{t_1}(u_1^2+u_2^2)dt$. We apply Pontryagin's maximum principle to the resulting problem [4, § 3; 5, § 5.2; 6, § 12.4; 7, § 3.2.2]. The abnormal trajectories are constant and non-strictly anormal. To parameterize the normal extremal trajectories, we put $X_3=\frac{\partial}{\partial y}$ and denote the Hamiltonians linear on the layers of the cotangent bundle $T^*M : h_i(\lambda)=\langle\lambda,X_i\rangle$, $i=1,3$, $\lambda\in T^*M$. Then the maximized Hamiltonian of the Pontryagin maximum principle is $H=h_1^2+|x|^{2\alpha}h_3^2\equiv1$ and the Hamiltonian system for the normal extremals is of the form

$$\dot h_1=-\alpha\operatorname{sign}x|x|^{2\alpha-1}h_3^2,\quad \dot h_3=0,\quad \dot x=h_1,\quad \dot y=|x|^{2\alpha}h_3. \tag{7}$$

The Hamiltonian H is the first integral, so at each $h_3\neq0$ the independent subsystem of equations (7) for the variables h_1 and x has a phase portrait of type center.

If $h_3=0$, then $h_1\equiv\neq0$, $x=h_1t$, $y=0$. Let $h_3\neq0$. When integrating the system (7) by the method of separation of variables, we obtain the equation

$$\frac{dx}{\sqrt{H-h_3^2|x|^{2\alpha}}}=\pm dt,$$

in which the left part integrates in the general case into a hypergeometric function ${}_2F_1$. On the other hand, in [2], the system (7) is integrated in terms of some generalizations of trigonometric functions. However, we will not use explicit parametrization of the extremal trajectories and investigate the optimality of the extremal trajectories relying only on qualitative methods.

Considering the symmetry $(h_3,y)\mapsto(-h_3,-y)$ of the system (7), further we consider that $h_3>0$. After change of variables $X=xh_3^{1/\alpha}$, $Y=yh_3^{1+1/\alpha}$, $H_1=h_1$, $s=th_3^{1/\alpha}$, the Hamiltonian system (7) will take the form

$$H_1'=-\alpha\operatorname{sign}X|X|^{2\alpha-1},\quad X'=H_1,\quad Y'=|X|^{2\alpha} \tag{8}$$

with the first integral $H = H_1^2 + |X|^{2\alpha} \equiv 1$. Since $H = 1$, we have $H_1(0) = H_1^0 = \pm 1$. Taking advantage of the symmetry $(H_1, X) \mapsto (-H_1, -X)$, we obtain $H_1^0 = 1$. The first two equations of system (8) have a phase portrait of type center in the plane (H_1, X), so for any $\alpha \geq 1$, there exists a unique number $s_* = s_*(\alpha) > 0$ such that

$$X(s) > 0 \quad \text{at} \quad s \in (0, s_*), \quad X(s_*) = 0. \tag{9}$$

Then the first positive root of the function $x(t)$ is $t_* = s_* h_3^{-1/\alpha}$.

3. MAIN RESULTS

Theorem 1. 1. *If $h_3 = 0$, then the extremal trajectory $q(t)$ is optimal on any segment $[0, t_1]$, $t_1 > 0$.*

2. *If $h_3 \neq 0$, then the extremal trajectory $q(t)$ is optimal at any segment $[0, t_1]$, $t_1 \in (0, t_*]$, and not optimal for $t_1 > t_*$, where $t_* = s_* |h_3|^{-1/\alpha}$.*

Proof. Let us first study case 2. Let $h_3 \neq 0$. Consider an exponential mapping

$$\text{Exp} : (\lambda, t) \mapsto q(t), \quad \text{Exp} : \tilde{N} \to M, \quad \tilde{N} = (T_{q_0}^* M \cap \{H = 1\}) \times \mathbb{R}_+,$$

$$\lambda = (h_3, h_1^0), \quad h_3 \in \mathbb{R} \setminus \{0\}, h_1^0 = \pm 1.$$

At any $h_3 \neq 0$, the extremal trajectories $\text{Exp}(h_3, 1, t)$ and $\text{Exp}(h_3, -1, t)$ are symmetric w.r.t. the axis y and intersect this axis at $t = t_*$. Therefore, the intersection point $\text{Exp}(h_3, 1, t_*)$ is a Maxwell point [5, § 3.3.5] and these trajectories are not optimal under the condition $t > t_*$.

