DIFFERENTIAL EQUATIONS, 2025, Vol. 61, No. 1, pp. 83–95

CONTROL THEORY

STABLE SOLUTION OF PROBLEMS OF TRACKING AND DYNAMICAL RECONSTRUCTION UNDER MEASURING PHASE COORDINATES AT DISCRETE TIME MOMENTS

© 2025 V. I. Maksimov*

N.N. Krasovskii Institute of Mathematics and Mechanics of the Ural Branch of RAS, Yekaterinburg, Russia

**e-mail: maksimov@imm.uran.ru*

Recieved October 10, 2024
Revised Novermber 08, 2024
Accepted December 03, 2024

Abstract. The problem of dynamic reconstruction of input actions in a system of ordinary differential equations and the problem of tracking a trajectory of a system by some trajectory of another one influenced by an unknown disturbance are under consideration. An input action is assumed to be an unbounded function, namely, an element of the space of square integrable functions. Two solving algorithms, which are stable with respect to informational noises and computational errors and oriented to program realization, are designed. Upper estimates of their convergence rates are established. The algorithms are based on constructions from feedback control theory. They operate under conditions of (inaccurate) measuring the phase states of the given systems at discrete times.

Keywords: *problem of tracking, reconstruction*

DOI: 10.31857/S03740641250108e8

1. INTRODUCTION. PROBLEM STATEMENT

We consider a system of ordinary differential equations

$$\dot{y}(t) = f(t, y(t)) + Bu(t), \quad t \in T = [0, \vartheta], \tag{1}$$

with the initial condition

$$y(0) = y_0. \tag{2}$$

Here $0 < \vartheta < +\infty$, $y \in \mathbb{R}^N$, $u \in \mathbb{R}^r$ is the input influence, $f(t, y)$ is a Lipschitz (with Lipschitz constant L) vector function over a set of variables, B – a stationary matrix of dimension $N \times r$, $n, r \in \mathbb{N}$.

It is assumed that the system (1) is subjected to an unknown input influence $u(\cdot) \in L_2(T; \mathbb{R}^r)$. At discrete, sufficiently frequent, moments of time $\tau_i \in \Delta = \{\tau_i\}_{i=\overline{0,m}}$ ($\tau_0 = 0$, $\tau_m = \vartheta$, $\tau_{i+1} = \tau_i + \delta$) the phase states $y(\tau_i) = y(\tau_i; y_0, u(\cdot))$ of system (1) are measured. The states $y(\tau_i)$, $i = \overline{0, m-1}$, are measured with error. The measurement results are vectors $\xi_i^h \in \mathbb{R}^N$, satisfying the inequalities

$$|y(\tau_i) - \xi_i^h|_N \le h, \tag{3}$$

where $h \in (0, 1)$ is the level of measurement error, $|\cdot|_N$ denotes the Euclidean norm in the space $\mathbb{R}^N$.

It is required to specify an algorithm for approximate restoration of the input impact based on the results of inaccurate measurements $y(\tau_i)$. For this purpose, we consider the problem consisting in the construction of an algorithm that, based on the current measurements of values $y(\tau_i)$ in “real time”, forms (according to the feedback principle) the function $u = u^h(\cdot)$, that is an approximation (in the space metric $L_2(T;\mathbb{R}^r)$) of some input influence generating the solution $y(\cdot)$ of equation (1).

The formulated problem is a problem of dynamic recovery (reconstruction). One of the approaches to its solution was developed in [1, pp. 7–87; 2, pp. 400–415; 3, pp. 13–93; 4–12]. In [1–10], the case of instantaneous constraints on perturbations was considered; the case of absence of such constraints is described in [3, pp. 41–64; 6; 11; 12]. The approach is based on a combination of methods of the theory of positional control [13], according to which for dynamic, realized at the rate of "real time", restoration of the perturbation acting on the system (1), one proceeds as follows: some controlled system, quite often called a model, is introduced; after that, the restoration task is replaced by the task of forming the control of this model according to the feedback principle in such a way, that at a suitable matching of the measurement error h, the value of the measurement interval δ (as well as, perhaps, some other parameters, e.g., regulation parameter), the control u^h (.) — in one or some other metrics — approximates some input influence that induces a measured solution y (.) of system (1). Usually, when speaking of approximation, one means uniform (space metric C) or mean-square (space metric L_2) metrics. When implementing this approach, in many cases the right-hand side of the model has the same structure as the real system (system (1)). However, instead of the phase vector of the model in its right part there are the values ξ_i^h, i.e., the results of measurements of phase states of the real system instead of the states of the model. Quite often (see, for example, [1, p. 23; 4; 5]) the model has the following form:

$$\dot{y}^h(t) = f(\tau_i, \xi_i^h) + Bu_i^h \quad \text{at a.e.} \quad t \in \delta_i = [\tau_i, \tau_{i+1}), \quad i = \overline{0, m-1}. \tag{4}$$

In this case, the control $u^h(\cdot)$ in the model is formed according to some rule U in the form of feedback:

$$u^h(t) = u_i^h = U(\tau_i, \xi_i^h, y^h(\tau_i)) \quad \text{at a.e.} \quad t \in \delta_i, \quad i = \overline{0, m-1}. \tag{5}$$

In mathematical control theory, one of the "classical" problems is the so-called tracking problem, the study of which began in the fifties of the XX century and was caused by practical problems arising in aviation and astronautics. This problem has not lost its relevance nowadays, in particular, due to the needs of flight dynamics development. The tracking problem is also in demand when analyzing processes arising in control problems of mechanical systems [14, 15], as well as systems functioning under uncertainty [16]. It also plays an important role in the framework of positional differential games [13].

The essence of the tracking problem in the simplest case is as follows. There is a system (1) with an unknown input influence $u(\cdot)$, satisfying usually the instantaneous constraint $u(t) \in P$ at a.e. $t \in T$, where $P \subset \mathbb{R}^r$ is a compact set. Along with the system (1) there is another system of the same type

$$\dot{x}(t) = f(t, x(t)) + Bv(t), \quad t \in T, \tag{6}$$

$$x(0) = x_0$$

and control $v(\cdot)$, that obeys the same constraints as the function $u(\cdot)$. At moments τ_i, the phase states of systems (1) and (6), $y(\tau_i)$ and $x(\tau_i)$, respectively, are measured (with error). The measurement results are vectors $\xi_i^h \in \mathbb{R}^N$ and $\psi_i^h \in \mathbb{R}^N$, satisfying the inequalities

$$|\xi_i^h - y(\tau_i)|_N \le h, \quad |\psi_i^h - x(\tau_i)|_N \le h.$$

The essence of the tracking task consists in designing such an algorithm for forming the control of $v = v^h(\cdot)$ system (6) according to the feedback principle

