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BRIEF REPORTS

BASS-GURA FORMULA FOR LINEAR SYSTEM WITH DYNAMIC OUTPUT FEEDBACK

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Abstract. In this paper, we solve the problem of assigning the desired characteristic polynomial of a linear stationary dynamic system with one input and output dynamic feedback in the form of a first-order dynamic compensator. Necessary and sufficient conditions for the existence of the solution of the problem are considered. An explicit formula for the compensator parameters, analogous to the Bass–Gura formula for a state feedback system, is derived.

Keywords: *linear system, output feedback, Bass–Gura formula*

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1. INTRODUCTION. PROBLEM STATEMENT

The Ackerman and Bass-Gura formulas are known from mathematical control theory [1, p. 360], used to solve the problem of assigning the desired characteristic polynomial of a linear stationary system with one input and state feedback, whose behavior is described by equations

$$\dot{x} = Ax + bu, \quad u = -f^{\mathrm{T}}x, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}$ is the scalar control, $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$, $f \in \mathbb{R}^n$.

The characteristic polynomial of the system (1) is the characteristic polynomial of the matrix of the closed-loop system $A - bf^{\mathrm{T}}$. Let us denote by

$$a(\lambda) = \lambda^n + a_1\lambda^{n-1} + \dots + a_n, \quad d(\lambda) = \lambda^n + d_1\lambda^{n-1} + \dots + d_n$$

characteristic polynomial of the matrix A and the desired characteristic polynomial of the matrix $A - bf^{\mathrm{T}}$. Suppose that the matrix

$$X(A, b) = \begin{bmatrix} b & Ab & \dots & A^{n-1}b \end{bmatrix}$$

is nonsingular, which corresponds to the controllability condition of the system (1).

According to Ackerman's formula the required vector f is equal to

$$f^{\mathrm{T}} = \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix} X(A, b)^{-1} d(A).$$

According to the Bass-Gura formula

$$f^{\mathrm{T}} = (\bar{d} - \bar{a})^{\mathrm{T}} H^{-1} X(A, b)^{-1},$$

where

$$H = \begin{bmatrix} a_{n-1} & a_{n-2} & \dots & 1 \\ a_{n-2} & a_{n-3} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \dots & 0 \end{bmatrix}, \quad \bar{a} = \begin{bmatrix} a_n \\ a_{n-1} \\ \vdots \\ a_1 \end{bmatrix}, \quad \bar{d} = \begin{bmatrix} d_n \\ d_{n-1} \\ \vdots \\ d_1 \end{bmatrix}.$$

In papers [2, 3], the Ackerman formula and the Bass–Gura formula were generalized for systems with multiple inputs and state feedback. The purpose of this paper is to obtain a generalization of the Bass–Gura formula for a system with dynamic output feedback in the form of a first-order dynamic compensator.

It is known [4] that dynamic feedback significantly expands the possibilities of output feedback compared to static feedback. Dynamic output feedback can include state observers as well as dynamic compensators of the general kind. According to the seminal work [5], a dynamic compensator of order $\min\{p_c, p_o\}$, where p_c and p_o are the controllability and observability indices of the system, respectively, can be constructed for a fully controllable and fully observable system. In the case of a system with one input, the minimum order of the compensator is equal to the observability index p_o.

Let us consider a linear stationary system with one input

$$\dot{x} = Ax + bu, \quad y = Cx,$$

where $x \in \mathbb{R}^n$ is the state vector, $y \in \mathbb{R}^l$ is the measurement vector, $u \in \mathbb{R}$ is the scalar control, $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$, $C \in \mathbb{R}^{l\times n}$, $l < n$.

