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INTEGRAL EQUATIONS

ON THE SOLVABILITY OF A SYSTEM OF MULTIDIMENSIONAL INTEGRAL EQUATIONS WITH CONCAVE NONLINEARITIES

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Abstract. The work is devoted to the study of questions of existence and uniqueness of a continuous bounded and positive solution to one system of nonlinear multidimensional integral equations. The scalar analogue of the indicated system of integral equations, with different representations of the corresponding matrix kernel and nonlinearities, has important applied significance in a number of areas of physics and biology. This article proposes a special iterative approach for constructing a positive continuous and bounded solution to the system under study. It is possible to prove that the corresponding iterations uniformly converge to a continuous solution of the specified system. Using some a priori estimates for strictly concave functions, we also prove the uniqueness of the solution in a fairly wide subclass of continuous bounded and coordinately nonnegative vector functions. In the case when the integral of the matrix kernel has a unit spectral radius, it is proved that in a certain subclass of continuous bounded and coordinate-wise non-negative vector functions, this system has only a trivial solution, that is an eigenvector of the kernel integral matrix.

Keywords: *nonlinear integral equation, system of integral equations, positive solution, continuous solution, limited solution, trivial solution, iterative process*

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1. INTRODUCTION. PROBLEM STATEMENT

Consider a system of nonlinear multivariate integral equations

$$f_i(x_1,\ldots,x_n)=\sum_{j=1}^{N}\int_{\mathbb{R}^n}K_{ij}(x_1,\ldots,x_n,t_1,\ldots,t_n)G_j(f_j(t_1,\ldots,t_n))dt_1\ldots dt_n,\quad i=\overline{1,N},\tag{1}$$

with respect to the vector-function $f(x_1,\ldots,x_n)=(f_1(x_1,\ldots,x_n),\ldots,f_N(x_1,\ldots,x_n))^{\mathrm{T}}$ with non-negative continuous and bounded on the set $\mathbb{R}^n$ coordinates $f\,(x_1,\ldots,x_n)$, $i=\overline{1,N}$, where $(x_1,\ldots,x_n)\in\mathbb{R}^n$, $\mathbb{R}=(-\infty,+\infty)$, T is the transpose sign. In system (1) the matrix kernel

$$K(x,t):=(K_{ij}(x_1,\ldots,x_n,t_1,\ldots,t_n))_{i,j=\overline{1,N}}$$

satisfies the following conditions:

1) $K_{ij}(x_1,\ldots,x_n,t_1,\ldots,t_n)>0$, $(x_1,\ldots,x_n,t_1,\ldots,t_n)\in\mathbb{R}^{2n}$, $K_{ij}\in C(\mathbb{R}^{2n})$, $i,j=\overline{1,N}$;

2) there exist $a_{ij} := \sup_{(x_1,\dots,x_n)\in\mathbb{R}^n} \int_{\mathbb{R}^n} K_{ij}(x_1,\dots,x_n,t_1,\dots,t_n)\,dt_1\dots dt_n < +\infty$, $i,j=\overline{1,N}$, with $r(A)=1$, $A=(a_{ij})_{i,j=\overline{1,N}}$, where $r(A)$ is the spectral radius of the matrix A, i.e., the modulus of the largest modulo eigenvalue.

According to Perron's theorem (see [1, p. 260]), there exists a vector $\eta=(\eta_1,\dots,\eta_N)^{\mathrm{T}}$ with positive coordinates η_i such that

$$\sum_{j=1}^{N} a_{ij}\eta_j = \eta_i, \quad i=\overline{1,N}. \tag{2}$$

Let us fix the vector η and impose the following conditions on the nonlinearities of $\{G_j(u)\}_{j=\overline{1,N}}$ (Fig. 1):

a) $G_j \in C(\mathbb{R}^+), \mathbb{R}^+=[0,+\infty), G_j(u)$ are monotonically increasing on the set $\mathbb{R}^+, j=\overline{1,N}$;

b) $G_j(0)=0, G_j(\eta_j)=\eta_j, j=\overline{1,N}$;

c) $G_j(u), j=\overline{1,N}$, are strictly concave (convex upwards) on $\mathbb{R}^+$ and there exists a continuous mapping $\varphi : [0,1]\to[0,1]$ with properties

$$\varphi(0)=0,\ \varphi(1)=1,\ \varphi \text{monotonically increases on the interval } [0,1], \tag{3}$$

$$\varphi \text{ strictly concave on the segment } [0,1], \tag{4}$$

such that the following inequalities hold:

$$G_j(\sigma u) \ge \varphi(\sigma) G_j(u), \quad u\in[0,\eta_j], \sigma\in[0,1], j=\overline{1,N};$$

d) there exists a number $r>0$ such that the functional equations $G_i(u)=u/\varepsilon_i(r), i=\overline{1,N}$, have positive solutions d_i, where

$$\varepsilon_i(r) := \min_{j=\overline{1,N}} \left\{ \inf_{(x_1,\dots,x_n)\in\mathbb{R}^n\setminus B_r} \int_{\mathbb{R}^n\setminus B_r} K_{ij}(x_1,\dots,x_n,t_1,\dots,t_n)dt_1\dots dt_n \right\} \in (0,1), \quad i=\overline{1,N},$$

$$B_r := \{x:=(x_1,\dots,x_n) : |x| = \sqrt{x_1^2+\dots+x_n^2} \le r\}.$$

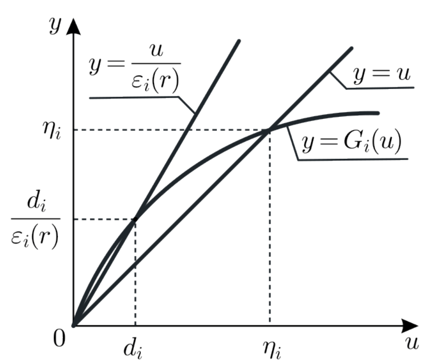

Fig. 1. Graph of the function $y=G_i(u)$

The main purpose of this paper is to investigate the existence and uniqueness of a continuous bounded and positive solution of system (1), as well as the uniform convergence to the solution of the corresponding iterative process with the rate of decreasing geometric progression.

The scalar analog of the system of nonlinear integral equations (1), besides purely theoretical interest, has a number of important applications to the study of various applied problems from physics and biology. In particular, under specific representations of the matrix kernel K and nonlinearities $\{G_j(u)\}_{j=\overline{1,N}}$, the scalar system (1) is encountered in problems from the dynamical theory of p-adic open, closed, and open-closed strings (see [2–5]) and in the mathematical theory of spatial and temporal pandemic propagation in the framework of the modified Atkinson–Roiter and Dickman–Kaper models (see [6, p. 318] and [7, p. 121], respectively).

Mathematical investigations of the system of the form (1) were mainly carried out in the one-dimensional case at $n = 1$. Thus, for example, in the case when $n = 1$ and the kernel K depends on the difference of its arguments, the system (1) is studied in [8–10]. The corresponding scalar analog of system (1) ($N = 1$) in the multidimensional case is studied in [5, 11–13], when the kernel K either depends on the difference of its arguments or is majorized by such a kernel. It should also be noted that the corresponding scalar one-dimensional equations under different restrictions on the kernel and on the nonlinearity have been studied (by different methods) in [2, 3, 14–17].