$$v^h(t) = v_i^h = V(\tau_i, \xi_i^h, \psi_i^h) \quad \text{at a.e.} \quad t \in \delta_i, \quad i = \overline{0, m-1}, \tag{7}$$

that, at appropriate coordination of values h and δ the solutions of systems (1) and (6), will be close, as a rule in uniform metric (in case of proximity of initial states of these systems), whatever the admissible realization of input influence $v(\cdot)$ is. Thus, when solving the tracking problem, it is necessary to construct such a law V of control formation (7), that whatever the number $\varepsilon > 0$, the numbers h_* and δ_* are specified, such that for all $h \in (0, h_*)$ and $\delta \in (0, \delta_*)$, the inequality is true

$$\sup_{t \in T} |x(t; x_0, v^h(\cdot)) - y(t; y_0, u(\cdot))|_N \le \varepsilon,$$

if the value $|x_0 - y_0|_N$ is small enough. Here $x(\cdot; x_0, v^h(\cdot))$ is the solution of the system (6) generated by the control $v^h(\cdot)$ of the form (7). Note that both in the reconstruction problem and in the tracking problem, the input influence of the given system is unknown.

If the algorithms for solving the reconstruction problem described in the papers cited above allowed us to obtain for an arbitrary measurable input influence $u(\cdot)$ (possibly constrained by some specified instantaneous constraints) estimates of the convergence rate (to $u(\cdot)$) of $u^h(\cdot)$ (in model (4) formed according to rule (5)) in a uniform or mean-square metric, then, while solving the reconstruction problem, we would simultaneously solve the tracking problem. Unfortunately, however, such estimates can be obtained only for special classes $u(\cdot)$, for example, for functions with bounded variation. In the case when $u(\cdot)$ is not such a function, the algorithms from these works guarantee only convergence of the controls $u^h(\cdot)$ to $u(\cdot)$.

A question naturally arises: can we choose not the system of the form (4), but the system of the form (6), i.e., a complete copy of the system (1), as a model in reconstruction algorithms? Then, while solving the reconstruction problem in accordance with the described approach, we would simultaneously solve the tracking problem. Unfortunately, for arbitrary f and B, even if smooth enough, it is not possible to give a positive answer to it. The purpose of this paper is to specify two classes of systems of the form (1), for which the answer to the question is positive. For each of these two classes, a different rule of control formation will be specified. The first class is a system being linear both in phase variables and perturbation; the second is a system with a monotonic function in phase variable f. It should be noted that the approach to solving problems of dynamic reconstruction developed, in this paper, was applied when solving problems of reconstruction of unknown structural characteristics of a bioreactor with recharge [3], the problem of formation of flight telemetry using indirect data [3], and problems of modeling of pollution spreading processes [17].

Thereafter, for each $h \in (0,1)$, we fix a family Δ_h of partitions of the segment T by control time instants $\tau_{h,i}$:

$$\Delta_h = \{\tau_{h,i}\}_{i=\overline{0,m_h}}, \quad \tau_{h,0} = 0, \quad \tau_{h,m_h} = \vartheta, \quad \tau_{h,i+1} = \tau_{h,i} + \delta(h), \quad \delta(h) \in (0,1). \tag{8}$$

It should be noted that the same solution of the system (1) can be conditioned by more than one influence. Let $\mathcal{U}(y(\cdot))$ be the set of all input influences from $L_2(T;\mathbb{R}^r)$, generating the solution $y(\cdot)$ of system (1), i.e.,

$$\mathcal{U}(y(\cdot)) = \{\tilde{u}(\cdot) \in L_2(T;\mathbb{R}^r) : \dot{y}(t) - f(t, y(t)) = B\tilde{u}(t) \text{ at a.e. } t \in T\}.$$

By the symbol $u_*(\cdot)$ we denote the minimal element of the set $\mathcal{U}(y(\cdot))$, i.e., by $L_2(T;\mathbb{R}^r)$-norm.

$$u_*(\cdot) = \arg \min_{u(\cdot)\in U(y(\cdot))} |u(\cdot)|_{L_2(T;\mathbb{R}^r)}.$$

Such an element exists and is unique. Following the approach adopted in the theory of incorrect problems, we will recover $u_*(\cdot)$. Hereinafter $c^{(0)}, c^{(1)}, \ldots, c_0, c_1, \ldots, k^{(1)}, k^{(2)}, \ldots, k_1, k_2, \ldots$ denote positive constants that can be written out explicitly, $(\cdot,\cdot)$ is the scalar product in the corresponding finite-dimensional Euclidean space, and $|\cdot|$ is the modulus of a number.

2. SOLUTION ALGORITHM IN CASE OF A LINEAR SYSTEM

Let us consider the case when the system (1) is linear, i.e., has the form

$$\dot{y}(t) = Ay(t) + Bu(t) + f_1(t). \tag{9}$$

Here, A and B are constant matrices of corresponding dimensions, $f_1(\cdot) \in L_2(T;\mathbb{R}^N)$ is a given function. The model is a copy of the system (9):

$$\dot{y}^h(t) = Ay^h(t) + Bu^h(t) + f_1(t) \tag{10}$$

initialized

$$y^h(0) = \xi_0^h.$$

Let's fix the function $\alpha(h) : (0,1) \to (0.1)$. In the future we will need the following

Condition A. With $h \to 0$, we have $\alpha(h) \to 0$, $\delta(h)\alpha^{-2}(h) \to 0$, $h^2(\alpha(h)\delta(h))^{-1} \to 0$.

Let us denote by $\mathcal{Y}(t)$ the fundamental matrix of the system of equations $\dot{y}(t) = Ay(t)$. The inequality

$$\|\mathcal{Y}(t)\| \le \exp\{\chi t\}, \quad t \ge 0,$$

where $\chi = \|A\|$, $\|A\|$ is the Euclidean norm of the matrix A, is true.

Before the algorithm starts, we fix the value $h \in (0,1)$, the partition $\Delta_h = \{\tau_{h,i}\}_{i=\overline{0,m_h}}$ of the form (8) and the number $\alpha = \alpha(h)$. The algorithm operation is divided into a finite number of steps of the same type. At the i-th step, carried out at the time interval $\delta_i = [\tau_i, \tau_{i+1})$, $\tau_i = \tau_{h,i}$, the following operations are performed: at the moment τ_i, the vector u_i^h is calculated according to formula (5), in which

$$U(\tau_i, \xi_i^h, y^h(\tau_i)) = \alpha^{-1} \exp\{-2\chi\tau_{i+1}\} B'(\xi_i^h - y^h(\tau_i)) \tag{11}$$

(here dash means transpose); then the input of system (10) at all $t \in \delta_i$ is given control $u^h(t)$ of the form (5), (11), under the action of which the system (10) passes from the state $y^h(\tau_i)$ to the state $y^h(\tau_{i+1})$. The work of the algorithm ends at the moment ϑ.

