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CONTROL THEORY

ON THE PROBLEM OF PURSUING A GROUP OF COORDINATED EVADERS IN A GAME WITH FRACTIONAL DERIVATIVES

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Abstract. In a finite-dimensional Euclidean space, the problem of pursuing of a group of evaders by a group of pursuers is considered, described by a linear non-stationary system of differential equations with fractional Caputo derivatives. Sets of admissible players' controls — compacts, terminal sets — origin of coordinates. Sufficient conditions have been obtained for the capture of at least one evader and all evaders under the condition that the evaders use the same control. In the study, the method of matrix and scalar resolving functions is used as a basic one. It is shown that differential games described by equations with fractional derivatives have properties that are different from those of differential games described by ordinary differential equations.

Keywords: *differential game, group pursuit, pursuer, evader, fractional derivative*

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1. INTRODUCTION

One of the directions of development of the modern theory of differential games is the study of pursuit-evasion problems with participation of a group of participants [1–4], and besides deepening of classical methods of investigation the search of game problems to which previously developed methods are applicable is actively conducted.

Differential games with fractional derivatives were first considered in [5], where the method of scalar resolving functions was used for the study. Differential games with fractional derivatives based on the Hamilton-Jacobi equation were studied in [6]. In [7], the problem of pursuit by a group of pursuers of a single evader in differential games described by equations with fractional derivatives was considered. The problem of conflict interaction between a group of pursuers and a group of evaders in games with fractional dynamics was considered in [8], scalar resolving functions were used for analysis. A. A. Chikrii, in his paper [9], notes that scalar resolving functions attract a terminal set with images of some multivalued mappings that occur in a cone stretched over this set, which limits the possibilities for the pursuer's maneuver, and also proposes to use matrix resolving functions to analyze two-person pursuit games. In [10], matrix resolution functions were applied to study the problem of pursuit by a group of pursuers of a single evader described by a stationary linear system with fractional Caputo derivatives.

In [11], the problem of pursuit by a group of pursuers of a group of evaders in linear stationary differential games with simple matrices under the condition that all evaders use the same control was considered. Sufficient conditions for catching at least one evader were obtained. The pursuit problem in which all evaders use the same control will be referred to as the *coordinated evaders pursuit problem*.

In this paper we consider the problem of conflict interaction between a group of pursuers and a group of evaders in a differential game described by a nonstationary linear system of differential equations with fractional Caputo derivatives. Under the condition that the evaders use the same control, sufficient conditions for catching at least one evader are obtained, using matrix or scalar resolving functions. The study of the nonstationary case is supplemented by some results for games described by linear stationary systems with a simple matrix.

1. PROBLEM STATEMENT

In the space $\mathbb{R}^k$ ($k \geqslant 2$) we consider a differential game of $n+m$ persons: n pursuers $P_1, \dots, P_n$ and m evaders $E_1, \dots, E_m$, described by a system of the form

$$(D^{(\alpha)})z_{ij} = A_{ij}(t)z_{ij} + u_i - v, \quad z_{ij}(t_0) = z_{ij}^0, \quad u_i \in U_i, v \in V. \tag{1}$$

Here, $i \in I = \{1, \dots, n\}$, $j \in J = \{1, \dots, m\}$, $z_{ij}, u_i, v \in \mathbb{R}^k$, U_i, V are compact sets $\mathbb{R}^k$, $\alpha \in (0,1)$, $D^{(\alpha)}x$ is Caputo derivative of the function x of order α [12], $A_{ij}(t)$ are continuous matrix functions of order $k \times k$. Terminal sets M_{ij}^* of the form

$$M_{ij}^* = M_{ij} + M_{ij}^0,$$

where M_{ij} is a linear subspace of $\mathbb{R}^k$, M_{ij}^0 are convex compact sets from L_{ij} – the orthogonal complement of M_{ij} to $\mathbb{R}^k$. We consider $z_{ij}^0 \notin M_{ij}^*$ for all $i \in I$, $j \in J$.

The actions of the evaders can be interpreted as follows: there is a center that, for all evaders $E_1, \dots, E_m$, chooses the same control $v(\cdot)$.

Let $v : [t_0, +\infty) \to V$ be a measurable function, which we will call *admissible*. The *prehistory* of $v_t(\cdot)$, at the moment t of the function $v(\cdot)$, will be called the contraction of the function v at $[t_0, t]$.

2. SUFFICIENT CATCHING CONDITIONS

Definition 1. We will say that a quasi-strategy $\mathcal{U}_i$ of the pursuer P_i is defined, if a mapping U_i^0, that puts the measurable function $u_i(t)$ with values in U_i in accordance with the initial positions of $z^0 = (z_{ij}^0, i \in I, j \in J)$, the moment t, and an arbitrary control prehistory $v_t(\cdot)$ of the evader E_j, $j \in J$, is defined.

Let's denote this game by $G(n, m, z^0)$.

Definition 2. A capture of at least one evader occurs in the game $G(n, m, z^0)$, if there exist moment $T > 0$, quasi-strategies $\mathcal{U}_1, \dots, \mathcal{U}_n$ of pursuers $P_1, \dots, P_n$ such that for any measurable function $v(\cdot)$, $v(t) \in V$, $t \in [t_0, T]$, there exist moment $\tau \in [t_0, T]$ and numbers $p \in I$, $q \in J$, for which $z_{pq}(\tau) \in M_{pq}$.

Let us introduce the following notations: E^0 is a identity matrix of order $k \times k$, $\pi_{ij} : \mathbb{R}^k \to L_{ij}$ is an orthogonal projection operator,

$$\Gamma(\beta) = \int_0^{+\infty} s^{\beta-1}e^{-s}ds, \quad {}_\tau J_t f(t) = \frac{1}{\Gamma(\alpha)} \int_\tau^t (t-s)^{\alpha-1} f(s)ds,$$

$$G_{ij}^0(t,\tau) = \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} E^0,$$

$$G_{ij}^{l+1}(t,\tau) = {}_\tau J_t(A_{ij}(t)G_{ij}^l(t,\tau)), \quad l = 0, 1, \dots, \quad \Phi_{ij}(t,\tau) = \sum_{l=0}^{+\infty} G_{ij}^l(t,\tau),$$

$$\widetilde{G}_{ij}^0(t,\tau) = E^0, \quad \widetilde{G}_{ij}^{l+1}(t,\tau) = {}_\tau J_t(A_{ij}(t)\widetilde{G}_{ij}^l(t,\tau)), \quad l = 0, 1, \dots, \quad \Psi_{ij}(t,\tau) = \sum_{l=0}^{+\infty} \widetilde{G}_{ij}^l(t,\tau),$$

$$W_{ij}(t,\tau,v) = \pi_{ij}\Phi_{ij}(t,\tau)(U_i - v), \quad W_{ij}(t,\tau) = \bigcap_{v \in V} W_{ij}(t,\tau,v),$$

Int A, co A are the interior and the convex hull of the set A, respectively.