In this paper, under conditions 1), 2) and a)–d), we will first prove the constructive theorem of existence of a positive continuous and bounded solution of system (1). In the course of the proof of this theorem, we obtain a uniform estimate of the difference between the constructed solution and the corresponding successive approximations, with the right-hand side of the obtained inequality tending to zero as an infinitely decreasing geometric progression when the number of m-th approximation tends to infinity. Further, using some estimates for strictly concave and monotone functions, we prove the uniqueness of the solution of the system (1) in a sufficiently wide subclass of continuous bounded and coordinately nonnegative vector-functions. In the case when

$$C_{ij}(x_1,\ldots,x_n) := \int_{\mathbb{R}^n} K_{ij}(x_1,\ldots,x_n,t_1,\ldots,t_n)dt_1\ldots dt_n = a_{ij}$$

for all $(x_1,\ldots,x_n) \in \mathbb{R}^n$ and $i, j = \overline{1,N}$, we show that in the above mentioned subclass of vector-functions, the only solution of the system (1) is only the vector $\eta = (\eta_1,\ldots,\eta_N)_T$. In this paper, we give specific examples of the matrix kernel K and nonlinearities $\{G_j(u)\}_{j=\overline{1,N}}$, satisfying all conditions of the proved statements. Some of these examples have applications in the above-mentioned areas of physics and biology.

2. KEY NOTATIONS AND SUPPORTING RESULTS

The following lemma plays an important role in our further reasoning.

Lemma 1. *Let conditions* a), b), 1), 2) *be satisfied, and the graphs of the functions* $\{G_j(u)\}_{j=\overline{1,N}}$ *are strictly concave at* $\mathbb{R}^+$. *Then the inequality is true for any ordinally non-negative and bounded on* $\mathbb{R}^n$ *solution* $f^*(x_1,\ldots,x_n) = (f_1^*(x_1,\ldots,x_n),\ldots,f_N^*(x_1,\ldots,x_n))^{\mathrm{T}}$ *of the system* (1):

$$f_i^*(x_1,\ldots,x_n) \le \eta_i, \quad (x_1,\ldots,x_n) \in \mathbb{R}^n, \quad i = \overline{1,N},$$

where $\eta = (\eta_1,\ldots,\eta_N)^{\mathrm{T}}$ *is the fixed vector of the matrix* A (see (2)).

Proof. Let us denote $\gamma_i := \sup_{(x_1,\ldots,x_n)\in\mathbb{R}^n} f_i^*(x_1,\ldots,x_n), i = \overline{1,N}$. Then from system (1) by virtue of conditions 1), 2), a) and relation (2) we will have

$$f_i^*(x_1,\ldots,x_n) \le \sum_{j=1}^{N} a_{ij}G_j(\gamma_j) \le \max_{j=\overline{1,N}}\left\{\frac{G_j(\gamma_j)}{\eta_j}\right\}\sum_{j=1}^{N} a_{ij}\eta_j = \eta_i \max_{j=\overline{1,N}}\left\{\frac{G_j(\gamma_j)}{\eta_j}\right\},$$

$$(x_1,\ldots,x_n) \in \mathbb{R}^n,\ i = \overline{1,N}.$$

It follows that

$$\gamma_i \le \eta_i \max_{j=\overline{1,N}}\left\{\frac{G_j(\gamma_j)}{\eta_j}\right\}, \quad i = \overline{1,N}. \tag{5}$$

Obviously, there exists an index $j^* \in \{1,2,\ldots,N\}$ such that

$$\max_{j=\overline{1,N}}\left\{\frac{G_j(\gamma_j)}{\eta_j}\right\} = \frac{G_{j^*}(\gamma_{j^*})}{\eta_{j^*}}. \tag{6}$$

Replacing in inequality (5) the index i by the index j^*, we obtain $\gamma_{j^*} \le G_{j^*}(\gamma_{j^*})$. Let us see that the last inequality implies the evaluation of $\gamma_{j^*} \le \eta_{j^*}$. Assume the opposite: $\gamma_{j^*} > \eta_{j^*}$. By virtue of conditions a), b) and the strict concavity of the graph of $G_{j^*}(u)$, it follows that the function $\frac{G_{j^*}(u)}{u}$ is monotonically decreasing at $(0,+\infty)$. So $\frac{G_{j^*}(\gamma_{j^*})}{\gamma_{j^*}} < \frac{G_{j^*}(\eta_{j^*})}{\eta_{j^*}} = 1$. The latter inequality contradicts the inequality $\gamma_{j^*} \le G_{j^*}(\gamma_{j^*})$ obtained above. Thus, $\gamma_{j^*} \le \eta_{j^*}$. By virtue of this evaluation, relation (6) and conditions a), b), we arrive from (5) at the inequality $\gamma_i \le \eta_i, i = \overline{1,N}$. The lemma is proved.

The following is also useful

Lemma 2. *Let conditions* a), b), d), 1), *and* 2) *be satisfied and* $f(x_1,\dots,x_n)$ *be an arbitrary generically non-negative and continuous on* $\mathbb{R}^n$ *solution of system* (1). *Then if there exists an index* $j_0 \in \{1,2,\dots,N\}$ *such that* $\delta_{j_0} := \inf_{(x_1,\dots,x_n)\in\mathbb{R}^n\setminus B_r} f_{j_0}(x_1,\dots,x_n) > 0$*, then* $\inf_{(x_1,\dots,x_n)\in\mathbb{R}^n} f_i(x_1,\dots,x_n) > 0, i = \overline{1,N}$*, where the number* r *is defined under condition* d).

Proof. First of all, note that it follows from conditions a), b), d), 1) and, 2) that

$$f_i(x_1,\dots,x_n) \ge \sum_{j=1}^{N} \int_{\mathbb{R}^n\setminus B_r} K_{ij}(x_1,\dots,x_n,t_1,\dots,t_n) G_j(f_j(t_1,\dots,t_n))dt_1\dots dt_n \ge$$

$$\ge \int_{\mathbb{R}^n\setminus B_r} K_{ij_0}(x_1,\dots,x_n,t_1,\dots,t_n) G_{j_0}(f_{j_0}(t_1,\dots,t_n))dt_1\dots dt_n \ge$$

$$\ge G_{j_0}(\delta_{j_0}) \int_{\mathbb{R}^n\setminus B_r} K_{ij_0}(x_1,\dots,x_n,t_1,\dots,t_n)dt_1\dots dt_n, \quad (x_1,\dots,x_n)\in\mathbb{R}^n. \tag{7}$$

Next, let us consider the functions

$$\widetilde{C}_{ij_0}(x_1,\dots,x_n) := \int_{\mathbb{R}^n\setminus B_r} K_{ij_0}(x_1,\dots,x_n,t_1,\dots,t_n)dt_1\dots dt_n, \quad (x_1,\dots,x_n)\in\mathbb{R}^n, i=\overline{1,N},$$

and the following possible cases: A) $(x_1,\dots,x_n)\in\mathbb{R}^n\setminus B_r$, B) $(x_1,\dots,x_n)\in B_r$.