Let's introduce the functional

$$\lambda(t) = \exp\{-2\chi t\} |y^h(t) - y(t)|_N^2.$$

In the future, we'll need the following

Lemma 1 (Gronwall's discrete inequality [18, p. 311]). *Let $\phi_j \ge 0$, $f_j \ge 0$ at $j = \overline{0,m}$ and $f_j \le f_{j+1}$ at $j = \overline{0, m-1}$. Then from the inequalities*

$$\phi_{j+1} \le c_0\delta \sum_{i=1}^{j} \phi_i + f_j, \quad j = \overline{1, m-1},$$

inequalities follow

$$\phi_{j+1} \le f_j \exp\{c_0 j\delta\}, \quad j = \overline{0, m-1},$$

if $c_0 > 0$, $\phi_1 \le f_0$.

Lemma 2. *Let condition* A *be satisfied. Then it is possible to specify such a number $h_* \in (0,1)$, that for all $h \in (0, h_*)$ the inequalities are true.*

$$\max_{i \in \overline{0, m_h - 1}} \lambda(\tau_{i+1}) \le d_1\{\alpha + \delta + h^2\delta^{-1}\}, \tag{12}$$

$$\int_0^{\vartheta} |u^h(s)|_r^2\, ds \le (1 + d_2\delta\alpha^{-2}) \int_0^{\vartheta} |u_*(s)|_r^2\, ds + d_3 h^2(\alpha\delta)^{-1}, \tag{13}$$

where d_j, $j = 1, 2, 3$ are positive constants independent of h, δ and α.

Proof. Let's estimate the change in the value of

$$\varepsilon(t) = \lambda(t) + \alpha \int_0^t (|u^h(\tau)|_r^2 - |u_*(\tau)|_r^2) d\tau.$$

Here $\alpha = \alpha(h)$, $\delta = \delta(h)$. It is easy to see that the inequality is true

$$\varepsilon(\tau_{i+1}) \le \varepsilon(\tau_i) + \lambda_{1i} + \mu_{1i} + \alpha \int_{\tau_i}^{\tau_{i+1}} (|u^h(\tau)|_r^2 - |u_*(\tau)|_r^2) d\tau, \tag{14}$$

where

$$\lambda_{1i} = 2\left(S_i^h, \int_{\tau_i}^{\tau_{i+1}} Y(\tau_{i+1} - \tau) B(u^h(\tau) - u_*(\tau)) d\tau\right),$$

$$\mu_{1i} = \delta \exp\{-2\chi\tau_{i+1}\} \int_{\tau_i}^{\tau_{i+1}} |Y(\tau_{i+1} - \tau) B(u^h(\tau) - u_*(\tau))|_N^2 d\tau,$$

$$S_i^h = \exp\{-2\chi\tau_{i+1}\} Y(\delta) s_i^h, \quad s_i^h = y^h(\tau_i) - y(\tau_i).$$

Note that at $t \in [0, \delta_*]$, $\delta_* \in (0,1)$,

$$\|\mathcal{Y}(t) - I\| \le c_* t, \quad c_* = c_*(\delta_*),$$

where I is a unit matrix of dimension $N \times N$. Therefore

$$|S_i^h - \exp\{-2\chi\tau_{i+1}\}s_i^h|_N \le \delta c_* \exp\{-2\chi\tau_{i+1}\}|s_i^h|_N \le \delta c_*|s_i^h|_N. \tag{15}$$

In this case, taking into account (15) and the inequality $|S_i^h|_N \le |s_i^h|_N$, we have

$$|(S_i^h, \mathcal{Y}(\delta)Bu) - \exp\{-2\chi\tau_{i+1}\}(s_i^h, Bu)| \le$$
$$\le |S_i^h|_N|\mathcal{Y}(\delta) - I|_N|Bu|_N + |(S_i^h, Bu) - \exp\{-2\chi\tau_{i+1}\}(s_i^h, Bu)| \le 2\delta c^{(0)}|s_i^h|_N|Bu|_N. \tag{16}$$

Further, by virtue of (16), the inequality is true

$$\lambda_{1i} \le 2\exp\{-2\chi\tau_{i+1}\}\left(y^h(\tau_i) - y(\tau_i), \int_{\tau_i}^{\tau_{i+1}} B\{u_i^h - u_*(\tau)\}d\tau\right) + I_{1i},$$

where

$$I_{1i} = \delta c^{(1)}|s_i^h|_N \int_{\tau_i}^{\tau_{i+1}} |u_i^h - u_*(\tau)|_r d\tau.$$

It is not difficult to see that there is an estimation

$$I_{1i} \le \delta^2\lambda(\tau_i) + c^{(2)}\delta \int_{\tau_i}^{\tau_{i+1}} (|u_i^h|_r^2 + |u_*(\tau)|_r^2)d\tau. \tag{17}$$

Considering (17) and the rule for choosing the control $u^h(\cdot)$ (see (5), (11)), we obtain

$$\lambda_{1i} + \alpha\int_{\tau_i}^{\tau_{i+1}} (|u^h(s)|_r^2 - |u_*(s)|_r^2)ds \le$$
$$\le \delta^2\lambda(\tau_i) + c^{(3)}h\int_{\tau_i}^{\tau_{i+1}} (|u_i^h|_r + |u_*(s)|_r)ds + c^{(2)}\delta\int_{\tau_i}^{\tau_{i+1}} (|u_i^h|_r^2 + |u_*(s)|_r^2)ds. \tag{18}$$

In addition, the estimates are correct

$$\mu_{1i} \le \delta c^{(4)}\int_{\tau_i}^{\tau_{i+1}} (|u_i^h|_r^2 + |u_*(\tau)|_r^2)d\tau,$$
$$c^{(3)}h\int_{\tau_i}^{\tau_{i+1}} (|u_i^h|_r + |u_*(s)|_r)ds \le \delta c^{(5)}\int_{\tau_i}^{\tau_{i+1}} (|u_i^h|_r^2 + |u_*(s)|_r^2)ds + c^{(6)}h^2. \tag{19}$$

From (14), using (18), (19), we establish the validity of the inequality

$$\gamma(\tau_{i+1}) = \lambda(\tau_{i+1}) + \alpha\int_{\tau_i}^{\tau_{i+1}} |u^h(s)|_r^2 ds \le$$
$$\le (1+\delta^2)\lambda(\tau_i) + \alpha\int_{\tau_i}^{\tau_{i+1}} |u_*(\tau)|_r^2 d\tau + \delta c^{(7)}\int_{\tau_i}^{\tau_{i+1}} (|u_*(\tau)|_r^2 + |u_i^h|_r^2)d\tau + c^{(6)}h^2. \tag{20}$$