Assumption 1. *There exists a mapping $q : I \to J$, such that for all $i \in I$, $t \geqslant t_0$, $\tau \in [t_0, t]$ the following condition is satisfied*

$$W_{iq(i)}(t,\tau) \neq \emptyset.$$

Remark 1. Fulfillment of assumption 1 will allow further organizing the pursuit of evaders, so that each pursuer will carry out the capture of "its" evader.

It follows from the measurable choice theorem [13, Theorem 8.1.3], that for every $i \in I$ for any $\geqslant t_0$, there exists at least one measurable selector $\gamma_{iq(i)}(t,\tau) \in W_{iq(i)}(t,\tau)$ for all $t \geqslant t_0, \tau \in [t_0,t]$. Let us choose arbitrary measurable selectors $\gamma_{iq(i)}(t,\tau)$, fix them and denote

$$\xi_{iq(i)}(t) = \pi_{iq(i)}\Psi_{iq(i)}(t,t_0)z^0_{iq(i)} + \int_{t_0}^{t} \gamma_{iq(i)}(t,\tau)d\tau.$$

Theorem 1. *Let Assumption* 1 *be satisfied, and there exist* $T > t_0, l \in I$ *such that* $\xi_{lq(l)}(T) \in M^0_{lq(l)}$. *Then a capture occurs in the game* $G(n,m,z^0)$.

Proof. Let's consider the multivalued mapping ($\tau \in [t_0,T], v \in V$):

$$U_l(T,\tau,v) = \{u_l \in U_l : \pi_{lq(l)}\Phi_{lq(l)}(T,\tau)(u-v) - \gamma_{lq(l)}(T,\tau) = 0\}.$$

By assumption 1, $U_l(T,\tau,v) \neq \emptyset$ for all $\tau \in [t_0,T], v \in V$. It follows from the measurable choice theorem [13, Theorem 8.1.3], that there exists a measurable selector $u_l^*(\tau,v) \in U_l(T,\tau,v)$. We assume the control of the pursuer P_l is equal to

$$u_l(\tau) = u_l^*(\tau, v(\tau)), \quad \tau \in [t_0,T].$$

The controls of the other pursuers are set arbitrarily. The solution of the Cauchy problem for the system (1) is represented as [14]

$$z_{lq(l)}(T) = \Psi_{lq(l)}(T,t_0)z^0_{lq(l)} + \int_{t_0}^{T} \Phi_{lq(l)}(T,s)(u_l(s) - v(s))ds,$$

therefore

$$\pi_{lq(l)}z_{lq(l)}(T) = \xi_{lq(l)}(T) + \int_{t_0}^{T} (\pi_{lq(l)}\Phi_{lq(l)}(T,s)(u_l(s)-v(s)) - \gamma_{lq(l)}(T,s))ds = \xi_{lq(l)}(T) \in M^0_{lq(l)}.$$

This means that a capture of at least one evader occurs in the game $G(n,m,z^0)$. The theorem is proven.

In the following, we will assume that $\xi_{iq(i)}(t) \notin M^0_{iq(i)}$ is for all $i \in I, t \geqslant t_0$.

Consider an arbitrary diagonal square matrix Λ_i of order $k_i \times k_i$, where k_i is the dimension of $L_{iq(i)}$, of the form

$$\Lambda_i = \begin{pmatrix} \lambda_{i1} & 0 & \dots & 0 \\ 0 & \lambda_{i2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_{ik_i} \end{pmatrix} = \text{diag}(\lambda_{i1}, \lambda_{i2}, \dots, \lambda_{ik_i}).$$

We will identify the matrix Λ_i with the vector $(\lambda_{i1},\dots,\lambda_{ik_i})$. We will understand the inequality $\Lambda_i \geqslant 0$ coordinatewise. Let us introduce multivalued mappings

$$M_i(t,\tau,v) = \{\Lambda_i : \Lambda_i \geqslant 0, \Lambda_i(M^0_{iq(i)} - \xi_{iq(i)}(t)) \cap (W_{iq(i)}(t,\tau,v) - \gamma_{iq(i)}(t,\tau)) \neq \emptyset\}.$$

Due to the properties of the parameters of the conflict-controlled process, the mappings $M_i(t,\tau,v)$ are (τ,v) measurable mappings [15]. Let us define the scalar functions

$$\lambda_i^0(t,\tau,v) = \sup_{\Lambda_i \in M_i(t,\tau,v)} \min_{l \in J_i} \lambda_{il}(t,\tau,v), \quad J_i = \{1,\dots,k_i\}. \tag{2}$$

Assumption 2. *For all* $t \geqslant t_0, \tau \in [t_0,t], v \in V$, *an exact upper bound is achieved in* (2).

We consider this assumption to be satisfied. Let us define the set

$$M_i^*(t,\tau,v) = \{\Lambda_i(t,\tau,v) \in M_i(t,\tau,v) : \lambda_i^0(t,\tau,v) = \min_{l \in J_i} \lambda_{il}(t,\tau,v)\}.$$

It follows from [15], that under the assumptions made, $M_i^*(t,\tau,v)$ is measurable by (τ,v) and closed-valued at any $t \geqslant 0$. By the measurable choice theorem [13, Theorem 8.1.3], for each $i \in I$ in $M_i^*(t,\tau,v)$, there exists at least one selector measurable by (τ,v) at any $t \geqslant 0$. Let us fix these selectors and denote them by $\Lambda_i^*(t,\tau,v) = \text{diag}(\lambda^*_{i1}(t,\tau,v),\dots,\lambda^*_{ik_i}(t,\tau,v))$. Let further

$$\delta(t,\tau) = \inf_{v \in V} \max_{i \in I} \min_{l \in J_i} \lambda^*_{il}(t,\tau,v).$$

Lemma 1. *Let assumptions* 1, 2 *be satisfied,*

$$\lim_{t\to+\infty}\int_{t_0}^{t}\delta(t,s)ds=+\infty. \tag{3}$$

Then, there exists a moment $T>t_0$ such that for each measurable function $v(\cdot)$, $v(t)\in V$, $t\in[t_0,T]$, there exists a number $l\in I$, such that for all $p\in J_l$ the inequalities are true:

$$\int_{t_0}^{T}\lambda_{lp}^*(T,s,v(s))ds\geqslant 1.$$

Proof. Let $v(\cdot)$ be an arbitrary admissible function. Then for all $t\geqslant t_0$, $s\in[t_0,t]$, $l\in I$, $p\in J_l$, the inequalities are true:

$$\lambda_{lp}^*(t,s,v(s))\geqslant\lambda_l^*(t,s,v(s)). \tag{4}$$

In addition, relations are true,

$$\max_{l\in I}\int_{t_0}^{t}\lambda_l^*(t,s,v(s))ds\geqslant\frac{1}{n}\int_{t_0}^{t}\sum_{l\in I}\lambda_l^*(t,s,v(s))ds\geqslant\frac{1}{n}\int_{t_0}^{t}\max_{l\in I}\lambda_l^*(t,s,v(s))ds\geqslant\frac{1}{n}\int_{t_0}^{t}\delta(t,s)ds.$$

It follows from condition (3) that there exists a number $T>t_0$, for which

$$\frac{1}{n}\int_{t_0}^{T}\delta(T,s)ds\geqslant 1.$$

Hence,

$$\max_{l\in I}\int_{t_0}^{T}\lambda_l^*(T,s,v(s))ds\geqslant 1,$$

so there is a number $l\in I$, for which

$$\int_{t_0}^{T}\lambda_l^*(T,s,v(s))ds\geqslant 1.$$

From the last inequality and inequality (4), the validity of the statement of the lemma follows.