We will search for the control in the form of a dynamic compensator of the first order

$$u = -f^{\mathsf{T}}y - z, \quad \dot{z} + pz = q^{\mathsf{T}}y,$$

where $f \in \mathbb{R}^l$, $p \in \mathbb{R}$, $q \in \mathbb{R}^l$ are the compensator parameters. The system with compensator is described by the equations

$$\dot{x} = (A - bf^{\mathsf{T}}C)x - bz, \quad \dot{z} = q^{\mathsf{T}}Cx - pz. \tag{2}$$

The characteristic polynomial of the system (2) is the characteristic polynomial of the matrix of the closed-loop system

$$D = \begin{bmatrix} A - bf^\top C & -b \\ q^\top C & -p \end{bmatrix}.$$

We will search for the compensator parameters taking into account the properties of the given characteristic polynomial of the matrix D. For this purpose, it is necessary to obtain an explicit formula for the feedback parameters similar to the Bass–Gura formula.

2. KEY FINDINGS

Let's denote by

$$a(\lambda) = \det(\lambda E - A) = \lambda^n + a_1\lambda^{n-1} + \dots + a_n$$

the characteristic polynomial of the matrix A. Let us introduce a column vector $g(\lambda) = C(\lambda E - A)^*b$, where $(\lambda E - A)^*$ is the adjoint matrix to $\lambda E - A$.

Lemma. *The characteristic polynomial of the matrix D is*

$$\det(\lambda E - D) = (\lambda + p)a(\lambda) + (f^\top(\lambda + p) + q^\top)g(\lambda). \tag{3}$$

Proof. Following simple transformations in the determinant of the matrix $\lambda E - D$, we obtain

$$\det(\lambda E - D) = \det\begin{bmatrix} \lambda E - A + bf^\top C & b \\ -q^\top C & \lambda + p \end{bmatrix} = \det\begin{bmatrix} \lambda E - A + b(f^\top + (\lambda + p)^{-1}q^\top)C & b \\ 0 & \lambda + p \end{bmatrix} =$$

$$= (\lambda + p)a(\lambda) + (f^\top(\lambda + p) + q^\top)C(\lambda E - A)^*b = (\lambda + p)a(\lambda) + (f^\top(\lambda + p) + q^\top)g(\lambda).$$

Here the $\det(A + bc^\top) = \det A + c^\top A^* b$ is applied, where A is a square matrix, A^* is a adjoint matrix to A, b is a column vector, $c^\top$ is a row vector [6, p. 133]. The lemma is proved.

The following theorem formulates necessary and sufficient conditions for the existence of a solution to the problem and simultaneously describes the algorithm for calculating the compensator parameters.

Theorem. *The characteristic polynomial of the matrix D can be arbitrarily set by choosing the compensator parameters f, p, q, only when*

$$\operatorname{rank} X(A, b) = n, \quad \operatorname{rank} Y(A, C) = n,$$

where

$$X(A,b) = \begin{bmatrix} b & Ab & \dots & A^{n-1}b \end{bmatrix}, \quad Y(A,C) = \begin{bmatrix} C & CA \end{bmatrix}^{\mathsf{T}}.$$

Proof. Let's denote by

$$d(\lambda) = \lambda^{n+1} + d_1\lambda^n + \dots + d_{n+1} \tag{4}$$

the desired characteristic polynomial of the matrix D. We will search for the compensator parameters from the condition of coincidence of polynomials (3) and (4).

Let's denote

$$\pi_k(\lambda) = \begin{bmatrix} \lambda^k & \lambda^{k-1} & \dots & \lambda & 1 \end{bmatrix}^\top, \quad \bar a = \begin{bmatrix} a_2 & a_3 & \dots & a_n \end{bmatrix}, \quad \bar d = \begin{bmatrix} d_2 & d_3 & \dots & d_{n+1} \end{bmatrix}.$$

Then

$$a(\lambda) = \lambda^n + \begin{bmatrix} a_1 & \bar a \end{bmatrix}\pi_{n-1}(\lambda), \quad \lambda a(\lambda) = \lambda^{n+1} + a_1\lambda^n + \begin{bmatrix} \bar a & 0 \end{bmatrix}\pi_{n-1}(\lambda),$$
$$d(\lambda) = \lambda^{n+1} + d_1\lambda^n + \bar d\pi_{n-1}(\lambda).$$