In turn, by virtue of (3), (11) we have

$$|u_i^h|_r^2 \le \alpha^{-2}c^{(8)}(h^2 + |y^h(\tau_i) - y(\tau_i)|_N^2)^2 \le \alpha^{-2}c^{(9)}(\lambda(\tau_i) + h^2) \le \alpha^{-2}c^{(9)}(\gamma(\tau_i) + h^2). \tag{21}$$

From (20), (21) follows the estimation of

$$\gamma(\tau_{i+1}) \le (1+\delta^2)\gamma(\tau_i) + (\alpha + c^{(7)}\delta)\int_{\tau_i}^{\tau_{i+1}} |u_*(s)|_r^2 ds + c^{(6)}h^2 + c^{(9)}\delta^2\alpha^{-2}(\gamma(\tau_i) + h^2). \tag{22}$$

Taking into account condition A, we conclude that it is possible to specify the number $h_1 \in (0,1)$ such that the inequality holds

$$\sup_{h\in(0,h_1)} \delta(h)\alpha^{-2}(h) \le 1.$$

From (22), we derive in the standard way (see, e.g., [13, p. 59–64]) the relation

$$\gamma(\tau_{i+1}) \le \left((\alpha + c^{(7)}\delta)\int_{\tau_i}^{\tau_{i+1}} |u_*(s)|_r^2 ds + c^{(6)}h^2\delta^{-1} + c^{(9)}h^2\right)\exp\{\delta(1 + c^{(9)}\alpha^{-2})\tau_{i+1}\}. \qquad (23)$$

Note that $\delta(h)\alpha^{-2}(h) \to 0$ at $h \to 0$. Therefore, we can specify a number $c^{(10)} > 0$, such that for all $h \in (0, h_1)$ the inequality is true

$$\exp\{\delta(1 + c^{(9)}\alpha^{-2})\vartheta\} \le 1 + \delta c^{(10)}(1 + \alpha^{-2}).$$

Then from (23) follows the relation

$$\int_0^\vartheta |u^h(s)|_r^2 ds \le (1 + c^{(7)}\delta\alpha^{-1})(1 + c^{(10)}\delta\alpha(1 + \alpha^{-2}))\int_0^\vartheta |u_*(s)|_r^2 ds + c^{(11)}h^2(\delta\alpha)^{-1}. \qquad (24)$$

By virtue of condition A, there is such a number $h_* \in (0, h_1)$ such that for all $h \in (0, h_*)$

$$(1 + c^{(7)}\delta\alpha^{-1})(1 + c^{(10)}\delta(1 + \alpha^{-2})) \le 1 + d_2\delta\alpha^{-2}. \qquad (25)$$

Inequality (13) follows from (24) and (25). In turn, inequality (12) follows from (23). The lemma is proved.

Remark. If $\delta(h) = d_4 h$, $\alpha(h) = d_5 h^{1/2-\varepsilon}$, where d_4 and d_5 are positive constants, $\varepsilon \in (0, 1/2)$, then the inequalities hold

$$\max_{i=0,m_h-1} \lambda(\tau_{i+1}) \le d_6 h^{1/2-\varepsilon},$$

$$\int_0^\vartheta |u^h(s)|_r^2 ds \le (1 + d_7 h^{2\varepsilon})\int_0^\vartheta |u_*(s)|_r^2 ds + d_8 h^{1/2+\varepsilon}.$$

It follows from lemma 2

Theorem 1. *Let the conditions of lemma 2 be satisfied. Then there is convergence $u^h(\cdot) \to u_*(\cdot)$ at $h \to 0$.*

The proof of this theorem follows the standard scheme (see, for example, the proof of Theorem 1.2.3 in [3, pp. 21-27]).

Under some additional conditions, an estimate of the convergence rate of the algorithm can be obtained. To justify it, we need the following

Lemma 3 [3, p. 29]. *Let $x_1(\cdot) \in L_\infty(T_*; \mathbb{R}^n)$, $y_1(\cdot) \in W(T_*; \mathbb{R}^n)$, $T_* = [a, b]$, $-\infty < a < b < +\infty$,*

$$\left|\int_a^t x_1(\tau)d\tau\right|_n \le \varepsilon, \quad |y_1(t)|_n \le K, \quad t \in T_*.$$

Then the inequality is true for all $t \in T_*$:

$$\left|\int_a^t (x_1(\tau), y_1(\tau))d\tau\right| \le \varepsilon(K + \mathrm{var}(T_*; y_1(\cdot))).$$

Here, $\mathrm{var}(T_*; y_1(\cdot))$ denotes the variation of the function $y_1(\cdot)$ on the segment T_*, and $W(T_*; \mathbb{R}^n)$ denotes the set of functions $y(\cdot) : T_* \to \mathbb{R}^n$ with bounded variation.

Lemma 4. *Let $u_*(\cdot)$ be a function of bounded variation, B be a matrix independent of t and y (stationary) matrix, $N \ge r$,* rank $B = r$. *Let the conditions of Lemma 2 also be satisfied. Then we can specify a number $d_9 > 0$ such that for all $h \in (0, h_*)$ the inequality is true.*

$$\int_0^\vartheta |u^h(\tau) - u_*(\tau)|_r^2 d\tau \le d_9(\alpha^{1/2} + h^2(\alpha\delta)^{-1} + \delta\alpha^{-2} + h^{1/2} + h\delta^{-1/2}). \qquad (26)$$

Proof. Note that for any $t_1, t_2 \in T$, $t_1 < t_2$, the following relation is true

$$\left|\int_{t_1}^{t_2} B\{u^h(t) - u_*(t)\}dt\right|_N = \left|\int_{t_1}^{t_2} [\dot{y}^h(\tau) - \dot{y}(\tau) - A(y^h(\tau) - y(\tau))]d\tau\right|_N \le$$

$$\le |\mu_h(t_2) - \mu_h(t_1)|_N + k^{(1)}\int_{t_1}^{t_2} |\mu_h(\tau)|_N d\tau,$$

where $\mu_h(t) = y^h(t) - y(t)$. It is not difficult to see that the inequalities are true at $t \in \delta_i$

$$|\mu_h(t)|_N^2 \le k^{(2)}\lambda(\tau_i) + k^{(3)} \left| \int_{\tau_i}^t Y(t-s)B(u^h(s) - u_*(s))ds \right|_N \le$$

$$\le k^{(2)}\lambda(\tau_i) + k^{(4)} \int_{\tau_i}^t (|u^h(s)|_r + |u_*(s)|_r)ds. \tag{27}$$