Let's find the number

$$T_0=\inf\left\{t\geqslant t_0:\inf_{v(\cdot)}\max_{l\in I}\min_{p\in J_l}\int_{t_0}^{t}\lambda_{lp}^*(t,s,v(s))ds\geqslant 1\right\}.$$

Consider the sets ($i\in I$, $p\in J_l$)

$$T_{ip}(v(\cdot))=\left\{t\geqslant t_0:\int_{t_0}^{t}\lambda_{ip}^*(T_0,s,v(s))ds\geqslant 1\right\}.$$

Let's determine the values

$$t_{ip}^*(v(\cdot))=\begin{cases}\inf\{t:t\in T_{ip}(v(\cdot))\}, & \text{if } T_{ip}(v(\cdot))\neq\emptyset,\\ +\infty, & \text{if } T_{ip}(v(\cdot))=\emptyset.\end{cases}$$

Assumption 3. 1) *For all $\tau\in[t_0,T_0]$, $v\in V$, $l\in I$, $J_l^0\subset J_l$, selectors $B_l(T_0,\tau,v)=diag(\beta_{l1}(T_0,\tau,v),\ldots$ $\ldots,\beta_{lk_l}(T_0,\tau,v))$ where*

$$\beta_{lp}(T_0,\tau,v)=\begin{cases}\lambda_{lp}^*(T_0,\tau,v), & \text{if } p\in J_l^0,\\ 0, & \text{if } p\notin J_l^0,\end{cases}$$

satisfy the condition $B_l(T_0,\tau,v)\subset M_l(T_0,\tau,v)$.

2) $\int_{t_0}^{T_0}B_l(T_0,s,v(s))M_{lq(l)}^0ds\subset M_{lq(l)}^0$.

Theorem 2. *Let assumptions* 1–3 *and condition* (19) *be satisfied. Then, at least one evader is captured in the game* $G(n,m,z^0)$.

Proof. It follows from lemma 1, that $T_0 < +\infty$. Let $v : [t_0, T_0] \to V$ be an arbitrary admissible function. Let us introduce the functions $B_l^*(T_0, t, v) = \text{diag}(\beta_{l1}^*(T_0, t, v), \ldots, \beta_{lk_l}^*(T_0, t, v))$, where

$$\beta_{lp}^*(T_0, t, v) = \begin{cases} \lambda_{lp}^*(T_0, t, v), & \text{if } t \in [t_0, t_{lp}^*(v(\cdot))), \\ 0, & \text{if } t \in [t_{lp}^*(v(\cdot)), T_0]. \end{cases}$$

By assumption 3, $B_i^*(T_0, t, v)$ is a measurable selector of $M_i(T_0, t, v)$. Consider multivalued mappings

$$U_i(T_0, t, v) = \{u_i \in U_i : \pi_{iq(i)}\Phi_{iq(i)}(T_0, t)(u_i - v) - \gamma_{iq(i)}(T_0, t) \in B_i^*(T_0, t, v)(M_{iq(i)}^0 - \xi_{iq(i)}(T_0))\}.$$

Then $U_i(T_0, t, v) \neq \emptyset$ for all $i \in I$, $t \in [t_0, T_0]$, $v \in V$, and hence by the measurable choice theorem [13, Theorem 8.1.3], $U_i(T_0, t, v)$ has at least one measurable selector $u_i^*(T_0, t, v)$. We define the pursuers' controls by assuming $u_i(t) = u_i^*(T_0, t, v(t))$. We'll show that this evaders' control guarantees the capture of at least one evader.

The solution of the Cauchy problem of the system (1) has the form [14]

$$z_{iq(i)}(t) = \Psi_{iq(i)}(t, t_0)z_{iq(i)}^0 + \int_{t_0}^{t} \Phi_{iq(i)}(t, s)(u_i(s) - v(s))ds,$$

therefore

$$\begin{aligned}
\pi_{iq(i)}z_{iq(i)}(T_0) &= \pi_{iq(i)}\Psi_{iq(i)}(T_0, t_0)z_{iq(i)}^0 + \int_{t_0}^{T_0} \gamma_{iq(i)}(T_0, s)ds + \\
&+ \int_{t_0}^{T_0} (\pi_{iq(i)}\Phi_{iq(i)}(T_0, s)(u_i(s) - v(s)) - \gamma_{iq(i)}(T_0, s))ds = \\
&= \xi_{iq(i)}(T_0) + \int_{t_0}^{T_0} (\pi_{iq(i)}\Phi_{iq(i)}(T_0, s)(u_i(s) - v(s)) - \gamma_{iq(i)}(T_0, s))ds \in \\
&\in \xi_{iq(i)}(T_0) + \int_{t_0}^{T_0} B_i^*(T_0, s, v(s))(M_{iq(i)}^0 - \xi_{iq(i)}(T_0))ds = \\
&= \xi_{iq(i)}(T_0)\left(E^0 - \int_{t_0}^{T_0} B_i^*(T_0, s, v(s))ds\right) + \int_{t_0}^{T_0} B_i^*(T_0, s, v(s))M_{iq(i)}^0 ds.
\end{aligned}$$

From the definition of $B_i^*(T_0, s, v)$ and lemma 1, it follows that there exists a number $l \in I$, for which

$$\int_{t_0}^{T_0} B_l^*(T_0, s, v(s))ds = E^0.$$

Then,

$$\pi_{lq(l)}z_{lq(l)}(T_0) = \int_{t_0}^{T_0} B_l^*(T_0, s, v(s))M_{lq(l)}^0 ds \subset M_{lq(l)}^0.$$

The theorem is proved.

Remark 2. Scalar resolving functions are a special case of matrix resolving functions, since they are represented in the form λE^0, where λ is a non-negative real number.