Let us write the matrix $(\lambda E - A)^*$ as a matrix polynomial [7, p. 91]

$$(\lambda E - A)^* = E\lambda^{n-1} + A_1\lambda^{n-2} + \dots + A_{n-1},$$

where

$$A_1 = A + a_1E, \quad A_2 = A^2 + a_1A + a_2E = AA_1 + a_2E, \quad \dots$$
$$\dots, \quad A_{n-1} = A^{n-1} + a_1A^{n-2} + \dots + a_{n-1}E = AA_{n-2} + a_{n-1}E.$$

Note that by the Cayley-Hamilton theorem,

$$A_n = A^n + a_1A^{n-1} + \dots + a_nE = AA_{n-1} + a_nE = 0.$$

Let's introduce the matrix

$$G = \begin{bmatrix} 1 & a_1 & \dots & a_{n-1} \\ 0 & 1 & \dots & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

Vectors $g(\lambda)$ and $\lambda g(\lambda)$ can be written in the form

$$g(\lambda) = C(\lambda E - A)^*b = C\begin{bmatrix} b & A_1b & \dots & A_{n-1}b \end{bmatrix}\pi_{n-1}(\lambda) = CX(A,b)G\pi_{n-1}(\lambda),$$
$$\lambda g(\lambda) = Cb\lambda^n + C\begin{bmatrix} A_1b & A_2b & \dots & A_nb \end{bmatrix}\pi_{n-1}(\lambda) =$$
$$= Cb\lambda^n + CAX(A,b)G\pi_{n-1}(\lambda) + Cb\begin{bmatrix} a_1 & \bar a \end{bmatrix}\pi_{n-1}(\lambda).$$

The characteristic polynomial (3) of the matrix D is equal to

$$\det(\lambda E - D) = \lambda^{n+1} + a_1\lambda^n + \begin{bmatrix} \bar a & 0 \end{bmatrix}\pi_{n-1}(\lambda) + p\lambda^n + p\begin{bmatrix} a_1 & \bar a \end{bmatrix}\pi_{n-1}(\lambda) +$$
$$+ f^{\mathsf{T}}Cb\lambda^n + f^{\mathsf{T}}CAX(A,b)G\pi_{n-1}(\lambda) + f^{\mathsf{T}}Cb\begin{bmatrix} a_1 & \bar a \end{bmatrix}\pi_{n-1} + (fp+q)^{\mathsf{T}}CX(A,b)G\pi_{n-1}(\lambda). \tag{5}$$

The given polynomial (4) and the polynomial (5) coincide if and only if

$$a_1 + p + f^\top Cb = d_1, \tag{6}$$

$$\begin{bmatrix} \bar a & 0 \end{bmatrix} + p\begin{bmatrix} a_1 & \bar a \end{bmatrix} + f^\top CAX(A,b)G + f^\top Cb\begin{bmatrix} a_1 & \bar a \end{bmatrix} + (fp+q)^\top CX(A,b)G = \bar d. \tag{7}$$

Let us denote $r = fp + q$. From equation (6) express p and substitute it into (7). Then (7) takes the form

$$\begin{bmatrix} r^\top & f^{\mathsf{T}} \end{bmatrix} Y(A,C)X(A,b)G = \bar d - \begin{bmatrix} \bar a & 0 \end{bmatrix} - (d_1 - a_1)\begin{bmatrix} a_1 & \bar a \end{bmatrix}. \tag{8}$$

Let f and r be solutions of equation (8). Then from relation (6), we obtain $p = d_1 - a_1 - fCb$, and $q = r - fp$.