In turn, by virtue of (12) and (21) at $t \in \delta_i$, we have

$$\int_{\tau_i}^t |u^h(s)|_r ds \le k^{(5)}\delta\alpha^{-1}(\lambda^{1/2}(\tau_i) + h) \le k^{(6)}\delta\alpha^{-1}(\alpha^{1/2} + \delta^{1/2} + h\delta^{-1/2}). \tag{28}$$

Given the convergence of $\delta(h)\alpha^{-2}(h) \to 0$ at $h \to 0$, we conclude that at $h \in (0, h_*)$, the following estimates are valid

$$\delta\alpha^{-1/2} \le k^{(7)}\alpha^{3/2}, \quad \delta^{3/2}\alpha^{-1} \le k^{(8)}\alpha^2, \quad h\delta^{1/2}\alpha^{-1} \le k^{(9)}h. \tag{29}$$

Moreover, in view of (28) and (29) at $t \in \delta_i$, the following estimates are true

$$\int_{\tau_i}^t |u^h(s)|_r ds \le k^{(10)}(h + \alpha^{3/2}),$$

$$\int_{\tau_i}^t |u_*(s)|_r ds \le k^{(11)}\delta^{1/2} \le k^{(12)}\alpha. \tag{30}$$

From (27), taking into account (30), we derive the following relation, which is valid at $t \in \delta_i$

$$|\mu_h(t)|_N^2 \le k^{(2)}\lambda(\tau_i) + k^{(13)}(h + \alpha). \tag{31}$$

In this case, by virtue of (12), from (31) we obtain

$$\sup_{t \in T} |\mu_h(t)|_N \le k^{(14)}(\alpha + h + h^2\delta^{-1})^{1/2}.$$

Hence we deduce

$$\left| \int_{t_1}^{t_2} (u^h(t) - u_*(t))dt \right|_r \le k^{(15)} \left| \int_{t_1}^{t_2} B(u^h(t) - u_*(t))dt \right|_N \le k^{(16)}(\alpha^{1/2} + h^{1/2} + h\delta^{-1/2}). \tag{32}$$

Again using lemma 2 (see (13)), we set

$$\int_0^\vartheta |u^h(\tau) - u_*(\tau)|_r^2 d\tau = \int_0^\vartheta |u^h(\tau)|_r^2 d\tau - 2\int_0^\vartheta (u^h(\tau), u_*(\tau))d\tau + \int_0^\vartheta |u_*(\tau)|_r^2 d\tau \le$$

$$\le (2 + d_2\alpha^{-2}\delta) \int_0^\vartheta |u_*(\tau)|_r^2 d\tau - \int_0^\vartheta (u^h(\tau), u_*(\tau))d\tau + d_3 h^2(\alpha\delta)^{-1} =$$

$$= 2\int_0^\vartheta (u_*(\tau) - u^h(\tau), u_*(\tau))d\tau + d_2\alpha^{-2}\delta \int_0^\vartheta |u_*(\tau)|_r^2 d\tau + d_3 h^2(\alpha\delta)^{-1}. \tag{33}$$

Considering lemma 3 and also (32), we obtain

$$\sup_{t \in T} \left| \int_0^t (u_*(\tau) - u^h(\tau), u_*(\tau))d\tau \right| \le k^{(17)}(\alpha^{1/2} + h^{1/2} + h\delta^{-1/2}). \tag{34}$$

Thus, inequality (26) is true for all $h \in (0, h_*)$, $t \in T$, by virtue of (33), (34). The lemma is proved.

3. SOLUTION ALGORITHM IN CASE OF NONLINEAR SYSTEM

Let us specify the algorithm for solving the problem under consideration in the case when the system is nonlinear in phase variable. Let the system (1) have the following form:

$$\dot{y}(t) = f(t, y(t)) + Bu(t), \tag{35}$$

where B is a constant matrix of dimension $N \times r$. Let us assume that the function f is continuous on t, monotone on x, i.e., at some $\omega \geq 0$ the inequality is satisfied

$$(f(t,x) - f(t,y), x - y) \leq -\omega|x - y|_N^2, \quad t \in T, x, y \in \mathbb{R}^N,$$

and satisfies the growth condition

$$|f(t,x)|_N \leq c(1 + |x|_N), \quad t \in T, x \in \mathbb{R}^N,$$

where $c => 0$. If these conditions are satisfied, it is known that at any $u(\cdot) \in L_2(T; \mathbb{R}^r)$, there exists a single solution of the system (35), understood in the sense of Carathéodory. As a model, we take a copy of (35), namely the system

$$\dot{y}^h(t) = f(t, y^h(t)) + Bu^h(t) \tag{36}$$

with initial state of

$$y^h(0) = \xi_0^h.$$

The algorithm for solving the problem, in this case, is similar to the algorithm described above for the linear system. First of all, we select some family Δ_h (8) of partitions of the segment T, as well as the function $\alpha(h) : (0,1) \to (0,1)$.

The values $h \in (0,1)$, $\alpha = \alpha(h)$ and the partition $\Delta_h = \{\tau_{h,i}\}_{i=\overline{0,m_h}}$ of the form (8) are fixed before the algorithm starts. The work of the algorithm is divided into $m - 1$, $m = m_h$ steps of the same type. At i-th step, carried out at the time interval $\delta_i = [\tau_i, \tau_{i+1})$, $\tau_i = \tau_{h,i}$, the following operations are performed. First (at the moment τ_i), the vector u_i^h is calculated according to formula (5), in which

$$U(\tau_i, \xi_i^h, y^h(\tau_i)) = \alpha^{-1}B'(\xi_i^h - y^h(\tau_i)). \tag{37}$$

Then, the control $u^h(t)$ of the form (5), (37) is applied to the input of the system (36). Under the action of this control, the system (36) changes from the state $y^h(\tau_i)$ to the state $y^h(\tau_{i+1})$. The operation of the algorithm ends at the moment ϑ.

As in the linear case, it turns out that at a certain agreement of the values h, $\delta(h)$ and $\alpha(h)$ the function $u^h(\cdot)$ is an approximation of $u_*(\cdot)$. Before proceeding to the proof of this fact, we give a lemma that will be needed later.

Lemma 5. *It is possible to specify such a number $d_{10} > 0$, such that the inequality is satisfied uniformly over all $t \in T$, $y_0 \in \mathbb{R}^N$, $u(\cdot) \in L_2(T; \mathbb{R}^r)$.*

$$\int_0^t |\dot{y}(s; y_0, u(\cdot))|_N^2 ds \leq d_{10}\left(|y_0|_N^2 + \int_0^t |u(s)|_r^2 ds\right).$$

Here $y(\cdot; y_0, u(\cdot))$ is the solution of system (1) with initial state (2) generated by $u(\cdot) \in L_2(T; \mathbb{R}^r)$.