Example 1. Let the system (1) $k = 2$, $n = m = 1$, $t_0 = 0$, $A_{11}(t) = 0$ for all t, $V = \{0\}$, $z_{11}^0 = (2, 1)$, $M_{11}^* = \{0\}$, $U_1 = \{(u_1, u_2) : u_1 = 0, u_2 \in [-1, 1]\} \cup \{(u_1, u_2) : u_2 = 0,\ u_1 \in [-1, 1]\} \cup \{(u_1, u_2) : u_1 = u_2 \in [-1, 1]\}$. Then

$$\Psi_{11}(t, t_0) = E^0, \quad \Phi_{11}(t, s) = \frac{(t - s)^{\alpha - 1}}{\Gamma(\alpha)}, \quad W_{11}(t, s, v) = W_{11}(t, s) = \frac{(t - s)^{\alpha - 1}}{\Gamma(\alpha)}U_1.$$

Let's take $\gamma_{11}(t, s) = 0$ for all (t, s), then $\xi_{11}(t) = z_{11}^0$,

$$\begin{aligned}
M_1(t, s, v) &= \left\{\begin{pmatrix} 0 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \lambda_2 = \frac{\lambda(t - s)^{\alpha - 1}}{\Gamma(\alpha)}, \lambda \in [0, 1]\right\} \cup \\
&\cup \left\{\begin{pmatrix} \lambda_2/2 & 0 \\ 0 & 0 \end{pmatrix}, \lambda_2 = \frac{\lambda(t - s)^{\alpha - 1}}{\Gamma(\alpha)}, \lambda \in [0, 1]\right\} \cup \left\{\begin{pmatrix} \lambda_2/2 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \lambda_2 = \frac{\lambda(t - s)^{\alpha - 1}}{\Gamma(\alpha)}, \lambda \in [0, 1]\right\},
\end{aligned}$$

$$\lambda_1^*(t,s,v) = \sup_{\Lambda\in M_1(t,s,v)} \min_{l\in J_1} \lambda_{1l}(t,s,v) = \frac{(t-s)^{\alpha-1}}{2\Gamma(\alpha)}.$$

Hence,

$$M_1^*(t,s,v) = \mathrm{diag}\left(\frac{(t-s)^{\alpha-1}}{2\Gamma(\alpha)}, \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\right), \quad \delta(t,s) = \frac{(t-s)^{\alpha-1}}{2\Gamma(\alpha)}.$$

We have $\lim_{t\to+\infty}\int_0^t \delta(t,s)ds = +\infty$, so $T_0 = (2\Gamma(\alpha+1))^{1/\alpha}$. Let $T_1 = T_0 - (\Gamma(\alpha+1))^{1/\alpha}$. The control of pursuer P_1 has the form

$$u_1(t) = \begin{cases} (-1,-1), & t\in[0,T_1), \\ (-1,0), & t\in[T_1,T_0], \end{cases}$$

then [14]

$$z_{11}(T_0) = z_{11}^0 + \frac{1}{\Gamma(\alpha)}\int_0^{T_0}(T_0-s)^{\alpha-1}u_1(s)ds = 0.$$

Note that the use of scalar resolving functions, i.e., functions of the form

$$\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix},$$

does not allow us to prove the solvability of the pursuit problem, since in this case the condition $-\Lambda z_{11}^0 \in U_1 - v$ is satisfied only for the zero matrix Λ.

Example 2. Consider the game $G(n,1,z^0)$, in which the system (1) has the form

$$\begin{cases} (D^{(\alpha)})z_{i1} = tz_{i2}, \\ (D^{(\alpha)})z_{i2} = u_i - v, \end{cases} \qquad z_i(0) = z_i^0. \tag{5}$$

Here $z_i = (z_{i1}, z_{i2}) \in \mathbb{R}^{2k}$, $U_i = V = \{v\in\mathbb{R}^k : \|v\| \leqslant 1\}$, $M_{i1}^* = \{(z_{i1},z_{i2})\in\mathbb{R}^{2k} : z_{i1} = 0\}$, so $(i\in I)$

$$M_{i1}^0 = \{(z_{i1},z_{i2})\in\mathbb{R}^{2k} : z_{i1} = z_{i2} = 0\}, \quad M_{i1} = \{(z_{i1},z_{i2})\in\mathbb{R}^{2k} : z_{i1} = 0\},$$

$$L_{i1} = \{(z_{i1},z_{i2})\in\mathbb{R}^{2k} : z_{i2} = 0\}, \quad \pi_{i1} = \begin{pmatrix} E^0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Let's denote

$$p(t,\tau) = \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}, \quad q(t,\tau) = \frac{\alpha(t-\tau)^{2\alpha-1}(t+\tau)}{\Gamma(2\alpha+1)}, \quad r(t,\tau) = \frac{(t-\tau)^{\alpha}(t+\alpha\tau)}{\Gamma(\alpha+2)}.$$

Then [14]

$$\Psi_i(t,\tau) = \begin{pmatrix} E^0 & r(t,\tau)E^0 \\ 0 & E^0 \end{pmatrix}, \quad \Phi_i(t,\tau) = \begin{pmatrix} p(t,\tau)E^0 & q(t,\tau)E^0 \\ 0 & p(t,\tau)E^0 \end{pmatrix}.$$

Hence,

$$W_i(t,\tau,v) = q(t,\tau)(V-v), \quad W_i(t,\tau) = \{0\}, \quad \gamma_i(t,\tau) = 0, \quad \xi_i(t) = \pi_i\Psi_i(t,0)z_i^0 = z_{i1}^0 + r(t,0)z_{i2}^0,$$

$$\lambda_i(t,\tau,v) = q(t,\tau)\frac{(\xi_i(t),v) + \sqrt{(\xi_i(t),v)^2 + \|\xi_i(t)\|^2(1-\|v\|^2)}}{\|\xi_i(t)\|^2}.$$

Assertion. *Let $z_{i2}^0 = 0$ for all $i\in I$ and $0\in \operatorname{Int}\operatorname{co}\{z_{i1}^0, i\in I\}$. Then a capture occurs in the game $G(n,1,z^0)$.*

Proof. In this case, $\xi_{i1}(t) = z_{i1}^0$ for all $t>0$. It follows from [16], that

$$\delta(t,\tau) = \min_v \max_i \lambda_i(t,\tau,v) \geqslant q(t,\tau)\delta_0$$

for all t,τ with some $\delta_0 > 0$. Therefore, all conditions of Theorem 2 are satisfied and, hence there is a capture in the game $G(n,1,z^0)$. The assertion is proved.

Note that in [14], the problem of pursuit by one pursuer of one evader described by system (5), in which the pursuer has an advantage over the evader, was considered in the space $\mathbb{R}^2$.

3. SUFFICIENT CAPTURE CONDITIONS IN THE LINEAR STATIONARY CASE WITH SIMPLE MATRICES

Theorem 3. *Let in the system (1) for all* i, j $A_{ij}(t) = a_{ij}E^0$ *for any* t, $M_{ij}^* = \{0\}$, $t_0 = 0$, $U_i = V = \{v : \|v\| \leqslant 1\}$, *there exists a mapping* $q : I \to J$ *such that* $a_{iq(i)} < 0$ *for all* $i \in I$ *and*

$$0 \in \operatorname{Int} \operatorname{co}\{z_{iq(i)}^0, i \in I\}. \tag{6}$$

Then a capture of at least one evader occurs in the game $G(n, m, z^0)$.