Let us now prove that any trajectory $\text{Exp}(h_3, 1, t)$ is optimal at $t \in [0, t_1]$, $t_1 \in (0, t_*)$. Given the symmetries of the problem, we will assume that $h_1^0 = 1$ and $h_3 > 0$, and denote by $\text{Exp}(h_3, t) := \text{Exp}(h_3, 1, t)$. Let $N = \{(h_3, t) \in \mathbb{R}^2 : h_3 > 0, t \in (0, t_*)\}$, $D = \{(x, y) \in M : xy > 0\}$. Let us show that $\text{Exp} : N \to D$ is a diffeomorphism by using the following theorem of Adamar on global diffeomorphism.

Theorem 2 [8; 9, § 6.2]. *Let $F : X \to Y$ be a smooth mapping between smooth manifolds of the same dimension such that X, Y are connected, Y is one-connected, F is nondegenerate and proper. Then F is a diffeomorphism.*

Let us first prove that $\text{Exp}(N) \subset D$. Since $h_3 > 0$ and $t \in (0, t_*)$, then $x(t) > 0$ by virtue of inequality (9). It follows from the ordinary differential equation (8) that $y(t) > 0$. Therefore $\text{Exp}(N) \subset D$.

Obviously, N and D are connected, and D is one-connected. Let us show that $\text{Exp}|_N$ is nondegenerate, i.e., the Jacobian of $\partial(x, y)/\partial(t, h_3)$ is different from zero in the region N. We have $\frac{\partial x}{\partial t} = H_1$, $\frac{\partial y}{\partial t} = h_3^{-1} X^{2\alpha}$, $\frac{\partial x}{\partial h_3} = -\alpha^{-1} h_3^{-1-1/\alpha} X + (\frac{\partial s}{\partial h_3}) H_1 h_3^{-1/\alpha}$, $\frac{\partial y}{} \partial h_3 = -(1 + \frac{1}{\alpha})\, h_3^{-2-1/\alpha}\, Y + (\frac{\partial s}{\partial h_3})\, X^{2\alpha} h_3^{-1-1/\alpha}$, whence $J = h_3^{-2-1/\alpha} \alpha^{-1} J_1$, $J_1 = X^{2\alpha+1} - (\alpha + 1) Y H_1$. Differentiating by virtue of (8), we obtain $J_1' = \alpha X^{2\alpha-1} J_2$, $J_2 = H_1 X + (\alpha + 1) Y$. Differentiating again, we have $J_2' = H_1^2 + X^{2\alpha} > 0$, so $J|_N > 0$, i.e., $\text{Exp}|_N$ is nondegenerate.

Now let us show that the mapping $\text{Exp} : N \to D$ is proper. This is equivalent to the following condition: if the sequence $\{(h_3^n, t^n) : n \in \mathbb{N}\} \subset N$ is not contained in any compact set in N, then its image $q^n = \text{Exp}(h_3^n, t^n)$ is not contained in any compact set in D. Let the sequence $\{(h_3^n, t^n) : n \in \mathbb{N}\} \subset N$ be not contained in any compact set in N, we'll denote $s^n = (h_3^n)^{1/\alpha} t^n \in (0, s_*)$. Then it contains a subsequence for which one of the following conditions is satisfied: 1) $h_3^n \to \bar{h}_3 \in (0, +\infty)$, $s^n \to 0$; 2) $h_3^n \to 0$, $s^n \to 0$; 3) $h_3^n \to 0$, $s^n \to \bar{s} \in (0, s_*)$; 4) $h_3^n \to 0$, $s^n \to s_*$; 5) $h_3^n \to \bar{h}_3 \in (0, +\infty)$, $s^n \to s_*$; 6) $h_3^n \to +\infty$, $s^n \to s_*$; 7) $h_3^n \to +\infty$, $s^n \to \bar{s} \in (0, s_*)$; 8) $h_3^n \to +\infty$, $s^n \to 0$.

