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BRIEF REPORTS

OPTIMAL TRAJECTORIES IN THE GRUSHIN α-PLANE

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Abstract. For the Grushin α-plane, optimal trajectories, cutting time, and cutting set are described.

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1. INTRODUCTION. PROBLEM STATEMENT

In sub-Riemannian geometry [1,§ 9.2], the Grushin plane is well known as the simplest example of an almost Riemannian manifold (such a manifold is Riemannian in complement to a special submanifold of dimension one). A natural generalization of this example is α-the Grushin plane when the degeneracy on a special set is of order $\alpha \geq 1$. Extremal trajectories for such a case were parameterized in [2], and their optimality was investigated based on this in [3]. In this paper, an independent study of the optimality of extremal trajectories is carried out using a qualitative approach that does not use the parameterization of these trajectories.

The optimal control problem for the classical Grushin plane is posed as follows [1, § 9.2]:

$$\dot{q} = u_1 X_1 + u_2 X_2, \quad q = (x, y) \in M = \mathbb{R}^2, \quad u = (u_1, u_2) \in \mathbb{R}^2, \tag{1}$$

$$q(0) = q_0, \quad q(t_1) = q_1, \quad l = \int_0^{t_1} (u_1^2 + u_2^2)^{1/2} dt \to \min, \tag{2}$$

where $X_1 = \frac{\partial}{\partial x}$, $X_2 = x\frac{\partial}{\partial y}$.

A natural generalization of this problem (α-Grushin plane) [2, 3] is posed similarly, but for vector fields:

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = |x|^\alpha \frac{\partial}{\partial y}, \quad \alpha \in \mathbb{R}, \alpha \geq 1. \tag{3}$$

Problem (1)–(3) is called an *almost Riemannian problem on the α-plane of Grushin*.

Let us denote the cost function in this problem – the almost Riemannian distance – as $d(q_0, q_1) = \inf\{l(q(\cdot)) : q(\cdot)$ the trajectory of system (1), (2)$\}$. A special set is the set of points $q \in M$, where the set of admissible velocities $\{\dot{q} = u_1 X_1 + u_2 X_2\}$ is not full-dimensional: $Z = \{q = (x, y) \in M : x = 0\}$. If $q_0 \in M \setminus Z$, then the problem locally becomes Riemannian, so the case $q_0 \in Z$, that is considered in this paper, is of special interest.

2. BASIC CONCEPTS AND PROPERTIES

2.1. Symmetries

The problem (1)–(3) has obvious symmetries – reflections

$$(x, y) \mapsto (-x, y), \quad (x, y) \mapsto (x, -y), \quad (x, y) \mapsto (-x, -y). \tag{4}$$

Vector fields X_1, X_2 are independent of y, so parallel translations are also symmetries

$$(x, y) \mapsto (x, y + a), \quad a \in \mathbb{R}. \tag{5}$$

Another one-parameter group of symmetries is given by the flow of the vector field

$$V = x\frac{\partial}{\partial x} + (1+\alpha)y\frac{\partial}{\partial y}, \quad (x, y) \mapsto e^{tV}(x, y) = (e^t x, e^{(1+\alpha)t}y), \quad t \in \mathbb{R}, \tag{6}$$

since $[V, X_1] = -X_1$, $[V, X_2] = -X_2$. So the optimal synthesis and, in particular, the distance d are invariant with respect to symmetries (4), (5) and homogeneous of order 1 with respect to symmetry (6): $d(e^{tV}(q_0), e^{tV}(q_1)) = e^t d(q_0, q_1)$, $q_i \in M$, $t \in \mathbb{R}$. Given the symmetry (5), we will further assume $q_0 = (0, 0)$.

2.2. Existence of solutions

System (1) is completely controllable in each of the Riemannian half-planes $\{q \in M : \operatorname{sign} x = \pm 1\}$, since in them, the set of admissible velocities is full-dimensional. It is possible to move between these half-planes along the fields $\pm X_1$, so system (1) is quite controllable. Note that at the points $q \in Z$, the condition of the Rashevskii-Chow theorem [4, § 5.3; 5, § 2.2.4] is satisfied only at $\alpha \in 2\mathbb{N}$. All conditions of Filippov's theorem [4,§ 10.3; 5,§ 3.1.2] are satisfied, so optimal trajectories exist.