In case A), considering the definition of numbers $\varepsilon_i(r)$ in condition d) and inequality (7), we obtain

$$f_i(x_1,\dots,x_n) \ge G_{j_0}(\delta_{j_0})\varepsilon_i(r), \quad (x_1,\dots,x_n)\in\mathbb{R}^n\setminus B_r, i=\overline{1,N}. \tag{8}$$

Let us now consider the case B). It immediately follows from conditions 1), 2), that $\tilde{C}_{ij_0}\in C(\mathbb{R}^n)$, $\tilde{C}_{ij_0}(x_1,\dots,x_n) > 0$, $(x_1,\dots,x_n)\in\mathbb{R}^n$, $i=\overline{1,N}$. Given the compactness of the ball B_r, according to the Weierstrass theorem, we can assert that for each $i\in\{1,2,\dots,N\}$ there exists a point $x^i=(x_1^i,\dots,x_n^i)\in B_r$ such that

$$\min_{(x_1,\dots,x_n)\in B_r}\{\widetilde{C}_{ij_0}(x_1,\dots,x_n)\} = \widetilde{C}_{ij_0}(x_1^i,\dots,x_n^i) > 0. \tag{9}$$

From (7)–(9) we conclude that

$$\inf_{(x_1,\dots,x_n)\in\mathbb{R}^n} f_i(x_1,\dots,x_n) \ge \min\{\varepsilon_i(r), \widetilde{C}_{ij_0}(x_1^i,\dots,x_n^i)\} G_{j_0}(\delta_{j_0}), \quad (x_1,\dots,x_n)\in\mathbb{R}^n, i=\overline{1,N}.$$

The lemma is proved.

Now consider the functions $C_{ij}(x_1,\dots,x_n), i,j=\overline{1,N}$ and suppose that

e) there exist a point $(x_1,\dots,x_n)\in\mathbb{R}^n$ and indices $i_1,j_1\in\{1,2,\dots,N\}$ such that

$$C_{i_1,j_1}(x_1,\dots,x_n) < a_{i_1 j_1}.$$

Lemma 3. *Let the conditions of Lemma* 1 *and* e) *be satisfied. Then, any continuous bounded and coordinate non-negative solution* $f(x_1,\dots,x_n)$ *of system* (1) *satisfies the inequalities* $f_i(x_1,\dots,x_n) < \eta_i, (x_1,\dots,x_n)\in\mathbb{R}^n$, $i=\overline{1,N}$.

Proof. According to lemma 1, the solution is $f_i(x_1,\dots,x_n)\le\eta_i, i=\overline{1,N}$. Let us verify that $f_i(x_1,\dots,x_n)\not\equiv\eta_i$, $i=\overline{1,N}$. Indeed, otherwise, from (1) with condition b) we obtain

$$\sum_{j=1}^{N} C_{ij}(x_1,\dots,x_n)\eta_j \equiv \eta_i, \quad i=\overline{1,N}.$$

Taking into account (2), we come to the equality

$$\sum_{j=1}^{N} \eta_j (C_{ij}(x_1, \ldots, x_n) - a_{ij}) \equiv 0, \quad i = \overline{1, N}. \tag{10}$$

Since $C_{ij}(x_1, \ldots, x_n) \le a_{ij}, \eta_j > 0, i, j = \overline{1, N}$, we arrive at a contradiction in (10) by virtue of condition e). Hence, there exists a point $(x_1^*, \ldots, x_n^*) \in \mathbb{R}^n$ and an index $j^* \in \{1, 2, \ldots, N\}$ such that $f_{j^*}(x_1^*, \ldots, x_n^*) < \eta_{j^*}$. Hence, by continuity of the function f_{j^*} it follows. That there exists a neighborhood $O_\delta(x_1^*, \ldots, x_n^*)$ of the point $(x_1^*, \ldots, x_n^*)$ such that

$$f_{j^*}(x_1, \ldots, x_n) < \eta_{j^*}, \quad (x_1, \ldots, x_n) \in O_\delta(x_1^*, \ldots, x_n^*). \tag{11}$$

By virtue of (11), relation (2) and inequality $C_{ij}(x_1, \ldots, x_n) \le a_{ij}$ from (1), taking into account conditions a), b) we will have

$$f_i(x_1, \ldots, x_n) = \sum_{j \ne j^*} \int\limits_{\mathbb{R}^n} K_{ij}(x_1, \ldots, x_n, t_1, \ldots, t_n) G_j(f_j(t_1, \ldots, t_n))\, dt_1 \ldots dt_n +$$

$$+ \int\limits_{\mathbb{R}^n} K_{ij^*}(x_1, \ldots, x_n, t_1, \ldots, t_n) G_{j^*}(f_{j^*}(t_1, \ldots, t_n))\, dt_1 \ldots dt_n \le$$

$$\le \sum_{j \ne j^*} C_{ij}(x_1, \ldots, x_n)\eta_j + \int\limits_{\mathbb{R}^n \setminus O_\delta(x_1^*, \ldots, x_n^*)} K_{ij^*}(x_1, \ldots, x_n, t_1, \ldots, t_n) G_{j^*}(f_{j^*}(t_1, \ldots, t_n))\, dt_1 \ldots dt_n +$$

$$+ \int\limits_{O_\delta(x_1^*, \ldots, x_n^*)} K_{ij^*}(x_1, \ldots, x_n, t_1, \ldots, t_n) G_{j^*}(f_{j^*}(t_1, \ldots, t_n))\, dt_1 \ldots dt_n \le$$

$$\le \sum_{j \ne j^*} C_{ij}(x_1, \ldots, x_n)\eta_j + \eta_{j^*} \int\limits_{\mathbb{R}^n \setminus O_\delta(x_1^*, \ldots, x_n^*)} K_{ij^*}(x_1, \ldots, x_n, t_1, \ldots, t_n)\, dt_1 \ldots dt_n +$$

$$+ \int\limits_{O_\delta(x_1^*, \ldots, x_n^*)} K_{ij^*}(x_1, \ldots, x_n, t_1, \ldots, t_n) G_{j^*}(f_{j^*}(t_1, \ldots, t_n))\, dt_1 \ldots dt_n <$$

$$< \sum_{j \ne j^*} C_{ij}(x_1, \ldots, x_n)\eta_j + \eta_{j^*} \int\limits_{\mathbb{R}^n \setminus O_\delta(x_1^*, \ldots, x_n^*)} K_{ij^*}(x_1, \ldots, x_n, t_1, \ldots, t_n)\, dt_1 \ldots dt_n +$$

$$+ \eta_{j^*} \int\limits_{O_\delta(x_1^*, \ldots, x_n^*)} K_{ij^*}(x_1, \ldots, x_n, t_1, \ldots, t_n) dt_1 \ldots dt_n =$$

$$= \sum_{j \ne j^*} C_{ij}(x_1, \ldots, x_n)\eta_j + C_{ij^*}(x_1, \ldots, x_n)\eta_{j^*} \le \sum_{j=1}^{N} a_{ij}\eta_j = \eta_i, \quad i, j = \overline{1, N}.$$

The lemma is proved.