Lemma 6. *Let $\alpha(h) \to 0$, $\delta(h)\alpha^{-2}(h) \to 0$ at $h \to 0$. Then we can specify such a number $h_1 \in (0,1)$, such that for all $h \in (0, h_1)$, $t \in T$ for some positive d_{11}, d_{12}, d_{13}, the inequalities are true.*

$$\max_{i=\overline{0,m_h-1}} \varepsilon_1(\tau_i) \leq d_{11}(\alpha + \delta + h^2\delta^{-1}), \tag{38}$$

$$\int_0^\vartheta |u^h(\tau)|_r^2 d\tau \leq (1 + d_{12}\delta\alpha^{-2}) \int_0^\vartheta |u_*(\tau)|_r^2 d\tau + d_{13}(h^2(\alpha\delta)^{-1} + \delta\alpha^{-1}), \tag{39}$$

where $\varepsilon_1(t) = |y^h(t) - y(t)|_N^2$, $\alpha = \alpha(h)$, $\delta = \delta(h)$.

Proof. Consider the change in the value of $\varepsilon_1(t)$ at $t \in T$. For $t \in \delta_i = [\tau_i, \tau_{i+1})$, $i = \overline{0, m-1}$, we have

$$\frac{d\varepsilon_1(t)}{dt} = 2(y^h(t) - y(t), f(t, y^h(t)) - f(t, y(t)) + B(u_i^h - u_*(t))) \leq$$

$$\leq -2\omega\varepsilon_1(t) + 2(y^h(t) - y(t), B(u_i^h - u_*(t))) \leq -2\omega\varepsilon_1(t) + \sum_{j=1}^{3} I_{ji}(t), \tag{40}$$

where

$$I_{1i}(t) = 2(y^h(\tau_i) - \xi_i^h, B(u_i^h - u_*(t))),$$
$$I_{2i}(t) = 2\|B\|h(|u_i^h|_r + |u_*(t)|_r),$$
$$I_{3i}(t) = 2\|B\|(|u_i^h|_N + |u_*(t)|_N)\int_{\tau_i}^{\tau_{i+1}} |\dot{y}^h(s) - \dot{y}(s)|_N ds.$$

From (40) follows the inequality

$$\varepsilon_1(\tau_{i+1}) \leq \varepsilon_1(\tau_i) - 2\omega\int_{\tau_i}^{\tau_{i+1}} \varepsilon_1(s)ds + \int_{\tau_i}^{\tau_{i+1}} \sum_{j=1}^{3} I_{ji}(s)ds. \tag{41}$$

Further, at $t \in \delta_i$ we have

$$\varepsilon_1(\tau_i) = \left| y^h(t) - y(t) - \int_{\tau_i}^{t} (\dot{y}^h(s) - \dot{y}(s))ds \right|_N^2 \leq 2\varepsilon_1(t) + 2\delta\int_{\tau_i}^{t} |\dot{y}^h(s) - \dot{y}(s)|_N^2 ds,$$

therefore

$$-\omega\varepsilon_1(\tau_i) \geq -2\omega\varepsilon_1(t) - 2\omega\delta\int_{\tau_i}^{t} |\dot{y}^h(s) - \dot{y}(s)|_N^2 ds.$$

Thus, at $t \in \delta_i$ the inequality is true

$$-2\omega\varepsilon_1(t) \leq -\omega\varepsilon(\tau_i) + 2\omega\delta\int_{\tau_i}^{t} |\dot{y}^h(s) - \dot{y}(t)|_N^2 ds.$$

Hence, after integration at $t \in [\tau_i, \tau_{i+1}]$, we obtain

$$-2\omega\int_{\tau_i}^{t} \varepsilon_1(s)ds \leq -\omega\delta\varepsilon_1(\tau_i) + 2\omega\delta^2\int_{\tau_i}^{t} |\dot{y}^h(s) - \dot{y}(s)|_N^2 ds. \tag{42}$$

From (41), (42), considering in (42) $t = \tau_{i+1}$, we deduce

$$\varepsilon_1(\tau_{i+1}) \leq (1 - \omega\delta)\varepsilon_1(\tau_i) + \tilde{I}_{1i} + \sum_{j=1}^{3}\int_{\tau_i}^{\tau_{i+1}} I_{ji}(s)ds, \tag{43}$$

where

$$\tilde{I}_{1i} = 4\omega\delta^2\int_{\tau_i}^{\tau_{i+1}} (|\dot{y}^h(s)|_N^2 + |\dot{y}(s)|_N^2)ds.$$

Further, taking into account the definition of u_i^h (see (5), (37)), we conclude that the following inequality holds

$$\int_{\tau_i}^{\tau_{i+1}} (I_{1i}(t) + \alpha(|u_i^h|_r^2 - |u_*(t)|_r^2))dt \leq 0. \tag{44}$$

It's not hard to see that

$$\int_{\tau_i}^{\tau_{i+1}} I_{2i}(t)dt \leq c_0h^2 + \tilde{I}_{2i}, \tag{45}$$

where

$$\tilde{I}_{2i} = \delta\int_{\tau_i}^{\tau_{i+1}} (|u_i^h|_r^2 + |u_*(t)|_r^2)dt.$$

In turn, by virtue of (5), (37) and (3), the inequality is true

$$|u_i^h|_r \leq \alpha^{-1}c_1(h + \varepsilon_1(\tau_i)),$$

therefore

$$\delta \int_{\tau_i}^{\tau_{i+1}} |u^h(s)|_r^2 ds \le 2\delta^2\alpha^{-2}c_1^2(h^2 + \varepsilon_1(\tau_i)), \tag{46}$$

hence,

$$\delta \int_0^{\tau_{i+1}} |u^h(s)|_r^2 ds \le 2\delta^2\alpha^{-2}c_1^2 \left(\sum_{j=0}^{i} \varepsilon_1(\tau_j) + \vartheta h^2\delta^{-1}\right). \tag{47}$$

Considering (47), we obtain

$$\sum_{j=0}^{i} \tilde{I}_{2j} \le \delta \int_0^{\tau_{i+1}} |u_*(s)|_r^2 ds + 2\vartheta c_1^2 \delta h^2 \alpha^{-2} + 2c_1^2\delta^2\alpha^{-2} \sum_{j=0}^{i} \varepsilon_1(\tau_j). \tag{48}$$