2.3. Extreme trajectories

As usual in sub-Riemannian geometry, we pass from length minimization (2) to energy minimization $J = 0.5\int_0^{t_1}(u_1^2 + u_2^2)dt$. We apply Pontryagin's maximum principle to the resulting problem [4, § 3; 5, § 5.2; 6, § 12.4; 7, § 3.2.2]. The abnormal trajectories are constant and non-strictly anormal. To parameterize the normal extremal trajectories, we put $X_3 = \frac{\partial}{\partial y}$ and denote the Hamiltonians linear on the layers of the cotangent bundle $T^*M : h_i(\lambda) = \langle \lambda, X_i \rangle$, $i = 1, 3$, $\lambda \in T^*M$. Then the maximized Hamiltonian of the Pontryagin maximum principle is $H = h_1^2 + |x|^{2\alpha}h_3^2 \equiv 1$ and the Hamiltonian system for the normal extremals is of the form

$$\dot{h}_1 = -\alpha \operatorname{sign} x|x|^{2\alpha-1}h_3^2, \quad \dot{h}_3 = 0, \quad \dot{x} = h_1, \quad \dot{y} = |x|^{2\alpha}h_3. \tag{7}$$

The Hamiltonian H is the first integral, so at each $h_3 \neq 0$ the independent subsystem of equations (7) for the variables h_1 and x has a phase portrait of type center.

If $h_3 = 0$, then $h_1 \equiv\neq 0$, $x = h_1 t$, $y = 0$. Let $h_3 \neq 0$. When integrating the system (7) by the method of separation of variables, we obtain the equation

$$\frac{dx}{\sqrt{H - h_3^2|x|^{2\alpha}}} = \pm dt,$$

in which the left part integrates in the general case into a hypergeometric function ${}_2F_1$. On the other hand, in [2], the system (7) is integrated in terms of some generalizations of trigonometric functions. However, we will not use explicit parametrization of the extremal trajectories and investigate the optimality of the extremal trajectories relying only on qualitative methods.

Considering the symmetry $(h_3, y) \mapsto (-h_3, -y)$ of the system (7), further we consider that $h_3 > 0$. After change of variables $X = xh_3^{1/\alpha}$, $Y = yh_3^{1+1/\alpha}$, $H_1 = h_1$, $s = th_3^{1/\alpha}$, the Hamiltonian system (7) will take the form

$$H_1' = -\alpha \operatorname{sign} X|X|^{2\alpha-1}, \quad X' = H_1, \quad Y' = |X|^{2\alpha} \tag{8}$$

with the first integral $H = H_1^2 + |X|^{2\alpha} \equiv 1$. Since $H = 1$, we have $H_1(0) = H_1^0 = \pm 1$. Taking advantage of the symmetry $(H_1, X) \mapsto (-H_1, -X)$, we obtain $H_1^0 = 1$. The first two equations of system (8) have a phase portrait of type center in the plane (H_1, X), so for any $\alpha \geq 1$, there exists a unique number $s_* = s_*(\alpha) > 0$ such that

$$X(s) > 0 \quad \text{at} \quad s \in (0, s_*), \quad X(s_*) = 0. \tag{9}$$

Then the first positive root of the function $x(t)$ is $t_* = s_* h_3^{-1/\alpha}$.

3. MAIN RESULTS

Theorem 1. 1. *If* $h_3 = 0$*, then the extremal trajectory* $q(t)$ *is optimal on any segment* $[0, t_1]$, $t_1 > 0$.

2. *If* $h_3 \neq 0$*, then the extremal trajectory* $q(t)$ *is optimal at any segment* $[0, t_1]$, $t_1 \in (0, t_*]$*, and not optimal for* $t_1 > t_*$*, where* $t_* = s_*|h_3|^{-1/\alpha}$.