3. THEOREM OF EXISTENCE OF BOUNDED SOLUTION

Let us now consider the following successive approximations for system (1):

$$f_i^{(m+1)}(x_1, \ldots, x_n) = \sum_{j=1}^{N} \int_{\mathbb{R}^n} K_{ij}(x_1, \ldots, x_n, t_1, \ldots, t_n) G_j(f_j^{(m)}(t_1, \ldots, t_n)) dt_1 \ldots dt_n,$$

$$f_i^{(0)}(x_1, \ldots, x_n) \equiv \eta_i, \quad (x_1, \ldots, x_n) \in \mathbb{R}^n, i = \overline{1, N}, m = 0, 1, 2, \ldots \tag{12}$$

Suppose that conditions a)–d), 1), and 2) are satisfied. By induction on m, it is not difficult to check the validity of the following statements:

$$f_i^{(m)}(x_1,\dots,x_n) \text{ monotonically decreasing on } m,\ m=0,1,2,\dots,\ i=\overline{1,N}, \tag{13}$$

$$f_i^{(m)}\in C(\mathbb{R}^n),\ \ i=\overline{1,N}, \tag{14}$$

$$f_i^{(m)}(x_1,\dots,x_n)>0,\ \ m=0,1,2,\dots,\ i=\overline{1,N}. \tag{15}$$

Let us prove that for all $(x_1,\dots,x_n)\in\mathbb{R}^n\backslash B_r$ the following lower bound estimates hold:

$$f_i^{(m)}(x_1,\dots,x_n)\ge d_i,\quad m=0,1,2,\dots,i=\overline{1,N}, \tag{16}$$

where the numbers d_i are defined under condition d).

Let us check inequality (16) at $m=0$. Indeed, since the functions $G_i(u)/u$ are monotonically decreasing at $(0,+\infty)$, $i=\overline{1,N}$, then from the estimation of

$$1=\frac{G_i(\eta_i)}{\eta_i}<\frac{1}{\varepsilon_i(r)}=\frac{G_i(d_i)}{d_i}$$

we get that $d_i<\eta_i=f_i^{(0)}(x_1,\dots,x_n), i=\overline{1,N}$.

Suppose now that for $(x_1,\dots,x_n)\in\mathbb{R}^n\backslash B_r$, inequality (16) holds for some natural m. Then, using the conditions a), b), d), 1), and 2), from (12) and (15) we will have

$$f_i^{(m+1)}(x_1,\dots,x_n)\ge\sum_{j=1}^N\int_{\mathbb{R}^n\backslash B_r}K_{ij}(x_1,\dots,x_n,t_1,\dots,t_n)G_j(f_j^{(m)}(t_1,\dots,t_n))dt_1\dots dt_n\ge$$

$$\ge\sum_{j=1}^N G_j(d_j)\int_{\mathbb{R}^n\backslash B_r}K_{ij}(x_1,\dots,x_n,t_1,\dots,t_n)dt_1\dots dt_n\ge G_i(d_i)\varepsilon_i(r)=d_i,\quad i=\overline{1,N}.$$

If condition e) is satisfied, by analogy with the proof of Lemma 3, we can also verify that the inequalities hold

$$f_i^{(m)}(x_1,\dots,x_n)<\eta_i,\quad m=1,2,\dots,i=\overline{1,N},(x_1,\dots,x_n)\in\mathbb{R}^n. \tag{17}$$

Taking into account (14), (15) and the compactness of the ball B_r, we can say that for every $i\in\{1,2,\dots,N\}$ and $m\in\{0,1,2,\dots\}$, there exists a point $(x_1^{(m,i)},\dots,x_n^{(m,i)})\in B_r$ such that

$$\min_{(x_1,\dots,x_n)\in B_r}f_i^{(m)}(x_1,\dots,x_n)=f_i^{(m)}(x_1^{(m,i)},\dots,x_n^{(m,i)})>0,\quad (x_1,\dots,x_n)\in B_r. \tag{18}$$

Thus, it follows from (16) and (18) for $(x_1,\dots,x_n)\in\mathbb{R}^n$, that

$$f_i^{(m)}(x_1,\dots,x_n)\ge\min\{f_i^{(m)}(x_1^{(m,i)},\dots,x_n^{(m,i)}),d_i\}>0,\quad m=0,1,2,\dots,i=\overline{1,N}. \tag{19}$$

Let us now consider the functions $\chi_i(x_1,\dots,x_n)=\frac{f_i^{(2)}(x_1,\dots,x_n)}{f_i^{(1)}(x_1,\dots,x_n)}, i=\overline{1,N}$, on the set $\mathbb{R}^n$. From (13), (14), and (19) we have

$$\chi_i\in C(\mathbb{R}^n),\quad i=\overline{1,N},$$

$$\frac{\alpha_i}{\eta_i}\le\chi_i(x_1,\dots,x_n)\le1,\quad (x_1,\dots,x_n)\in\mathbb{R}^n,i=\overline{1,N}, \tag{20}$$

where by virtue of (17), (19),

$$0<\alpha_i:=\min\{f_i^{(2)}(x_1^{(2,i)},\dots,x_n^{(2,i)}),d_i\}<\eta_i,\quad i=\overline{1,N}.$$

Let us denote by $\sigma_0=\min_{i=\overline{1,N}}(\alpha_i\eta_i)$. Obviously, $\sigma_0\in(0,1)$.

Consequently, considering (20) and (12) and conditions 1), a), we will have

$$\sum_{j=1}^{N}\int_{\mathbb{R}^n} K_{ij}(x_1,\ldots,x_n,t_1,\ldots,t_n)G_j(\sigma_0 f_j^{(1)}(t_1,\ldots,t_n))dt_1\ldots dt_n \le$$

$$\le f_i^{(3)}(x_1,\ldots,x_n) \le \sum_{j=1}^{N}\int_{\mathbb{R}^n} K_{ij}(x_1,\ldots,x_n,t_1,\ldots,t_n)G_j(f_j^{(1)}(t_1,\ldots,t_n))dt_1\ldots dt_n =$$

$$= f_i^{(2)}(x_1,\ldots,x_n), \quad (x_1,\ldots,x_n)\in\mathbb{R}^n, i=\overline{1,N}.$$

Hence, by virtue of condition c), we arrive at the inequalities

$$\varphi(\sigma_0)f_i^{(2)}(x_1,\ldots,x_n) \le f_i^{(3)}(x_1,\ldots,x_n) \le f_i^{(2)}(x_1,\ldots,x_n), \quad i=\overline{1,N}. \tag{21}$$

Now, using (21), (12), conditions 1), a), and c), let us write down

$$\varphi(\varphi(\sigma_0))f_i^{(3)}(x_1,\ldots,x_n) \le f_i^{(4)}(x_1,\ldots,x_n) \le f_i^{(3)}(x_1,\ldots,x_n), \quad i=\overline{1,N}.$$

Continuing this reasoning, at m-step we obtain the following estimate:

$$F_m(\sigma_0)f_i^{(m+1)}(x_1,\ldots,x_n) \le f_i^{(m+2)}(x_1,\ldots,x_n) \le f_i^{(m+1)}(x_1,\ldots,x_n),$$

$$(x_1,\ldots,x_n)\in\mathbb{R}^n, m=1,2,\ldots, i=\overline{1,N}, F_m(\sigma):=\underbrace{\varphi(\varphi\ldots\varphi(\sigma))}_{m\text{ times}}, \sigma\in[0,1]. \tag{22}$$

Then, using properties (3) and (4) of the function φ, we prove the validity of the inequality

$$F_m(\sigma_0) \ge k^m\sigma_0 + 1 - k^m, \quad m=1,2,\ldots, \tag{23}$$

where

$$k := \frac{1-\varphi(\frac{\sigma_0}{2})}{1-\frac{\sigma_0}{2}} \in (0,1), \quad \sigma_0 = \min_{i=\overline{1,N}}\left\{\frac{\alpha_i}{\eta_i}\right\} \in (0,1). \tag{24}$$

For this purpose, consider the line $y = ku + 1 - k$, passing through the points $(1,1)$ and $(\frac{\sigma_0}{2}, \varphi(\frac{\sigma_0}{2}))$, where the number k is given according to formula (24). From properties (3) and (4), it immediately follows that (Fig. 2)

$$\varphi(\sigma_0) \ge k\sigma_0 + 1 - k. \tag{25}$$

Since $k\sigma_0 + 1 - k \in (0,1)$, then taking into account the properties of concavity of the graph and monotonicity of the function φ from (25) we will have

$$F_2(\sigma_0) = \varphi(\varphi(\sigma_0)) \ge \varphi(k\sigma_0+1-k) \ge k(k\sigma_0+1-k)+1-k = k^2\sigma_0+1-k^2.$$

Continuing this process, at m-th step we obtain inequality (23).