We'll show that for each of them, the sequence $q^n = (x^n, y^n)$ contains a subsequence on which one of the following conditions is satisfied: $x^n \to 0$, $x^n \to +\infty$, $y^n \to 0$, $y^n \to +\infty$, i.e., q^n is not contained in any compact set in D.

Given 1) we have $X(s^n) \to X(0) = 0$, so $x^n = X(s^n)/(h_3^n)^{1/\alpha} \to 0$.

If condition 2) is satisfied, the sequence $t^n = s^n/(h_3^n)^{1/\alpha} > 0$ contains a subsequence of one of the following kinds: $t^n \to 0$, $t^n \to \bar{t} \in (0, +\infty)$, $t^n \to +\infty$. If $t^n \to 0$, then $x^n = x(h_3^n, t^n) \to x(0, 0) = 0$. If $t^n \to \bar{t} \in (0, +\infty)$, then $y^n = y(h_3^n, t^n) \to y(0, \bar{t}) = 0$. Let $t^n \to +\infty$. Passing to the subsequence if necessary, we can assume that $\{s^n\}$ is decreasing. There exists a number $K \in \mathbb{N}$ such that $s^K < s_*/2$, so $H_1(s) > 0$ for all $s \in [0, s^K]$. Hence, $H_1|_{[0,s^K]} \geq \varepsilon := \min_{[0,s^K]} H_1 > 0$ and

$$X(s^n) = \int_0^{s^n} H_1(s) ds \geq \varepsilon s^n = \varepsilon t^n (h_3^n)^{1/\alpha}, \quad x^n = \frac{X(s^n)}{(h_3^n)^{1/\alpha}} \geq \varepsilon t^n \to +\infty.$$

For the remaining conditions, we have: 3) $X(s^n) \to X(\bar{s}) \in (0, +\infty)$ and $x^n = \frac{X(s^n)}{(h_3^n)^{1/\alpha}} \to +\infty$; 4) $Y(s^n) \to Y(s_*) = \int_0^{s_*} |X(s)|^{2\alpha} ds \in (0, +\infty)$ and $y^n = \frac{Y(s^n)}{(h_3^n)^{1+1/\alpha}} \to +\infty$; 5) $X(s^n) \to X(s_*) = 0$, from where $x^n = \frac{X(s^n)}{(h_3^n)^{1/\alpha}} \to +0$; 6) $X(s^n) \to X(s_*) = 0$, from where $x^n = \frac{X(s^n)}{(h_3^n)^{1/\alpha}} \to +0$; 7) $X(s^n) \to X(\bar{s}) \in (0, +\infty)$, from where $x^n = \frac{X(s^n)}{(h_3^n)^{1/\alpha}} \to +0$; 8) $X(s^n) \to X(0) = 0$, from where $x^n = \frac{X(s^n)}{(h_3^n)^{1/\alpha}} \to +0$. Therefore, the mapping $\mathrm{Exp} : N \to D$ is proper. By Theorem 2, this mapping is a diffeomorphism. By the existence of optimal trajectories, any trajectory $\mathrm{Exp}(h_3, t)$, $h_3 \neq 0$, $t \in [0, t_1]$, is optimal for any $t_1 \in (0, t_*)$.

At $t = t_*$, two trajectories arrive at the point $\mathrm{Exp}(h_3, t_*)$ that are symmetric about the axis y and have the same value of the time functional, so both are optimal.