Proof. Let us first study case 2. Let $h_3 \neq 0$. Consider an exponential mapping

$$\text{Exp} : (\lambda, t) \mapsto q(t), \quad \text{Exp} : \tilde{N} \to M, \quad \tilde{N} = (T^*_{q_0}M \cap \{H = 1\}) \times \mathbb{R}_+,$$

$$\lambda = (h_3, h_1^0), \quad h_3 \in \mathbb{R} \setminus \{0\}, h_1^0 = \pm 1.$$

At any $h_3 \neq 0$, the extremal trajectories $\text{Exp}(h_3, 1, t)$ and $\text{Exp}(h_3, -1, t)$ are symmetric w.r.t. the axis y and intersect this axis at $t = t_*$. Therefore, the intersection point $\text{Exp}(h_3, 1, t_*)$ is a Maxwell point [5, § 3.3.5] and these trajectories are not optimal under the condition $t > t_*$.

Let us now prove that any trajectory $\text{Exp}(h_3, 1, t)$ is optimal at $t \in [0, t_1]$, $t_1 \in (0, t_*)$. Given the symmetries of the problem, we will assume that $h_1^0 = 1$ and $h_3 > 0$, and denote by $\text{Exp}(h_3, t) := \text{Exp}(h_3, 1, t)$. Let $N = \{(h_3, t) \in \mathbb{R}^2 : h_3 > 0, t \in (0, t_*)\}$, $D = \{(x, y) \in M : xy > 0\}$. Let us show that $\text{Exp} : N \to D$ is a diffeomorphism by using the following theorem of Adamar on global diffeomorphism.

Theorem 2 [8; 9, § 6.2]. *Let* $F : X \to Y$ *be a smooth mapping between smooth manifolds of the same dimension such that* X, Y *are connected,* Y *is one-connected,* F *is nondegenerate and proper. Then* F *is a diffeomorphism.*

Let us first prove that $\text{Exp}(N) \subset D$. Since $h_3 > 0$ and $t \in (0, t_*)$, then $x(t) > 0$ by virtue of inequality (9). It follows from the ordinary differential equation (8) that $y(t) > 0$. Therefore $\text{Exp}(N) \subset D$.

Obviously, N and D are connected, and D is one-connected. Let us show that $\text{Exp}|_N$ is nondegenerate, i.e., the Jacobian of $\partial(x, y)/\partial(t, h_3)$ is different from zero in the region N. We have $\frac{\partial x}{\partial t} = H_1$, $\frac{\partial y}{\partial t} = h_3^{-1} X^{2\alpha}$, $\frac{\partial x}{\partial h_3} = -\alpha^{-1} h_3^{-1-1/\alpha} X + (\frac{\partial s}{\partial h_3}) H_1 h_3^{-1/\alpha}$, $\frac{\partial y}{} \partial h_3 = -(1 + \frac{1}{\alpha})\, h_3^{-2-1/\alpha}\, Y + (\frac{\partial s}{\partial h_3})\, X^{2\alpha} h_3^{-1-1/\alpha}$, whence $J = h_3^{-2-1/\alpha} \alpha^{-1} J_1$, $J_1 = X^{2\alpha+1} - (\alpha + 1) Y H_1$. Differentiating by virtue of (8), we obtain $J_1' = \alpha X^{2\alpha-1} J_2$, $J_2 = H_1 X + (\alpha + 1) Y$. Differentiating again, we have $J_2' = H_1^2 + X^{2\alpha} > 0$, so $J|_N > 0$, i.e., $\text{Exp}|_N$ is nondegenerate.