Thus, in view of (22), (23), (17) and (13) we arrive at the following uniform estimate for successive approximations of (12):

$$0 \le f_i^{(m+1)}(x_1,\ldots,x_n) - f_i^{(m+2)}(x_1,\ldots,x_n) < \eta_i(1-\sigma_0)k^m,$$

$$(x_1,\ldots,x_n)\in\mathbb{R}^n, m=1,2,\ldots, i=\overline{1,N}. \tag{26}$$

From (26), we obtain uniform convergence of the sequence of continuous vector functions $f^{(m)}(x_1,\ldots,x_n) = (f_1^{(m)}(x_1,\ldots,x_n),\ldots,f_N^{(m)}(x_1,\ldots,x_n))^T, m=0,1,2,\ldots$, on the set $\mathbb{R}^n$:

$$\lim_{m\to\infty} f_i^{(m)}(x_1,\ldots,x_n) = f_i(x_1,\ldots,x_n), \quad (x_1,\ldots,x_n)\in\mathbb{R}^n, i=\overline{1,N},$$

and $f_i \in C(\mathbb{R}^n)$, $i=\overline{1,N}$.

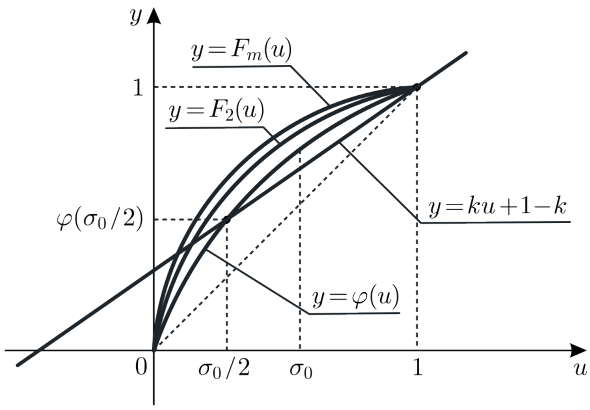

Fig. 2. Graph of the function $y = \varphi(u)$

By virtue of (13), conditions 1), 2), a), (14), (16), (26), and B. Levi's theorem (see [18, p. 303]), the limit vector function $f(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_N(x_1, \ldots, x_n))^{\mathrm{T}}$ satisfies the system (1) and the evaluation from below

$$f_i(x_1, \ldots, x_n) \geq d_i, \quad (x_1, \ldots, x_n) \in \mathbb{R}^n \backslash B_r, i = \overline{1, N}. \tag{27}$$

Given the estimate (27) and lemma 2, we conclude that

$$\inf_{(x_1,\ldots,x_n)\in\mathbb{R}^n} f_i(x_1, \ldots, x_n) > 0, \quad i = \overline{1, N}. \tag{28}$$

Then, taking into account condition e), the statement of lemma 3, and the monotonicity property (13), we arrive at the strict inequality

$$f_i(x_1, \ldots, x_n) < \eta_i, \quad (x_1, \ldots, x_n) \in \mathbb{R}^n, i = \overline{1, N}. \tag{29}$$

Now in evaluation (26), instead of m, we take $m+1, m+2, \ldots, m+p$. As a result, we obtain the following inequalities:

$$0 \leq f_i^{(m+2)}(x_1, \ldots, x_n) - f_i^{(m+3)}(x_1, \ldots, x_n) < \eta_i(1-\sigma_0)k^{m+1},$$
$$(x_1, \ldots, x_n) \in \mathbb{R}^n, m = 1, 2, \ldots, i = \overline{1, N},$$
$$0 \leq f_i^{(m+3)}(x_1, \ldots, x_n) - f_i^{(m+4)}(x_1, \ldots, x_n) < \eta_i(1-\sigma_0)k^{m+2},$$
$$(x_1, \ldots, x_n) \in \mathbb{R}^n, m = 1, 2, \ldots, i = \overline{1, N},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$0 \leq f_i^{(m+p+1)}(x_1, \ldots, x_n) - f_i^{(m+p+2)}(x_1, \ldots, x_n) < \eta_i(1-\sigma_0)k^{m+p},$$
$$(x_1, \ldots, x_n) \in \mathbb{R}^n, p, m = 1, 2, \ldots, i = \overline{1, N}.$$

Summarizing them with inequality (26), we arrive at a two-sided estimator

$$0 \leq f_i^{(m+1)}(x_1, \ldots, x_n) - f_i^{(m+p+2)}(x_1, \ldots, x_n) < \eta_i(1-\sigma_0)(k^m + k^{m+1} + \cdots + k^{m+p}),$$
$$(x_1, \ldots, x_n) \in \mathbb{R}^n, p, m = 1, 2, \ldots, i = \overline{1, N}. \tag{30}$$

From (30), in particular, it follows that

$$0 < f_i^{(m+1)}(x_1, \ldots, x_n) - f_i^{(m+p+2)}(x_1, \ldots, x_n) < \eta_i(1-\sigma_0)\frac{k^m}{1-k}. \tag{31}$$

Fixing the index m and decreasing $p \to \infty$ in (31), we obtain

$$0 < f_i^{(m+1)}(x_1, \dots, x_n) - f_i(x_1, \dots, x_n) < \eta_i(1 - \sigma_0)\frac{k^m}{1-k}. \tag{32}$$

Note also that if the functions $\{C_{ij}(x_1, \dots, x_n)\}_{i,j=\overline{1,N}}$ satisfy the additional condition

$$a_{ij} - C_{ij}(x_1, \dots, x_n) \in L_1(\mathbb{R}^n), \quad i, j = \overline{1, N}, \tag{33}$$

then, reasoning similarly to the proof of the main theorem (on the integral asymptotics of the solution) from [13], we can assert that there exist positive constants $D_1, D_2, \dots, D_N$ such that

$$0 \le \int_{\mathbb{R}^n} (\eta_i - f_i^{(m)}(x_1, \dots, x_n))dx_1 \dots dx_n \le D_i, \quad m = 0, 1, 2, \dots, i = \overline{1, N}.$$

Hence, according to the theorem of B. Levi, we conclude that $\eta_i - f_i \in L_1(\mathbb{R}^n), i = \overline{1, N}$, and

$$\int_{\mathbb{R}^n} (\eta_i - f_i(x_1, \dots, x_n))dx_1 \dots dx_n \le D_i, \quad i = \overline{1, N}.$$

Based on the above, the following is true

Theorem 1. *If conditions* a)–e), 1), 2) *are satisfied, the system of nonlinear multivariate integral equations* (1) *has an ordinally positive continuous and bounded on* $\mathbb{R}^n$ *solution* $f(x_1, \dots, x_n) = (f_1(x_1, \dots, x_n), \dots, f_N(x_1, \dots, x_n))^T$, *that is a uniform limit of successive approximations* (12). *Moreover, the estimates* (27)–(29) *and* (32) *hold. If in addition condition* (33) *is satisfied, then* $\eta_i - f_i \in L_1(\mathbb{R}^n), i = \overline{1, N}$.