Equation (8) has a solution with respect to the unknowns f and r for any vector $\bar{d}$, if and only if $\operatorname{rank} Y(A,C)X(A,b)G = n$. The matrix G is nonsingular. The matrices $X(A,b)$ and $Y(A,C)$ have the dimensions $n \times n$ and $2l \times n$, respectively. Hence, $\operatorname{rank} Y(A,C)X(A,b)G = n$ if and only if $\operatorname{rank} X(A,b) = n$ and $\operatorname{rank} Y(A,C) = n$. The theorem is proved.

Remark. It follows from the theorem, that the necessary condition for the existence of a solution to the problem is the condition $2l \geq n$. Consequently, the problem has a solution at a sufficiently large number of output variables. For example, at $n = 5$ the number of output variables should be at least 3. This is a significant limitation of the considered output feedback.

If the conditions of the theorem are satisfied and $2l = n$, then the solution of equation (8) is unique. If $2l > n$, then equation (8) has infinitely many solutions.

In the case of a unique solution

$$\begin{bmatrix} r^{\mathrm{T}} & f^{\mathrm{T}} \end{bmatrix} = \left(\bar{d} - \begin{bmatrix} \bar{a} & 0 \end{bmatrix} - (d_1 - a_1)\begin{bmatrix} a_1 & \bar{a} \end{bmatrix}\right) G^{-1} X(A,b)^{-1} Y(A,C)^{-1}. \tag{9}$$

Formula (9) can be considered as an analog of the Bass-Gura formula for a system with state feedback.

Let the conditions of the theorem be satisfied and $2l > n$. Then we can find a partial solution of equation (8):

$$\begin{bmatrix} r^{\mathrm{T}} & f^{\mathrm{T}} \end{bmatrix} = \left(\bar{d} - \begin{bmatrix} \bar{a} & 0 \end{bmatrix} - (d_1 - a_1)\begin{bmatrix} a_1 & \bar{a} \end{bmatrix}\right) G^{-1} X(A,b)^{-1} (Y(A,C)^{\mathrm{T}} Y(A,C))^{-1} Y(A,C)^{\mathrm{T}}.$$

3. NUMERICAL EXAMPLE

Let $n = 6$, $l = 3$,

$$A = \begin{bmatrix} -1.68 & 0.64 & 1.53 & -1.5 & -1.45 & -0.22 \\ 0.89 & 1.48 & 2.35 & 0.78 & -2.21 & -0.08 \\ -0.74 & 0.96 & 1.28 & -2.04 & 1.61 & 1.6 \\ 0.35 & -1.78 & 0.74 & -1.54 & -0.16 & -0.06 \\ 0.15 & -1.05 & -1.19 & 0.65 & -0.22 & -0.54 \\ -0.53 & 0.37 & 0.7 & -0.09 & 0.15 & -0.41 \end{bmatrix},$$

$$b = \begin{bmatrix} -0.47 \\ -0.53 \\ 1.87 \\ 0.79 \\ -0.56 \\ 0.46 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let us set the desired characteristic polynomial of the system (2):

$$d(\lambda) = (\lambda + 0.3)(\lambda + 0.4)(\lambda + 0.5)(\lambda + 0.2 + 0.7i)(\lambda + 0.2 - 0.7i)(\lambda + 0.1 + 0.3i)(\lambda + 0.1 - 0.3i) =$$

$$= \lambda^7 + 1.8\lambda^6 + 1.9\lambda^5 + 1.34\lambda^4 + 0.5979\lambda^3 + 0.17482\lambda^2 + 0.03367\lambda + 0.00318.$$

The conditions of the theorem are fulfilled. The parameters of the compensator are determined uniquely:

$$f^{\mathrm{T}} = \begin{bmatrix} 0.0891861 & -1.5061263 & 14.434942 \end{bmatrix},$$

$$q^{\mathrm{T}} = \begin{bmatrix} 7.3744718 & -10.250088 & 53.52229 \end{bmatrix}, \quad p = -3.0716998.$$

The verification shows that the characteristic polynomial of the matrix D coincides with the given one.

CONFLICT OF INTERESTS

The author of this paper declares that he has no conflict of interests.

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