Then we have

$$\int_{\tau_i}^{\tau_{i+1}} I_{3i}(t)dt \le \tilde{I}_{3i} + \tilde{I}_{2i}, \tag{49}$$

where

$$\tilde{I}_{3i} = \|B\|^2\delta \int_{\tau_i}^{\tau_{i+1}} (|\dot{y}^h(s)|_N^2 + |\dot{y}(s)|_N^2)ds.$$

By virtue of lemma 5, for all $i = \overline{1,m}$, the following relation is correct

$$\int_0^{\tau_i} (|\dot{y}^h(s)|_N^2 + |\dot{y}(s)|_N^2)ds \le c_2 \left(1 + \int_0^{\tau_i} (|u^h(s)|_r^2 + |u_*(s)|_r^2)ds\right). \tag{50}$$

Then,

$$\sum_{j=0}^{i} \tilde{I}_{1j} \le c_3\delta \left(1 + \sum_{j=0}^{i} \tilde{I}_{2j}\right), \quad \sum_{j=0}^{i} \tilde{I}_{3j} \le c_4 \left(\delta + \sum_{j=0}^{i} \tilde{I}_{2j}\right).$$

In this case, taking into account (49), we conclude that the inequality holds

$$\sum_{j=0}^{i} \int_{\tau_j}^{\tau_{j+1}} I_{3j}(s)ds \le c_5\delta + c_6 \sum_{j=0}^{i} \tilde{I}_{2j}. \tag{51}$$

Then from (45), (47), (48), and (51), we obtain

$$\sum_{j=0}^{i} \left(\tilde{I}_{1j} + \int_{\tau_j}^{\tau_{j+1}} (I_{2j}(t) + I_{3j}(t))dt\right) \le c_7h^2\delta^{-1} + c_8\delta + c_9 \left(\delta^2\alpha^{-2} \sum_{j=0}^{i} \varepsilon_1(\tau_j) + \delta h^2\alpha^{-2}\right). \tag{52}$$

In turn, from (43), taking advantage of (44) and (52), we derive the estimation

$$\varepsilon_1(\tau_{i+1}) + \alpha \int_0^{\tau_{i+1}} (|u^h(s)|_r^2 - |u_*(s)|_r^2)ds \le$$

$$\le \varepsilon_1(0) + c_7h^2\delta^{-1} + c_8\delta + c_9\delta h^2\alpha^{-2} + c_9\delta^2\alpha^{-2} \sum_{j=0}^{i} \varepsilon_1(\tau_j). \tag{53}$$

By virtue of the discrete Gronwall inequality (see lemma 1) from (53), we have

$$\varepsilon_1(\tau_{i+1}) + \alpha \int_0^{\tau_{i+1}} |u^h(s)|_r^2 ds \le$$

$$\le \left(\varepsilon_0(0) + c_7h^2\delta^{-1} + c_8\delta + c_9\delta h^2\alpha^{-2} + \alpha \int_0^{\tau_{i+1}} |u_*(s)|_r^2 ds\right) \exp\{c_9(i+1)\delta^2\alpha^{-2}\}. \tag{54}$$

Note that

$$\varepsilon_1(0) \le h^2, \quad \exp\{c_9(i+1)\delta^2\alpha^{-2}\} \le \exp\{c_9\vartheta\delta\alpha^{-2}\}.$$

Furthermore, if $\delta(h)\alpha^{-2}(h) \to 0$ at $h \to 0$, then the inequalities are satisfied at $h \in (0, h_1)$, $h_1 \in (0,1)$

$$\exp\{c_9\vartheta\delta\alpha^{-2}\} \le 1 + c_{10}\delta\alpha^{-2}, \quad \delta\alpha^{-2} \le c_{11},$$

where $c_{10} = c_{10}(h_1) > 0$, $c_{11} = c_{11}(h_1) > 0$.

Thus, in view of (54) at $h \in (0, h_1)$, $i = \overline{0, m-1}$, the inequality is true

$$\varepsilon_1(\tau_{i+1}) + \alpha\int_0^{\tau_{i+1}} |u^h(s)|_r^2 ds \le \alpha(1 + c_{12}\delta\alpha^{-2})\int_0^{\tau_{i+1}} |u_*(s)|_r^2 ds + c_{13}(h^2\delta^{-1} + \delta),$$

from which inequalities (38) and (39) follow. The lemma is proved.

By means of lemma 6, it can be proved that

Theorem 2. *Let the conditions of lemma* 6 *be satisfied. Suppose also* $h^2(\alpha(h)\delta(h))^{-1} \to 0$ *at* $h \to 0$. *Then, there is convergence of* $u^h(\cdot) \to u_*(\cdot)$ *at* $h \to 0$.

As in the case of a linear system, we can write out an estimate of the convergence rate of the algorithm.

Lemma 7. *Let the conditions of Theorem* 2 *be satisfied. Let also the function* $y \to f(t, y)$ *be a Lipschitz function,* $r \le N$, rank $B = r$. *Then at* $h \in (0, h_1)$*, the following estimate of the convergence rate of the algorithm takes place:*

$$\int_0^{\vartheta} |u^h(s) - u_*(s)|_r^2 ds \le d_{14}(\alpha^{1/2} + \delta^{1/2} + h\delta^{-1/2} + h\alpha^{-1/2} + \delta\alpha^{-2} + h^2(\alpha\delta)^{-1}). \tag{55}$$

Proof. The proof of the lemma is similar to the proof of Lemma 4. Indeed, let L be the Lipschitz constant of the function f. It is easy to see that at a.e. $t \in \delta_i$, the following relation is true

$$\dot\varepsilon_1(t) \le -2\omega\varepsilon_1(t) + I_{4i}(t) + I_{3i}(t) \le I_{4i}(t) + I_{3i}(t), \tag{56}$$

where

$$I_{4i}(t) = 2(y^h(\tau_i) - y(\tau_i), B(u_i^h - u_*(t))).$$

Note that the inequality is true at $t \in \delta_i$

$$\left|\int_{\tau_i}^t I_{4i}(s)ds\right|_N \le \varepsilon_1(\tau_i) + 2\|B\|^2\tilde I_{2i},$$

therefore (see (49)) for all $t \in \delta_i$, the estimate is true:

$$\left|\int_{\tau_i}^t (I_{4i}(s) + I_{3i}(s))ds\right|_N \le \varepsilon_1(\tau_i) + \tilde I_{3i} + (1 + 2\|B\|^2)\tilde I_{2i}. \tag{57}$$

Under the conditions of Theorem 2, we can consider that at $h \in (0, h_1)$, the following relations take place:

$$\max_{i=\overline{0,m_h}} \varepsilon_1(\tau_i) \le k_1, \quad \delta\alpha^{-2} \le k_2. \tag{58}$$