4. SINGULARITY OF THE SOLUTION OF THE SYSTEM (1)

Let us consider the following subclass of continuous nonnegative and bounded vector functions on $\mathbb{R}^n$:

$$\mathbb{H} := \Big\{ f(x_1, \dots, x_n) = (f_1(x_1, \dots, x_n), \dots, f_N(x_1, \dots, x_n))^{\mathrm{T}} \colon f_i \in C_M(\mathbb{R}^n),$$
$$f_i(x_1, \dots, x_n) \ge 0,\ (x_1, \dots, x_n) \in \mathbb{R}^n,\ i = \overline{1, N},$$
$$\text{there is such } j_0 \in \{1, 2, \dots, N\} \text{ that } \inf_{(x_1,\dots,x_n)\in\mathbb{R}^n\setminus B_r} f_{j_0}(x_1, \dots, x_n) > 0 \Big\}, \tag{34}$$

where the number $r > 0$ is defined in condition d), through $C_M(\mathbb{R}^n)$, the space of continuous and bounded functions on the set $\mathbb{R}^n$ is denoted. The following holds

Theorem 2. *If conditions* a)–e), 1), 2) *are satisfied, the system of nonlinear multivariate integral equations* (1) *has no other solutions in the class* $\mathbb{H}$ *except for the solution* f, *constructed by means of successive approximations* (13).

Proof. Suppose the converse: the system (1) besides the solution $f \in \mathbb{H}$, constructed by means of successive approximations (12), also possesses another solution $f^* \in \mathbb{H}$. Then, using lemmas 2 and 3, we conclude that

$$f_i^*(x_1, \dots, x_n) < \eta_i, \quad (x_1, \dots, x_n) \in \mathbb{R}^n, i = \overline{1, N}, \tag{35}$$
$$\alpha_i^* := \inf_{(x_1,\dots,x_n)\in\mathbb{R}^n} f_i^*(x_1, \dots, x_n) > 0, \quad i = \overline{1, N}. \tag{36}$$

Applying the method of induction by m, it is easy to verify the validity of the following inequalities:

$$f_i^*(x_1, \dots, x_n) < f_i^{(m)}(x_1, \dots, x_n), \quad (x_1, \dots, x_n) \in \mathbb{R}^n, m = 0, 1, 2, \dots, i = \overline{1, N}. \tag{37}$$

In (37) by decreasing $m \to \infty$, we arrive at the inequality

$$f_i^*(x_1, \dots, x_n) \le f_i(x_1, \dots, x_n), \quad (x_1, \dots, x_n) \in \mathbb{R}^n, i = \overline{1, N}. \tag{38}$$

Consider the functions $B_i(x_1,\dots,x_n) = f_i^*(x_1,\dots,x_n)/f_i(x_1,\dots,x_n)$, $(x_1,\dots,x_n) \in \mathbb{R}^n$, $i = \overline{1,N}$. . Since $f, f^* \in \mathbb{H}$, then by virtue of (28), (29), (35), (36), (38), we have that $B_i \in C(\mathbb{R}^n), i = \overline{1,N}$, and

$$\frac{\alpha_i^*}{\eta_i} \le B_i(x_1,\dots,x_n) \le 1, \quad (x_1,\dots,x_n) \in \mathbb{R}^n, i = \overline{1,N}.$$

Let us denote $\sigma^* = \min_{i\in\overline{1,N}}\{\alpha_i^*/\eta_i\}$. By virtue of (35) and (36), the number $\sigma^* \in (0,1)$. Thus, we obtain the inequality

$$\sigma^* f_i(x_1,\dots,x_n) \le f_i^*(x_1,\dots,x_n) \le f_i(x_1,\dots,x_n), \quad (x_1,\dots,x_n) \in \mathbb{R}^n, i = \overline{1,N}. \tag{39}$$

Then, reasoning as in the proof of Theorem 1, we obtain the following estimates from (39):

$$0 \le f_i(x_1,\dots,x_n) - f_i^*(x_1,\dots,x_n) \le \eta_i(1-\sigma^*)k_*^m, \quad (x_1,\dots,x_n) \in \mathbb{R}^n, m = 1,2,\dots, i = \overline{1,N}, \tag{40}$$

where $k_* = \frac{1-\varphi\left(\frac{\sigma^*}{2}\right)}{1-\frac{\sigma^*}{2}} \in (0,1)$.

In (40), by decreasing the number $m \to \infty$, we arrive at the equality $f_i(x_1,\dots,x_n) = f_i^*(x_1,\dots,x_n)$, $(x_1,\dots,x_n) \in \mathbb{R}^n, i = \overline{1,N}$. The theorem is proved.

Similarly, the following is proved

Theorem 3. *Let the conditions* a)–d), 1), 2) *be satisfied and the following relations hold*

$$C_{ij}(x_1,\dots,x_n) = a_{ij}, \quad (x_1,\dots,x_n) \in \mathbb{R}^n, i, j = \overline{1,N}.$$

Then the system (1) *in the class* $\mathbb{H}$ *possesses only a trivial solution* $\eta = (\eta_1,\dots,\eta_N)^{\mathrm{T}}$.

5. EXAMPLES

To illustrate the theoretical results obtained, we give examples of the matrix kernel K and nonlinearities $\{G_j(u)\}_{j=\overline{1,N}}$.

Core K examples:

p$_1$) $K_{ij}(x_1,\dots,x_n,t_1,\dots,t_n) = \mathring{K}_{ij}(x_1-t_1,x_2-t_2,\dots,x_n-t_n), (x_1,\dots,x_n),(t_1,\dots,t_n) \in \mathbb{R}^n, 2i, j = \overline{1,N}$, where $\mathring{K}_{ij}(\tau_1,\tau_2,\dots,\tau_n) > 0$, $\mathring{K}_{ij} \in C(\mathbb{R}^n)$, $\int_{\mathbb{R}^n}\mathring{K}_{ij}(\tau_1,\dots,\tau_n)\,d\tau_1\dots d\tau_n = a_{ij} < 1$, $i,j = \overline{1,N}$, $r(A) = 1$, $A = (a_{ij})_{i,j=\overline{1,N}}$, $(\tau_1,\dots,\tau_n) \in \mathbb{R}^n$.

p$_2$) $K_{ij}(x_1,\dots,x_n,t_1,\dots,t_n) = \lambda_{ij}(|x|)\mathring{K}_{ij}(x_1-t_1,x_2-t_2,\dots,x_n-t_n)$, $(x_1,\dots,x_n)$, $(t_1,\dots,t_n) \in \mathbb{R}^n$, $|x| = \sqrt{x_1^2+\dots+x_n^2}$, $0 < \inf_{v\ge 0}\lambda_{ij}(v) \le \lambda_{ij}(v) < 1$, $v \ge 0$, $1-\lambda_{ij} \in L_1(0,+\infty)$, $i,j = \overline{1,N}$.

p$_3$) $K_{ij}(x_1,\dots,x_n,t_1,\dots,t_n) = C_{ij}^*(x_1,\dots,x_n)\mathring{K}_{ij}(x_1-t_1,\dots,x_n-t_n), (x_1,\dots,x_n),(t_1,\dots,t_n) \in \mathbb{R}^n$, $\inf_{(x_1,\dots,x_n)\in\mathbb{R}^n} C_{ij}^*(x_1,\dots,x_n) > 0$, $C_{ij}^* \in C(\mathbb{R}^n)$, $\sup_{(x_1,\dots,x_n)\in\mathbb{R}^n} C_{ij}^*(x_1,\dots,x_n) = 1$, $i,j = \overline{1,N}$.