Proof. In this case

$$\Psi_{iq(i)}(t, t_0) = E_{1/\alpha}(a_{iq(i)}t^\alpha, 1), \quad \Phi_{iq(i)}(t, \tau) = (t-\tau)^{\alpha-1}E_{1/\alpha}(a_{iq(i)}(t-\tau)^\alpha, \alpha),$$

where $E_\rho(z, \mu) = \sum_{l=0}^{\infty} z^l/\Gamma(l\rho^{-1} + \mu)$ is the Mittag-Leffler function. Assumption 1 is fulfilled.

Let's take $\gamma_{iq(i)}(t, \tau) = 0$ as selectors for all $i \in I$, $t \geqslant 0$, $\tau \in [0, t]$. Then $\xi_{iq(i)}(t) = \pi_{iq(i)}E_{1/\alpha}(a_{iq(i)}t^\alpha, 1)z_{iq(i)}^0$. Let

$$\lambda(z, v) = \sup\{\lambda \geqslant 0 : -\lambda z \in V - v\}, \quad \delta = \min_{v\in V}\max_{i\in I}\lambda(z_{iq(i)}^0, v), \quad a = \min_{i\in I} a_{iq(i)}.$$

It follows from condition (6) and from [16], that $\delta > 0$. Let us show that there exists $T > 0$ such that for any admissible function $v(\cdot)$ there exists $l \in I$, for which

$$E_{1/\alpha}(a_{lq(l)}T^\alpha, 1) - \int_0^T (T-s)^{\alpha-1}E_{1/\alpha}(a_{lq(l)}(T-s)^\alpha, \alpha)\lambda(z_{lq(l)}^0, v(s))ds \leqslant 0. \tag{7}$$

Consider the functions

$$h_i(t, v(\cdot)) = E_{1/\alpha}(a_{iq(i)}t^\alpha, 1) - \int_0^t (t-s)^{\alpha-1}E_{1/\alpha}(a_{iq(i)}(t-s)^\alpha, \alpha)\lambda(z_{iq(i)}^0, v(s))ds.$$

It follows from [17], that for all $t \geqslant 0$, $\tau \in [0, t]$, $i \in I$ the inequalities hold

$$E_{1/\alpha}(a_{iq(i)}(t-\tau)^\alpha, \alpha) \geqslant E_{1/\alpha}(a(t-\tau)^\alpha, \alpha).$$

It follows from Theorem 4.1.1 of [18], that for all $t \geqslant 0$, $\tau \in [0, t]$, the inequality $E_{1,\alpha}(a(t-\tau)^\alpha, \alpha) \geqslant 0$ is true. From the last two inequalities we obtain

$$\begin{aligned}
&\sum_{i=1}^{n}\int_0^t (t-s)^{\alpha-1}E_{1/\alpha}(a_{iq(i)}(t-s)^\alpha, \alpha)\lambda(z_{iq(i)}, v(s))\,ds \geqslant \\
&\geqslant \int_0^t (t-s)^{\alpha-1}E_{1/\alpha}(a(t-s)^\alpha, \alpha)\max_{i\in I}\lambda(z_{iq(i)}, v(s))\,ds \geqslant \\
&\geqslant \delta\int_0^t (t-s)^{\alpha-1}E_{1/\alpha}(a(t-s)^\alpha, \alpha)\,ds = \delta t^\alpha E_{1/\alpha}(at^\alpha, \alpha+1),
\end{aligned}$$

hence

$$F(t) = \sum_{i=1}^{n} h_i(t, v(\cdot)) \leqslant \sum_{i=1}^{n} E_{1/\alpha}(a_{iq(i)}t^\alpha, 1) - \delta t^\alpha E_{1/\alpha}(at^\alpha, \alpha+1).$$

Since $a_{iq(i)} < 0$ for all $i \in I$, it follows from [18] that the asymptotic representation is valid at $t \to +\infty$

$$E_{1/\alpha}(a_{iq(i)}t^\alpha, 1) = -\frac{1}{a_{iq(i)}t^\alpha\Gamma(\alpha+1)} + O\left(\frac{1}{t^{2\alpha}}\right), \quad E_{1/\alpha}(at^\alpha, \alpha+1) = -\frac{1}{at^\alpha} + O\left(\frac{1}{t^{2\alpha}}\right),$$

$$F(t) = -\sum_{i=1}^{n}\frac{1}{a_{iq(i)}t^\alpha\Gamma(\alpha+1)} + \frac{1}{a} + O\left(\frac{1}{t^\alpha}\right),$$

hence $\lim_{t\to+\infty} F(t) < 0$. So $\lim_{t\to+\infty}\sum_{i=1}^{n} h_i(t, v(\cdot)) < 0$. Since $\sum_{i=1}^{n} h_i(0, v(\cdot)) > 0$, there exists $T > 0$, for which for any admissible function $v(\cdot)$ the inequality $\sum_{i=1}^{n} h_i(T, v(\cdot)) < 0$ is true. Thus, inequality (9) is proved.

Let

$$T_0 = \min\left\{t : \inf_{v(\cdot)}\min_{i\in I}\left(E_{1/\alpha}(a_{iq(i)}t^{\alpha}, 1) - \int_0^t (t-s)^{\alpha-1}E_{1/\alpha}(a_{iq(i)}(t-s)^{\alpha}, \alpha)\lambda(z^0_{iq(i)}, v(s))ds\right) \leqslant 0\right\}.$$

It follows from inequality (7), that $T_0 < +\infty$. Let $v(\cdot)$ be an admissible fleeing control. Consider the sets

$$T_i(v(\cdot)) = \left\{t \geqslant 0 : E_{1/\alpha}(a_{iq(i)}T_0^{\alpha}, 1) - \int_0^t (T_0-s)^{\alpha-1}E_{1/\alpha}(a_{iq(i)}(T_0-s)^{\alpha}, \alpha)\lambda(z^0_{iq(i)}, v(s))ds \leqslant 0\right\}.$$

Let the following be

$$t_i(v(\cdot)) = \begin{cases} \inf\{t : t \in T_i(v(\cdot))\}, & \text{if } T_i(v(\cdot)) \neq \emptyset,\\ +\infty, & \text{if } T_i(v(\cdot)) = \emptyset,\end{cases}$$

$$\beta_i(t, v(\cdot)) = \begin{cases} \lambda(z_{iq(i)}, v(t)), & t \in [0, t_i(v(\cdot))],\\ 0, & t \in [t_i(v(\cdot)), T_0].\end{cases}$$