Using (39), we obtain

$$\int_0^{\vartheta} |u^h(s)|_N^2 ds \le k_3(1 + \delta\alpha^{-2} + h^2\delta^{-1}\alpha^{-1}). \tag{59}$$

In turn, by virtue of (46), (50), (58), (59) and lemma 6, the inequalities are true at $h \in (0, h_1)$

$$\tilde I_{2i} \le k_4\delta + k_5\delta^2\alpha^{-2}(h^2 + \varepsilon_1(\tau_i)) \le k_6\delta, \tag{60}$$

$$\tilde I_{3i} \le k_7\delta + k_8\delta\int_0^{\tau_i} |u^h(s)|_r^2 ds \le k_9\delta + k_{10}(h^2\alpha^{-1} + \delta^2\alpha^{-2}) \le k_{11}(\delta + h^2\alpha^{-1}). \tag{61}$$

In view of (58)

$$\alpha^{-1} \le k_{12}\delta^{-1/2} \le k_{13}\delta^{-1}.$$

In this case, taking into account (57), (60), (61), from (56) we obtain the relations valid at $t \in \delta_i$

$$\varepsilon_1(t) \leq 2\varepsilon_1(\tau_i) + k_{11}(\delta + h^2\alpha^{-1}) \leq 2\varepsilon(\tau_i) + k_{14}(\delta + h^2\delta^{-1}). \tag{62}$$

Hence, by virtue of (38) and (62) at $t \in \delta_i$ there is a chain of inequalities

$$\left|\int_0^t (u^h(s) - u_*(s))ds\right|_r \leq k_{15}\left|\int_0^t (\dot{y}^h(s) - \dot{y}(s) - f(s, y^h(s)) + f(s, y(s)))ds\right|_N \leq$$

$$\leq k_{15}\left(\varepsilon_1^{1/2}(t) + \varepsilon_1^{1/2}(0) + L\int_0^t \varepsilon_1^{1/2}(s)ds\right) \leq k_{16}(\alpha + \delta + h^2\delta^{-1} + h^2\alpha^{-1})^{1/2}.$$

In addition, similarly to (33), (34), the estimates are established

$$\int_0^\vartheta |u^h(s) - u_*(s)|_r^2 ds \leq$$

$$\leq 2\int_0^\vartheta (u_*(s) - u^h(s), u_*(s))ds + d_{12}\delta\alpha^{-2}\int_0^\vartheta |u_*(s)|_r^2 ds + d_{13}(h^2(\alpha\delta)^{-1} + \delta\alpha^{-1}), \tag{63}$$

$$\sup_{t\in T}\left|\int_0^t (u^h(s) - u_*(s), u_*(s))ds\right| \leq k_{18}(\alpha + \delta + h^2\delta^{-1} + h^2\alpha^{-1})^{1/2}. \tag{64}$$

Lemma 3 is used to derive inequality (64). Inequality (55) follows from inequalities (63) and (64). The lemma is proved.

CONFLICT OF INTERESTS

The author of this paper declares that he has no conflict of interests.

REFERENCES

1. *Osipov Yu.S., Vasiliev F.P., and Potapov M.M.* Osnovy metoda dinamicheskoy regulyarizatsii (Basics of the Dynamic Regularization Method), Moscow: MSU Press, 1999.

2. *Osipov Yu.S. and Kryazhimskii A.V.* Inverse Problems for Ordinary Differential Equations: Dynamical Solutions, Basel: Gordon and Breach, 1995.

3. *Osipov Yu.S., Kryazhimskii A.V., and Maksimov V.I.* Metody dinamicheskogo vosstanovleniya vkhodov upravlyayemykh sistem (Methods of Dynamic Restoration of Inputs of Controlled Systems), Yekaterinburg: Izd-vo Ural. Otd. Ross. Akad. Nauk, 2011.

4. *Kryazhimskii A.V. and Osipov Yu.S.* On modeling control in a dynamic system, Izv. RAS. Technicheskaya kibernetika, 1983, No. 2, pp. 51–68.

5. *Kryazhimskii A.V. and Osipov Yu.S.* Stable solutions of inverse problems of the dynamics of controllable systems, Proc. Steklov Inst. Math., 1990, Vol. 185, pp. 143–164.

6. *Maksimov V.I.* Dynamic reconstruction of system disturbances using inaccurate discrete measurements of phase coordinates, J. Comput. Syst. Sci. Int., 2018, Vol. 57, No. 3, pp. 358–373.

7. *Maksimov V.I.* The methods of dynamical reconstruction of an input in a system of ordinary differential equations, J. Inverse Ill-Posed Probl., 2021, Vol. 29, No. 1, pp. 125–156.

8. *Surkov P.G.* Dynamic right-hand side reconstruction problem for a system of fractional differential equations, Differ. Equat., 2019, Vol. 55, No. 6, pp. 849–858.

9. *Blizorukova M.S.* On the dynamic reconstruction of the input of a control system, Differ. Equat., 2014, Vol. 50, No. 7, pp. 847–853.

10. *Rozenberg V.L.* Reconstruction of external actions under incomplete information in a linear stochastic equation, Proc. Steklov Inst. Math., 2017, Vol. 296, suppl. 1, pp. 196–205.

11. *Maksimov V.* On a stable solution of the dynamical reconstruction and tracking control problems for coupled ordinary differential equation-heat equation, Math. Contr. Related Fields, 2024, Vol. 14, No. 1, pp. 322–345.

12. *Maksimov V.* Reconstruction of disturbances in a nonlinear system from measurements of some of the statevector coordinates, Comput. Math. Math. Phys., 2019, Vol. 59, No. 11, pp. 1771–1780.

13. *Krasovskii N.N. and Subbotin A.I.* Game-Theoretical Control Problems, New York; Berlin: Springer Verlag, 1988.

14. *Chernousko F.L., Ananievski I.M., and Reshmin S.A.* Control of Nonlinear Dynamical Systems: Methods and Applications, Berlin; Heidelberg: Springer, 2008.

15. *Ananevsky I.M. and Reshmin S.A.* The method of decomposition in the problem of tracking the trajectories of mechanical systems, J. Comput. Syst. Sci. Int., 2002, Vol. 41, No. 5, pp. 695–702.

16. *Utkin V.A. and Utkin A.V.* Problem of tracking in linear systems with parametric uncertainties under unstable zero dynamics, Autom. Remote Control, 2014, Vol. 75, pp. 1577–1592.

17. *Kryazhimskii A.V., Maksimov V.I., and Samarskai E.A.* On reconstruction inputs in parabolic systems, Math. Model., 1997, Vol. 9, No. 3, pp. 51–72.

18. *Samarskii A.A.* The Theory of Difference Schemes, New York: CRC Press, 2001.