Now let us show that the mapping $\text{Exp} : N \to D$ is proper. This is equivalent to the following condition: if the sequence $\{(h_3^n, t^n) : n \in \mathbb{N}\} \subset N$ is not contained in any compact set in N, then its image $q^n = \text{Exp}(h_3^n, t^n)$ is not contained in any compact set in D. Let the sequence $\{(h_3^n, t^n) : n \in \mathbb{N}\} \subset N$ be not contained in any compact set in N, we'll denote $s^n = (h_3^n)^{1/\alpha} t^n \in (0, s_*)$. Then it contains a subsequence for which one of the following conditions is satisfied: 1) $h_3^n \to \bar{h}_3 \in (0, +\infty)$, $s^n \to 0$; 2) $h_3^n \to 0$, $s^n \to 0$; 3) $h_3^n \to 0$, $s^n \to \bar{s} \in (0, s_*)$; 4) $h_3^n \to 0$, $s^n \to s_*$; 5) $h_3^n \to \bar{h}_3 \in (0, +\infty)$, $s^n \to s_*$; 6) $h_3^n \to +\infty$, $s^n \to s_*$; 7) $h_3^n \to +\infty$, $s^n \to \bar{s} \in (0, s_*)$; 8) $h_3^n \to +\infty$, $s^n \to 0$.

We'll show that for each of them, the sequence $q^n = (x^n, y^n)$ contains a subsequence on which one of the following conditions is satisfied: $x^n \to 0$, $x^n \to +\infty$, $y^n \to 0$, $y^n \to +\infty$, i.e., q^n is not contained in any compact set in D.

Given 1) we have $X(s^n) \to X(0) = 0$, so $x^n = X(s^n)/(h_3^n)^{1/\alpha} \to 0$.

If condition 2) is satisfied, the sequence $t^n = s^n/(h_3^n)^{1/\alpha} > 0$ contains a subsequence of one of the following kinds: $t^n \to 0$, $t^n \to \bar{t} \in (0, +\infty)$, $t^n \to +\infty$. If $t^n \to 0$, then $x^n = x(h_3^n, t^n) \to x(0, 0) = 0$. If $t^n \to \bar{t} \in (0, +\infty)$, then $y^n = y(h_3^n, t^n) \to y(0, \bar{t}) = 0$. Let $t^n \to +\infty$. Passing to the subsequence if necessary, we can assume that $\{s^n\}$ is decreasing. There exists a number $K \in \mathbb{N}$ such that $s^K < s_*/2$, so $H_1(s) > 0$ for all $s \in [0, s^K]$. Hence, $H_1|_{[0,s^K]} \geq \varepsilon := \min_{[0,s^K]} H_1 > 0$ and

$$X(s^n) = \int_0^{s^n} H_1(s) ds \geq \varepsilon s^n = \varepsilon t^n (h_3^n)^{1/\alpha}, \quad x^n = \frac{X(s^n)}{(h_3^n)^{1/\alpha}} \geq \varepsilon t^n \to +\infty.$$

For the remaining conditions, we have: 3) $X(s^n) \to X(\bar{s}) \in (0, +\infty)$ and $x^n = \frac{X(s^n)}{(h_3^n)^{1/\alpha}} \to +\infty$; 4) $Y(s^n) \to Y(s_*) = \int_0^{s_*} |X(s)|^{2\alpha} ds \in (0, +\infty)$ and $y^n = \frac{Y(s^n)}{(h_3^n)^{1+1/\alpha}} \to +\infty$; 5) $X(s^n) \to X(s_*) = 0$, from where $x^n = \frac{X(s^n)}{(h_3^n)^{1/\alpha}} \to +0$; 6) $X(s^n) \to X(s_*) = 0$, from where $x^n = \frac{X(s^n)}{(h_3^n)^{1/\alpha}} \to +0$; 7) $X(s^n) \to X(\bar{s}) \in (0, +\infty)$, from where $x^n = \frac{X(s^n)}{(h_3^n)^{1/\alpha}} \to +0$; 8) $X(s^n) \to X(0) = 0$, from where $x^n = \frac{X(s^n)}{(h_3^n)^{1/\alpha}} \to +0$. Therefore, the mapping $\mathrm{Exp} : N \to D$ is proper. By Theorem 2, this mapping is a diffeomorphism. By the existence of optimal trajectories, any trajectory $\mathrm{Exp}(h_3, t)$, $h_3 \neq 0$, $t \in [0, t_1]$, is optimal for any $t_1 \in (0, t_*)$.

At $t = t_*$, two trajectories arrive at the point $\mathrm{Exp}(h_3, t_*)$ that are symmetric about the axis y and have the same value of the time functional, so both are optimal.