Let's set the controls of the pursuers P_i, $i \in I$, assuming

$$u_i(t) = v(t) - \beta_i(t, v(\cdot))z^0_{iq(i)}.$$

The solution of the Cauchy problem of the system (1) is represented in the form [19]

$$z_{iq(i)}(T_0) = E_{1/\alpha}(a_{iq(i)}T_0^{\alpha}, 1)z^0_{iq(i)} + \int_0^{T_0}(T_0-s)^{\alpha-1}E_{1/\alpha}(a_{iq(i)}(T_0-s)^{\alpha}, \alpha)(u_i(s) - v(s))ds =$$

$$= \left(E_{1/\alpha}(a_{iq(i)}T_0^{\alpha}, 1) - \int_0^{T_0}(T_0-s)^{\alpha-1}E_{1/\alpha}(a_{iq(i)}(T_0-s)^{\alpha}, \alpha)\beta_i(s, v(s))ds\right)z^0_{iq(i)} =$$

$$= \left(E_{1/\alpha}(a_{iq(i)}T_0^{\alpha}, 1) - \int_0^{t_i(v(\cdot))}(T_0-s)^{\alpha-1}E_{1/\alpha}(a_{iq(i)}(T_0-s)^{\alpha}, \alpha)\beta_i(s, v(s))ds\right)z^0_{iq(i)}.$$

It follows from the previous proof that there exists a number $l \in I$, for which $z_{lq(l)}(T_0) = 0$. The theorem is proved.

Example 3. Let $k = 2$, $I = \{1, 2, 3, 4\}$, $J = \{1, 2\}$, $A_{ij}(t) = a_{ij}E^0$, $a_{ij} < 0$, $U_i = V = \{v : \|v\| \leqslant 1\}$, $z^0_{11} = (1, 3)$, $z^0_{21} = (-1, 3)$, $z^0_{31} = (-1, 1)$, $z^0_{41} = (1, 1)$, $z^0_{12} = (0, -1)$, $z^0_{22} = (-2, -1)$, $z^0_{32} = (-2, -3)$, $z^0_{42} = (0, -3)$. Define a mapping $q : I \to J$ as follows: $q(1) = 2$, $q(2) = q(3) = q(4) = 1$. The conditions of Theorem 3 are satisfied, and so a capture of at least one evader occurs in the game $G(4, 2, z^0)$. Note that $0 \notin \operatorname{Int}\operatorname{co}\{z^0_{i1}, i \in I\}$ and $0 \notin \operatorname{Int}\operatorname{co}\{z^0_{i2}, i \in I\}$.

We show that if $a_{iq(i)} > 0$, then condition (6) in Theorem 3 does not guarantee capture.

Example 4. Let $k = 2$, $n = 3$, $m = 1$, $I = \{1, 2, 3\}$, $M^*_{i1} = \{0\}$, $t_0 = 0$, $z^0_{11} = (0, 1)$, $z^0_{21} = (1/2, -\sqrt{3}/2)$, $z^0_{31} = (-1/2, -\sqrt{3}/2)$, $U_i = V = \{v : \|v\| \leqslant 1\}$. System (1) has the form

$$(D^{(1/2)})z_{i1} = z_{i1} + u_i - v.$$

Let's take $v(t) = 0$ for all $t \geqslant 0$. Then we have

$$z_{i1}(t) = E_2(\sqrt{t}, 1)z^0_{i1} + \int_0^t (t-s)^{-1/2}E_2((t-s)^{1/2}, 1/2)u_i(s)ds.$$

Suppose that there exist $T > 0$, function $u_l(\cdot)$, $l \in \{1, 2, 3\}$, for which $z_{l1}(T) = 0$. Then [20, p. 120, formula (1.15)]

$$E_2(\sqrt{T}, 1) = \|E_2(\sqrt{T}, 1)z^0_{l1}\| = \left\|\int_0^T (T-s)^{-1/2}E_2\big((T-s)^{1/2}, 1/2\big)u_l(s)\,ds\right\| \leqslant$$

$$\leqslant \int_0^T (T-s)^{-1/2}E_2\big((T-s)^{1/2}, 1/2\big)\,ds = \sqrt{T}E_2(\sqrt{T}, 3/2).$$

By virtue of [20, p. 118, formula (1.4)],

$$E_2(\sqrt{T}, 3/2) = \frac{1}{\sqrt{T}}(E_2(\sqrt{T}, 1) - 1).$$

Relation (7) entails the inequality

$$E_2(\sqrt{T}, 1) \leqslant E_2(\sqrt{T}, 1) - 1,$$

which is impossible. Consequently, in this game $G(3, 1, z^0)$, capture does not occur.

4. CAPTURE OF ALL EVADERS

In the space $\mathbb{R}^k$ ($k \geqslant 2$), we consider a differential game $G(1, m, z^0)$ involving $1 + m$ persons: one pursuer P_1 and m evaders $E_1, \ldots, E_m$. The law of motion of the pursuer P_1 has the form

$$(D^{(\alpha)})x_1 = ax_1 + u, \quad x_1(0) = x_1^0, \quad u \in V;$$

the law of motion of each of the evaders E_j is of the form

$$(D^{(\alpha)})y_j = ay_j + v_j, \quad y_j(0) = y_j^0, \quad v_j \in V.$$

Here $V = \{v : \|v\| \leqslant 1\}$, $\alpha \in (0, 1)$, $a \in \mathbb{R}^1$, $D^{(\alpha)} f$ is the Caputo derivative of the function f of order α, $j \in J = \{1, \ldots, m\}$. We consider $x_1^0 \neq y_j^0$ for all $j \in J$.

Let's denote

$$f(t) = E_{1/\alpha}(at^\alpha, 1), \quad F(t) = t^\alpha E_{1/\alpha}(at^\alpha, \alpha + 1), \quad z_j^0 = y_j^0 - x_1^0.$$

Lemma 2. *Let* $a < 0$*,* $T_2 > T_1 \geqslant 0$*,*

$$h(t) = \int_{T_1}^{T_2} (t - s)^{\alpha - 1} E_{1/\alpha}(a(t - s)^\alpha, \alpha) ds.$$

Then $\lim_{t \to +\infty} t^\alpha h(t) = 0$.

Proof. By substituting $t - s = \tau$ we get

$$h(t) = \int_{t - T_2}^{t - T_1} \tau^{\alpha - 1} E_{1/\alpha}(a\tau^\alpha, \alpha) d\tau.$$

By virtue of formula (2.32) from [20, p. 136], the inequality

$$|E_{1/\alpha}(a\tau^\alpha, \alpha)| \leqslant \frac{M}{\tau^\alpha}, \quad M > 0,$$

is true for all $t > T2$, therefore

$$|h(t)| = \left| \int_{t - T_2}^{t - T_1} \tau^{\alpha - 1} E_{1/\alpha}(a\tau^\alpha, \alpha) d\tau \right| \leqslant \int_{t - T_2}^{t - T_1} \frac{M\tau^{\alpha - 1}}{\tau^\alpha} d\tau = M(\ln(t - T_1) - \ln(t - T_2)).$$

Then

$$|t^\alpha h(t)| \leqslant Mt^\alpha(\ln(t - T_1) - \ln(t - T_2)) = Mt^\alpha \ln\left(1 + \frac{T_2 - T_1}{t - T_2}\right) \leqslant \frac{Mt^\alpha(T_2 - T_1)}{t - T_2}.$$

Since $\lim_{t \to +\infty} \frac{t^\alpha}{t - T_2} = 0$, then $\lim_{t \to +\infty} t^\alpha h(t) = 0$. The lemma is proved.