Here are also *examples of functions* $\mathring{K}_{ij}$, λ_{ij}, C_{ij}^*, $i,j = \overline{1,N}$:

q$_1$) $\mathring{K}_{ij}(\tau_1,\dots,\tau_n) = \pi^{-n/2}a_{ij}e^{-(\tau_1^2+\dots+\tau_n^2)}$, $r(A) = 1$, $A = (a_{ij})_{i,j=\overline{1,N}}$, $\tau_j \in \mathbb{R}$, $i,j = \overline{1,N}$,

q$_2$) $\mathring{K}_{ij}(\tau_1,\dots,\tau_n) = \int_a^b e^{-(|\tau_1|+\dots+|\tau_n|)s}\,dQ_{ij}(s)$, $\tau_j \in \mathbb{R}$, $i,j = \overline{1,N}$, where $Q_{ij}(s)$ — are monotonically increasing functions on $[a,b)$, $0 < a < b \le +\infty$, with

$$2^n\int_a^b \frac{1}{s^n}\,dQ_{ij}(s) = a_{ij}, \quad i,j = \overline{1,N};$$

q$_3$) $\lambda_{ij}(|x|) = 1 - \varepsilon_{ij}e^{-(x_1^2+\dots+x_n^2)}$, $0 < \varepsilon_{ij} < 1$ — are parameters, $(x_1,\dots,x_n) \in \mathbb{R}^n$, $i,j = \overline{1,N}$,

q$_4$) $C_{ij}^*(x_1,\dots,x_n) = 1 - \varepsilon_{ij}e^{-(|x_1|+\dots|x_n|)}$, $(x_1,\dots,x_n) \in \mathbb{R}^n$, $i,j = \overline{1,N}$.

Let us now turn to *examples of nonlinearities* $\{G_j(u)\}_{j=\overline{1,N}}$:

r$_1$) $G_j(u) = u^{\beta_j}\eta_j^{1-\beta_j}$, $u \in [0, +\infty)$, $\beta_j \in (0,1)$, $j = \overline{1, N}$;

r$_2$) $G_j(u) = \eta_j(u^{\beta_j} + u^{\delta_j})/(\eta_j^{\beta_j} + \eta_j^{\delta_j})$, $u \in [0, +\infty)$, $\beta_j, \delta_j \in (0,1)$, $j = \overline{1, N}$;

r$_3$) $G_j(u) = l_j(1 - e^{-u^{\beta_j}})$, $u \in [0, +\infty)$, $\beta_j \in (0,1)$, $l_j = \eta_j/(1 - \exp\{-\eta_j^{\beta_j}\})$, $j = \overline{1, N}$.

Let us elaborate on examples p$_3$), q$_1$), r$_3$) and verify that conditions 2) and d) are satisfied. First of all, note that in this case

$$\sup_{(x_1,\dots,x_n)\in\mathbb{R}^n} \int_{\mathbb{R}^n} K_{ij}(x_1,\dots,x_n,t_1,\dots,t_n)dt_1\dots dt_n =$$

$$= \sup_{(x_1,\dots,x_n)\in\mathbb{R}^n} \left(C^*_{ij}(x_1,\dots,x_n) \int_{\mathbb{R}^n} \mathring{K}_{ij}(x_1 - t_1,\dots,x_n - t_n)dt_1\dots dt_n \right) =$$

$$= \sup_{(x_1,\dots,x_n)\in\mathbb{R}^n} \left(C^*_{ij}(x_1,\dots,x_n) \int_{\mathbb{R}^n} \mathring{K}_{ij}(\tau_1,\dots,\tau_n)d\tau_1\dots d\tau_n \right) =$$

$$= a_{ij} \sup_{(x_1,\dots,x_n)\in\mathbb{R}^n} C^*_{ij}(x_1,\dots,x_n) = a_{ij}, \quad i,j = \overline{1, N}.$$

Since $r(A) = 1$ (see Example q$_1$)), condition 2) is satisfied. For completeness, let us give an example of the matrix $A = (a_{ij})_{i,j=\overline{1,N}}$ with unit spectral radius and with elements $a_{ij} \in (0,1)$, $i,j = \overline{1, N}$ (in the case when $N = 2$):

$$A = \begin{pmatrix} 7/9 & 1/3 \\ 1/3 & 1/2 \end{pmatrix}.$$

Let's check condition d). First evaluate the integral of the function $\mathring{K}_{ij}(x_1 - t_1,\dots,x_n - t_n)$ over the set $\mathbb{R}^n \backslash B_r$:

$$\int\limits_{\mathbb{R}^n\backslash B_r} \mathring{K}_{ij}(x_1 - t_1,\dots,x_n - t_n)\,dt_1\dots dt_n =$$

$$= \int\limits_{\mathbb{R}^n} \mathring{K}_{ij}(x_1 - t_1,\dots,x_n - t_n)\,dt_1\dots dt_n - \int\limits_{B_r} \mathring{K}_{ij}(x_1 - t_1,\dots,x_n - t_n)\,dt_1\dots dt_n =$$

$$= a_{ij} - \int\limits_{B_r} \mathring{K}_{ij}(x_1 - t_1,\dots,x_n - t_n)\,dt_1\dots dt_n \geq a_{ij} - \int\limits_{-r}^{r} \int\limits_{\mathbb{R}^{n-1}} \mathring{K}_{ij}(x_1 - t_1,\dots,x_n - t_n)\,dt_1\dots dt_n =$$

$$= a_{ij} - \int\limits_{-r}^{r} \Phi_{ij}(x_n - t_n)\,dt_n = a_{ij} - \int\limits_{x_n - r}^{x_n + r} \Phi_{ij}(\tau_n)\,d\tau_n,$$

where $\Phi_{ij}(\tau) := \int_{\mathbb{R}^{n-1}} \mathring{K}_{ij}(t_1,\dots,t_{n-1},\tau)dt_1\dots dt_{n-1}$.

Consider the functions $F_{ij}(x_n) := \int_{x_n - r}^{x_n + r} \Phi_{ij}(\tau_n)d\tau_n$, $i,j = \overline{1, N}$, $x_n \in \mathbb{R}$. Since $F_{ij}(x_n) \to 0$ at $|x_n| \to \infty$, for every fixed $i,j \in \{1,2,\dots,N\}$, there exists a number $r_0 > 0$ such that at $|x_n| > r_0$

$$F_{ij}(x_n) \leq \frac{a_{ij}}{2}.$$

But since $F_{ij} \in C(\mathbb{R})$ and $\mathring{K}_{ij}(t_1,\dots,t_n) > 0$, $(t_1,\dots,t_n) \in \mathbb{R}^n$, then for $x_n \in [-r_0, r_0]$

$$F_{ij}(x_n) \leq \max_{x_n\in[-r_0,r_0]} \left\{ \int_{x_n - r}^{x_n + r} \Phi_{ij}(\tau_n)d\tau_n \right\} =: \delta_{ij} < a_{ij}.$$

Hence, $F_{ij}(x_n) \leq \max\{a_{ij}/2, \delta_{ij}\} < a_{ij}$, $x_n \in \mathbb{R}$, $i,j = \overline{1, N}$.