Now consider case 1. If $h_3 = 0$, then the extremal trajectory is the line $q(t) = (h_1^0 t, 0)$. From the above proved inclusion of $\mathrm{Exp}(N) \subset D$, it follows that for $h_3 \neq 0$ and $t > 0$, the extremal trajectories do not intersect the coordinate axis $y = 0$, so at each point of this axis comes the only (up to reparameterization) extremal trajectory – the straight line $q(t) = (h_1^0 t, 0)$. By virtue of the existence of an optimal trajectory, it is optimal on any segment $[0, t_1]$, $t_1 > 0$. The theorem is proved.

Corollary. 1. *For any trajectory* $\mathrm{Exp}(\lambda, t)$, $\lambda = (h_3, h_1^0) \in T_{q_0}^* M \cap \{H = 1\}$*, the cut time (time to loss of optimality) is* $t_{cut} = t_* = |h_3|^{-1/\alpha} s_* \in (0, +\infty]$.

2. *The cut locus is*

$$\mathrm{Cut} = \{\mathrm{Exp}(\lambda, t_{cut}(\lambda)) : \lambda \in T_{q_0}^* M \cap \{H = 1\}\} = \{(x, y) \in M : x = 0, y \neq 0\}.$$

Remark. The optimality of extremal trajectories on α-Grushin plane was first investigated in [3] on the basis of similar reasoning, but using explicit parameterization of extremal trajectories obtained in [2]. The novelty of this study consists in the qualitative use of only the property of the Hamiltonian system (7), but not its explicit integration.

For the Grushin 2-plane, Fig. **??** shows an almost Riemannian sphere of radius 2: $\{q \in M : d(q_0, q) = 2\}$ and its radii (optimal trajectories arriving at points on this sphere), and Fig. 2 shows the wavefronts $\{\mathrm{Exp}(\lambda, R) : \lambda \in N\}$ for different values of R.

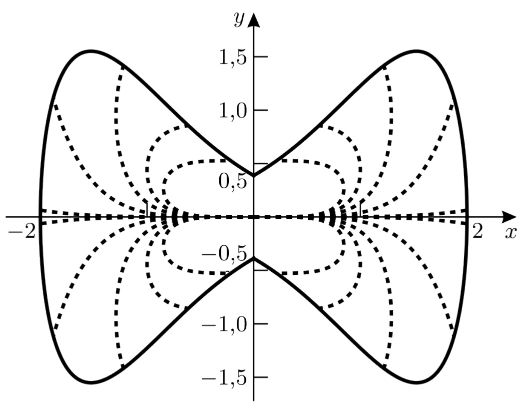

Fig. 1. Sphere of radius 2 and its radii

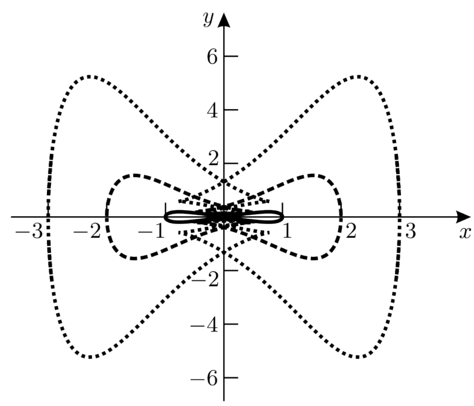

Fig. 2. Wavefronts

CONCLUSION

This paper presents a qualitative study of optimal trajectories on α-Grushin plane, that does not use explicit integration of the Hamiltonian system of Pontryagin's maximum principle. To the best of our knowledge, this study is the first one in optimal control theory. For example, even in the sub-Riemannian problem on the Heisenberg group, optimality is studied on the basis of explicit integration of the Hamiltonian system [1, §13.2]. We hope that the qualitative approach to the construction of optimal synthesis presented in this paper can be useful for other optimal control problems where explicit integration of the Hamiltonian system of the Pontryagin maximum principle is difficult or impossible. This approach can be applied to problems of small dimension and with a large symmetry group.

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The authors of this paper declare that they have no conflict of interests.

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