Theorem 4. *Let* $a < 0$*,* $M_{1j}^* = \{0\}$ *for all* $j \in J$*, there is* $v_0 \in V$*,* $\|v_0\| = 1$*, such that* $(y_j^0 - x_1^0, v_0) < 0$ *for all* $j \in J$*. All evaders use constant control* v_0*, the pursuer* P_1 *knows* v_0*. Then a capture of all evaders occurs in the game* $G(1, m, z^0)$*.*

Proof. 1. We show that there exist a moment T_m and a vector u_m, $\|u_m\| = 1$, for which the equality $x_1(T_m) = y_m(T_m)$ holds, where $x_1(t)$ is the trajectory of the pursuer P_1, using constant control u_m.

Let the pursuer P_1 uses the constant control u on the interval $[0, T_m]$. Then, by virtue of the Cauchy formula [19] and formula (1.15) from [20, p. 120], we have

$$x_1(t) = f(t)x_1^0 + \int_0^t (t-s)^{\alpha-1}E_{1/\alpha}(a(t-s)^\alpha, \alpha)ds \cdot u = f(t)x_1^0 + F(t)u,$$

$$y_m(t) = f(t)y_m^0 + F(t)v_0.$$

The $x_1(t) = y_m(t)$ can be represented as

$$F(t)u = f(t)z_m^0 + F(t)v_0.$$

Let us require that $\|u\| = 1$. For this purpose, consider the function

$$g_m(t) = \|f(t)z_m^0 + F(t)v_0\|^2 - F^2(t) = f^2(t)\|z_m^0\|^2 + 2f(t)F(t)(z_m^0, v_0),$$

where (a, b) is the scalar product of the vectors a and b. It follows from Theorem 4.1.1 [18], that $f(t) > 0$, $F(t) > 0$ for all $t > 0$. Therefore, the equation $g_m(t) = 0$ is equivalent to the equation

$$\frac{f(t)}{F(t)} = -\frac{2(z_m^0, v_0)}{\|z_m\|^2}. \tag{8}$$

Note that $\lim_{t\to+0} \frac{f(t)}{F(t)} = +\infty$. By virtue of Theorem 1.2.1 of [18], we have the asymptotic estimates

$$f(t) = -\frac{1}{at^\alpha\Gamma(1-\alpha)} + O(1/t^{2\alpha}), \quad F(t) = -\frac{1}{a} + O(1/t^\alpha), \tag{9}$$

therefore $\lim_{t\to+\infty} \frac{f(t)}{F(t)} = 0$. Hence, equation (8) has at least one positive root T_m. We now assume that the control of the pursuer P_1 on the interval $[0, T_m]$ is equal to

$$u_m = \frac{f(T_m)}{F(T_m)} z_m^0 + v_0.$$

We obtain that at time T_m, the pursuer P_1 will realize the capture of the evader E_m.

2. Let us further construct a control for the pursuer P_1, that guarantees the capture of E_{m-1}. Suppose that at $[T_m, T_{m-1}]$, the pursuer P_1 uses the constant control u (the moment T_{m-1} will be defined below). Then, by virtue of the Cauchy formula [19] ($t > T_m$),

$$x_1(t) = f(t)x_1^0 + \int_0^{T_m} (t-s)^{\alpha-1}E_{1/\alpha}(a(t-s)^\alpha, \alpha)ds \cdot u_m + \int_{T_m}^t (t-s)^{\alpha-1}E_{1/\alpha}(a(t-s)^\alpha, \alpha)ds \cdot u,$$

$$y_{m-1}(t) = f(t)y_{m-1}^0 + F(t)v_0.$$

Let's denote

$$H_m(t) = \int_{T_m}^t (t-s)^{\alpha-1}E_{1/\alpha}(a(t-s)^\alpha, \alpha)ds, \quad h_m(t) = \int_0^{T_m} (t-s)^{\alpha-1}E_{1/\alpha}(a(t-s)^\alpha, \alpha)ds.$$

Note that $H_m(t) + h_m(t) = F(t)$. Then the equality $x_1(t) = y_{m-1}(t)$ can be represented as

$$f(t)x_1^0 + h_m(t)u_m + H_m(t)u = f(t)y_{m-1}^0 + F(t)v_0$$

or

$$H_m(t)u = f(t)z_{m-1}^0 + F(t)v_0 - h_m(t)u_m.$$

Consider the function

$$g_{m-1}(t) = \|f(t)z_{m-1}^0 + F(t)v_0 - h_m(t)u_m\|^2 - H_m^2(t).$$

Then

$$g_{m-1}(T_m) = \|f(T_m)z^0_{m-1} + F(T_m)v_0 - h_m(T_m)u_m\|^2.$$

Since $F(T_m) = h_m(T_m)$ and $F(T_m)(v_0 - u_m) = -f(T_m)z^0_m$, then

$$g_{m-1}(T_m) = \|f(T_m)z^0_{m-1} - f(T_m)z^0_m\|^2 = f^2(T_m)\|z^0_{m-1} - z^0_m\|^2 > 0.$$

The function $t^\alpha g_{m-1}(t)$ can be written as

$$t^\alpha g_{m-1}(t) = t^\alpha f^2(t)\|z^0_{m-1}\|^2 + 2t^\alpha f(t)F(t)(z^0_{m-1}, v_0) - 2t^\alpha f(t)h_m(t)(z^0_{m-1}, u_m) - \\ - 2t^\alpha F(t)h_m(t)(v_0, u_m) + 2t^\alpha F(t)h_m(t).$$

By virtue of asymptotic estimates (9) and lemma 2, we obtain that the following relations are true

$$\lim_{t\to+\infty} t^\alpha f(t)F(t) = \frac{1}{a^2\Gamma(1-\alpha)}, \quad \lim_{t\to+\infty} t^\alpha f^2(t) = 0,$$

$$\lim_{t\to+\infty} t^\alpha f(t)h_m(t) = 0, \quad \lim_{t\to+\infty} t^\alpha F(t)h_m(t) = 0,$$

so it follows from the inequality $(z^0_{m-1}, v_0) < 0$, that $\lim_{t\to+\infty} t^\alpha g_{m-1}(t) = -\infty$, and hence there exists a moment $T_{m-1} > T_m$, for which $g_{m-1}(T_{m-1}) = 0$.

Choosing now on the interval $[T_m, T_{m-1}]$ control u_{m-1} of the form

$$u_{m-1} = f(T_{m-1})z^0_{m-1} + F(T_{m-1})v_0 - h_m(T_{m-1})u_m/H_m(T_{m-1}),$$

the pursuer P_1 at the moment T_{m-1} will catch the evader E_{m-1}.