Thus, we have

$$\inf_{(x_1,\ldots,x_n)\in\mathbb{R}^n\setminus B_r}\int_{\mathbb{R}^n\setminus B_r}\mathring{K}_{ij}(x_1-t_1,\ldots,x_n-t_n)dt_1\ldots dt_n\geq$$

$$\geq\inf_{(x_1,\ldots,x_n)\in\mathbb{R}^n}\int_{\mathbb{R}^n\setminus B_r}\mathring{K}_{ij}(x_1-t_1,\ldots,x_n-t_n)dt_1\ldots dt_n\geq a_{ij}-\max\{a_{ij}/2,\delta_{ij}\}>0,\quad i,j=\overline{1,N},$$

whence it follows that

$$\varepsilon_i(r)\geq\min_{j=\overline{1,N}}\{C^0_{ij}(a_{ij}-\max\{\frac{a_{ij}}{2},\delta_{ij}\})\}>0,$$

where $C^0_{ij}:=\inf_{(x_1,\ldots,x_n)\in\mathbb{R}^n}C^*_{ij}(x_1,\ldots,x_n)$.

On the other hand, it is obvious that $\varepsilon_i(r)\leq a_{ij}<1$, $i,j=\overline{1,N}$.

We now verify that, for Example p$_3$), the equations $G_i(u)=u/\varepsilon_i(r)$ have positive solutions d_i. Indeed, since $G_i\in C(\mathbb{R}^+)$, $G_i(\eta_i)=\eta_i$, $\lim_{u\to+0}G_i(u)/u=+\infty$, $\lim_{u\to+\infty}G_i(u)/u=0$, $i=\overline{1,N}$, and $\varepsilon_i(r)\in(0,1)$; and $G_i(u)/u$ decreases monotonically at $(0,+\infty)$, then for every $i\in\{1,2,\ldots,N\}$, there exists a single $d_i>0$ such that $G_i(d_i)/d_i=1/\varepsilon_i(r)$.

The verification of conditions 2) and d) for the rest of the examples is done in the same way.

Now let us give a specific example of a nonlinear multidimensional integral equation having an application in the theory of p-adic string (see [5]):

$$\varphi^p(x_1,\ldots,x_n)=\pi^{-n/2}\int_{\mathbb{R}^n}e^{-((x_1-t_1)^2+\cdots+(x_n-t_n)^2)}\varphi(t_1,\ldots,t_n)dt_1\ldots dt_n,\quad(x_1,\ldots,x_n)\in\mathbb{R}^n,$$

where $p>2$ is an odd number. Using the notation $f(x_1,\ldots,x_n)=\varphi^p(x_1,\ldots,x_n)$, this equation is reduced to a multivariate equation of the form (1) with concave nonlinearity with respect to the sought non-negative function $f(x_1,\ldots,x_n)$.

We also give an example of a one-dimensional convolutional integral equation with exponential nonlinearity arising in the mathematical theory of the geographical spread of an epidemic:

$$f(x)=a\int_{-\infty}^{\infty}K(x-t)(1-e^{-f(t)})dt,\quad x\in\mathbb{R},$$

where $a>1$ is a numerical parameter, the kernel $K(x)>0$, $x\in\mathbb{R}$, $\int_{-\infty}^{\infty}K(x)dx=1$ (see [6, p. 318] in the formulation of Theorem 1 ($f(x)=-\chi(x)$)).

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The author of this paper declares that he has no conflict of interests.

REFERENCES

1. *Lancaster P.* Theory of Matrices, New York; London: Academic Press, 1969.

2. *Vladimirov V.S. and Volovich Ya.I.* Nonlinear dynamics equation in p-adic string theory, Theor. Math. Phys., 2004, Vol. 138, No. 3, pp. 297–309.

3. *Khachatryan Kh.A.* On the solubility of certain classes of non-linear integral equations in p-adic string theory, Izv. Math., 2018, Vol. 82, No. 2, pp. 407–427.

4. *Aref'eva I.Ya.* Rolling tachyon on non-BPS branes and p-adic strings, Proc. Steklov Inst. Math., 2004, Vol. 245, pp. 40–47.

5. *Khachatryan A.Kh. and Khachatryan Kh.A.* Solvability of a class of nonlinear pseudo-differential equations in $\mathbb{R}^n$, p-Adic Numbers, Ultrametric Analysis and Applications, 2018, Vol. 10, No. 2, pp. 90–99.

6. *Atkinson C. and Reuter G.E.H.* Deterministic epidemic waves, Math. Proc. Cambridge Philos. Soc., 1976, Vol. 80, pp. 315–330.

7. *Diekmann O.* Threshold and travelling waves for the geographical spread of infection, J. Math. Biol., 1978, Vol. 6, No. 2, pp. 109–130.

8. *Petrosyan H.S., Terdzhyan Ts.E.*, and *Khachatryan Kh.A.* Uniqueness of the solution of one system of integral equations on the semi-axis with convex nonlinearity, Matematicheskie Trudy, 2020, Vol. 23, No. 2, pp. 187–203.

9. *Khachatryan Kh.A. and Petrosyan H.S.* Solvability of a certain system of singular integral equations with convex nonlinearity on the positive half-line, Russ. Math., 2021, Vol. 65, No. 1, pp. 27–46.

10. *Khachatryan Kh.A.* The solvability of a system of nonlinear integral equations of Hammerstein type on the whole line, Izvestiya of Saratov University. Mathematics. Mechanics. Informatics, 2019, Vol. 19, No. 2, pp. 164–181.

11. *Khachatryan Kh.A. and Petrosyan A.S.* Alternating bounded solutions of a class of nonlinear two-dimensional convolution-type integral equations, Trans. Moscow Math. Soc., 2021, Vol. 82, No. 2, pp. 259–271.

12. *Khachatryan Kh.A. and Petrosyan H.S.* On bounded solutions of a class of nonlinear integral equations in the plane and the Urysohn equation in a quadrant of the plane, Ukr. Math. J., 2021, Vol. 73, No. 5, pp. 811–829.

13. *Khachatryan Kh.A. and Petrosyan H.S.* On one class of multidimensional integral equations of convolution type with convex nonlinearity, *Differ. Equat.*, 2022, Vol. 58, No. 5, pp. 680–690.

14. *Arabadzhyan L.G.* Solutions of certain integral equations of the Hammerstein type, J. Contemp. Math. Anal., 1997, Vol. 32, No. 1, pp. 17–24.

15. *Zhukovskaya L.V.* Iterative method for solving nonlinear integral equations describing rolling solutions in string theory, Theor. Math. Phys., 2006, Vol. 146, No. 3, pp. 335–342.

16. *Khachatryan Kh.A.* Existence and uniqueness of solution of a certain boundary-value problem for a convolution integral equation with monotone non-linearity, Izv. Math., 2020, Vol. 84, No. 4, pp. 807–815.

17. *Khachatryan Kh.A.* Solvability of some nonlinear boundary value problems for singular integral equations of convolution type, Trans. Moscow Math. Soc., 2020, Vol. 81, No. 1, pp. 1–31.

18. *Kolmogorov A.N. and Fomin S.V.* Introductory Real Analysis, Englewood Cliffs: Prentice-Hall, 1970.