Now consider case 1. If $h_3 = 0$, then the extremal trajectory is the line $q(t) = (h_1^0 t, 0)$. From the above proved inclusion of $\mathrm{Exp}(N) \subset D$, it follows that for $h_3 \neq 0$ and $t > 0$, the extremal trajectories do not intersect the coordinate axis $y = 0$, so at each point of this axis comes the only (up to reparameterization) extremal trajectory – the straight line $q(t) = (h_1^0 t, 0)$. By virtue of the existence of an optimal trajectory, it is optimal on any segment $[0, t_1]$, $t_1 > 0$. The theorem is proved.

Corollary. 1. *For any trajectory* $\mathrm{Exp}(\lambda, t)$, $\lambda = (h_3, h_1^0) \in T_{q_0}^* M \cap \{H = 1\}$, *the cut time (time to loss of optimality) is* $t_{cut} = t_* = |h_3|^{-1/\alpha} s_* \in (0, +\infty]$.

2. *The cut locus is*

$$\mathrm{Cut} = \{\mathrm{Exp}(\lambda, t_{cut}(\lambda)) : \lambda \in T_{q_0}^* M \cap \{H = 1\}\} = \{(x, y) \in M : x = 0, y \neq 0\}.$$

Remark. The optimality of extremal trajectories on α-Grushin plane was first investigated in [3] on the basis of similar reasoning, but using explicit parameterization of extremal trajectories obtained in [2]. The novelty of this study consists in the qualitative use of only the property of the Hamiltonian system (7), but not its explicit integration.

For the Grushin 2-plane, Fig. **??** shows an almost Riemannian sphere of radius 2: $\{q \in M : d(q_0, q) = 2\}$ and its radii (optimal trajectories arriving at points on this sphere), and Fig. 2 shows the wavefronts $\{\mathrm{Exp}(\lambda, R) : \lambda \in N\}$ for different values of R.

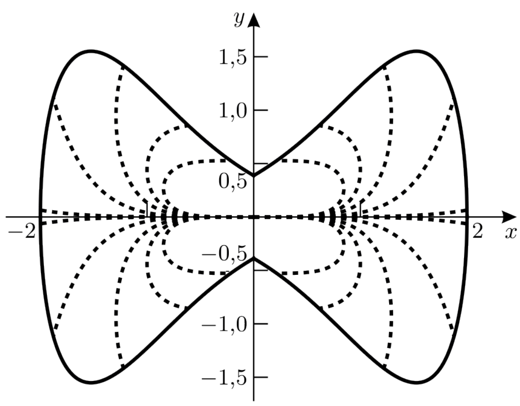

Fig. 1. Sphere of radius 2 and its radii

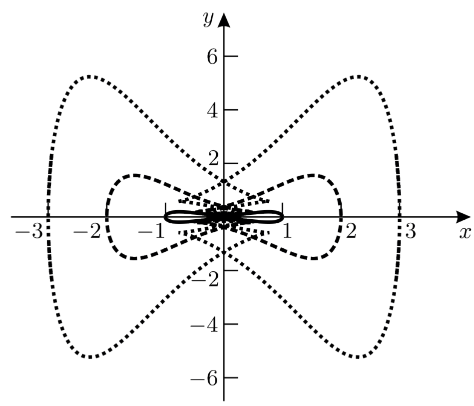

Fig. 2. Wavefronts

CONCLUSION

This paper presents a qualitative study of optimal trajectories on α-Grushin plane, that does not use explicit integration of the Hamiltonian system of Pontryagin's maximum principle. To the best of our knowledge, this study is the first one in optimal control theory. For example, even in the sub-Riemannian problem on the Heisenberg group, optimality is studied on the basis of explicit integration of the Hamiltonian system [1, §13.2]. We hope that the qualitative approach to the construction of optimal synthesis presented in this paper can be useful for other optimal control problems where explicit integration of the Hamiltonian system of the Pontryagin maximum principle is difficult or impossible. This approach can be applied to problems of small dimension and with a large symmetry group.

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CONFLICT OF INTERESTS

The authors of this paper declare that they have no conflict of interests.

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