3. Let's denote

$$h_l(t) = \int_{T_{l+1}}^{T_l} (t-s)^{\alpha-1}E_{1/\alpha}(a(t-s)^\alpha, \alpha)ds, \quad H_{k+1}(t) = \int_{T_{k+1}}^{t} (t-s)^{\alpha-1}E_{1/\alpha}(a(t-s)^\alpha, \alpha)ds,$$

$$s_l(t) = h_m(t)u_m + \dots + h_l(t)u_l, \quad \hat{s}_l(t) = h_m(t) + \dots + h_l(t), \quad l = m-1, \dots, k+1.$$

Suppose that the vectors $u_m, \dots, u_{k+1}$ and the moments of time $T_m < T_{m-1} < \dots < T_{k+1}$, guaranteeing the pursuer P_1 to catch the evaders $E_m, \dots, E_{k+1}$, are defined, and on the interval $[T_{k+2}, T_{k+1}]$ the vector u_{k+1} is equal to

$$u_{k+1} = f(T_{k+1})z^0_{k+1} + F(T_{k+1})v_0 - s_{k+2}(T_{k+1})/H_{k+2}(T_{k+1}). \quad (10)$$

Let us further construct a control of the pursuer P_1, which guarantees him to catch the evader E_k. Suppose that at $[T_{k+1}, T_k]$, the pursuer P_1 uses the constant control u (the moment T_k will be defined below). Then for $t > T_{k+1}$, by virtue of the Cauchy formula [19], we have

$$x_1(t) = f(t)x^0_1 + \int_0^{T_m} (t-s)^{\alpha-1}E_{1/\alpha}(a(t-s)^\alpha, \alpha)ds \cdot u_m + \int_{T_m}^{T_{m-1}} (t-s)^{\alpha-1}E_{1/\alpha}(a(t-s)^\alpha, \alpha)ds \cdot u_{m-1} + \dots$$

$$\dots + \int_{T_{k+2}}^{T_{k+1}} (t-s)^{\alpha-1}E_{1/\alpha}(a(t-s)^\alpha, \alpha)ds \cdot u_{k+1} + \int_{T_{k+1}}^{t} (t-s)^{\alpha-1}E_{1/\alpha}(a(t-s)^\alpha, \alpha)ds \cdot u,$$

$$y_k(t) = f(t)y^0_k + F(t)v_0.$$

The inequality $x_1(t) = y_k(t)$ can be represented as

$$f(t)x^0_1 + s_{k+1}(t) + H_{k+1}(t)u = f(t)y^0_k + F(t)v_0 \quad \text{или} \quad H_{k+1}(t)u = f(t)z^0_k - s_{k+1}(t) + F(t)v_0.$$

Consider the function

$$g_k(t) = \|f(t)z^0_k - s_{k+1}(t) + F(t)v_0\|^2 - H^2_{k+1}(t),$$

then

$$g_k(T_{k+1}) = \|f(T_{k+1})z^0_k - s_{k+1}(T_{k+1}) + F(T_{k+1})v_0\|^2.$$

It follows from the definition of the functions $H_{k+2}(\cdot)$ and $h_{k+2}(\cdot)$ that $H_{k+2}(T_{k+1}) = h_{k+1}(T_{k+1})$.

Since $s_{k+1}(T_{k+1}) = s_{k+2}(T_{k+1}) + h_{k+1}(T_{k+1})u_{k+1}$, then

$$s_{k+1}(T_{k+1}) = s_{k+2}(T_{k+1}) + H_{k+2}(T_{k+1})u_{k+1}. \tag{11}$$

Using formula (10), let us write equality (11) as

$$s_{k+1}(T_{k+1}) = f(T_{k+1})z^0_{k+1} + F(T_{k+1})v_0.$$

Then

$$g_k(T_{k+1}) = \|f(T_{k+1})z^0_k - f(T_{k+1})z^0_{k+1}\|^2 = f^2(T_{k+1})\|z^0_k - z^0_{k+1}\|^2 > 0.$$

Since $H_{k+1}(t) = F(t) - \hat{s}_{k+1}(t)$, the function $t^\alpha g_k(t)$ can be represented as

$$t^\alpha g_k(t) = t^\alpha f^2(t)\|z^0_k\|^2 + 2t^\alpha f(t)F(t)(z^0_k, v_0) + t^\alpha\|s_{k+1}(t)\|^2 -$$
$$-2t^\alpha F(t)(s_{k+1}(t), v_0) - 2t^\alpha f(t)(s_{k+1}(t), z^0_k) + 2t^\alpha F(t)\hat{s}_{k+1}(t) - t^\alpha \hat{s}^2_{k+1}(t).$$

It follows from lemma 2, that for any l and p

$$\lim_{t\to+\infty} t^\alpha h_l(t)h_p(t) = 0,$$

therefore

$$\lim_{t\to+\infty} t^\alpha\|s_{k+1}(t)\|^2 = \lim_{t\to+\infty} t^\alpha \hat{s}^2_{k+1}(t) = \lim_{t\to+\infty} t^\alpha f^2(t) = 0,$$

hence $\lim_{t\to+\infty} t^\alpha g_k(t) = -\infty$. Therefore, there is a moment $T_k > T_{k+1}$, for which $g_k(T_k) = 0$. Choosing its control u_k on the interval $[T_{k+1}, T_k]$ in the form of

$$u_k = f(T_k)z^0_k + F(T_k)v_0 - s_{k+1}(T_k)/H_{k+1}(T_k),$$

the pursuer P_1 at the moment T_k will catch the fleeing E_k. The theorem is proved.

Corollary. *Let $a < 0$, there exists a hyperplane H such that $y^0_j \in H$ for all $j \in J$, $x^0_1 \notin H$, v_0 the unit normal vector of the hyperplane H, directed into the half-space containing x^0_1. The evaders use constant control v_0. Then a capture of all evaders occurs in the game $G(1, m, z^0)$.*

The validity of this statement follows directly from Theorem 4, since $(y^0_j - x^0_1, v_0) < 0$ for all $j \in J$.

Remark 3. Let the corollary conditions be satisfied and the laws of motion of each participant have the form

$$\dot{x}_1 = ax_1 + u, \quad \dot{y}_j = ay_j + v_j, \quad u, v_j \in V, \quad j \in J. \tag{12}$$

In [2], the problem of evasion a group of evaders from a group of pursuers described by system (12) was considered, where it was shown that in the game $G(1, m, z^0)$, the pursuer P_1 will realize the capture of no more than one evader [2, Corollary 6.3.3, p. 333].

Thus, Theorem 4 shows that differential games described by equations with fractional derivatives have properties that differential games described by ordinary differential equations do not have.

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CONFLICT OF INTERESTS

The authors of this paper declare that they have no conflict of interests.

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