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# BLOW-UP OF THE SOLUTION AND GLOBAL SOLVABILITY OF THE CAUCHY PROBLEM FOR THE EQUATION OF VIBRATIONS OF A HOLLOW ROD

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**Abstract.** For a nonlinear partial differential equation of Sobolev type, generalizing the equation of oscillations of a hollow flexible rod, the Cauchy problem is studied in the space of continuous functions defined on the entire numerical axis and for which there are limits at infinity. The conditions for the existence of a global classical solution and the blow-up of the solution to the Cauchy problem on a finite time interval are considered.

**Keywords:** *equation of vibrations of a hollow flexible rod, nonlinear equation of Sobolev type, global solution, blow-up of the solution*

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## 1. INTRODUCTION. PROBLEM STATEMENT

The vibrations of a hollow flexible rod [1, Ch. 8, formula (8.230)] are modeled by a nonlinear differential equation of Sobolev type [2]

$$\delta u_{tt} - u_{ttxx} - \alpha_2 u_{txx} - \alpha_1 u_{tx} + \beta_2 u_{xxxx} + \beta_1 u_{xx} + \gamma u = u_{xx} f'(u_x), \tag{1}$$

where $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$, $\mathbb{R}_+ = (0,+\infty)$, $\mathbb{R} = (-\infty,+\infty)$; the dash in the equation denotes differentiation by $u_x = \partial_x u = \partial u/\partial x$; the coefficients $\alpha_i, \beta_i, i = 1,2, \gamma, \delta$ are non-negative constants; the nonlinearity $f$ is a twice continuously differentiable function $f(r)$, $r \in \mathbb{R}$, for which the modulus $|f(r)|$ at $r \geq 0$ is a non-decreasing function and the estimates are valid

$$\sup_{x\in\mathbb{R}} |f^{(i)}(g(x))| \leq \Big| f^{(i)}\Big(\sup_{x\in\mathbb{R}} |g(x)|\Big)\Big|, \quad i = 0,1, \quad g(x) \in C[\mathbb{R}],$$

$$|f(\xi r)| \leq \chi(\xi)|f(r)|, \quad \xi > 0, \quad r \geq 0, \tag{2}$$

$\chi$ – a continuous non-decreasing function (its simplest example is the power function, for other non-trivial examples see [3]).

We assume that the rod is infinite. This idealization is acceptable [4], if there are optimal damping devices at the rod boundaries, i.e., the parameters of the boundary clamping are such, that the perturbations falling on it are not reflected.

The Cauchy problem for equation (1) is investigated in the space $C[\mathbb{R}]$ [5, Ch. 8, § 1] of continuous functions $g = g(x)$, for which both limits exist at $x \to \pm\infty$ and the norm is $\|g\|_C = \sup_{x\in\mathbb{R}} |g(x)|$, with initial conditions

$$u|_{t=0} = \varphi(x), \quad u_t|_{t=0} = \psi(x), \quad x \in \mathbb{R}. \tag{3}$$

The sought classical solution $u = u(t,x)$, $(t,x) \in \overline{\mathbb{R}}_+ \times \mathbb{R}$, $\overline{\mathbb{R}}_+ = [0,+\infty)$, and its partial derivatives included in equation (1), for all values of the temporary variable $t$ on the variable $x$ belong to the space $C[\mathbb{R}]$. (By a *classical* *solution of the equation* we mean a sufficiently smooth function having all continuous derivatives of the desired order and satisfying the equation at every point in the domain of its setting.)

By $C^{(k)}[\mathbb{R}] = \{g(x) \in C[\mathbb{R}] : g'(x), \dots, g^{(k)}(x) \in C[\mathbb{R}]\}$, $k = 1, 2, \dots$, we denote subsets of differentiable functions in $C[\mathbb{R}]$.

Recall [5, Chap. 8, § 1; 6, § 2] that in the space $C[\mathbb{R}]$ the differential operator $\partial_x$ with domain of definition $D(\partial_x) = C^{(1)}[\mathbb{R}]$ generates a compressive strongly continuous group $U(\tau; \partial_x)g(x) = g(x+\tau)$, $\tau \in \mathbb{R}$, of left shifts, and the operator $\partial_x^2$ with domain of definition $D(\partial_x^2) = C^{(2)}[\mathbb{R}]$ is the derivative operator of the strongly continuous semigroup $U(t; \partial_x^2)g(x) = (2\sqrt{\pi t})^{-1} \int_0^{+\infty} e^{-\xi^2/(4t)} g(x+\xi)d\xi$, $t \in \overline{\mathbb{R}}_+$; and for the resolvents $(\lambda I - \partial_x)^{-1}$, $(\lambda I - \partial_x^2)^{-1}$ the estimates $\|(\lambda I - \partial_x)^{-1}\| \le 1/\lambda$ and $\|(\lambda I - \partial_x^2)^{-1}\| \le 1/\lambda$ are valid at $\lambda > 0$.

Let us investigate the Cauchy problem (1), (3) according to the following plan.

1. Let us make sure that the formulation of the Cauchy problem (1), (3) is correct and its classical solution exists locally in time. For this purpose, we find the solution of the Cauchy problem for the linear homogeneous equation corresponding to (1).

2. Let us introduce an auxiliary Cauchy problem

$$\delta v_{tt} - v_{ttxx} - \alpha_2 v_{txx} - \alpha_1 v_{tx} + \beta_2 v_{xxxx} + \beta_1 v_{xx} + \gamma v = \partial_x^2 f(v), \tag{4}$$

$$v|_{t=0} = \varphi'(x), \quad v_t|_{t=0} = \psi'(x), \quad x \in \mathbb{R}, \tag{5}$$

for which we find the time interval $[0, t_1]$ of existence and uniqueness of its classical solution and estimate the norm in $C[\mathbb{R}]$ of this local solution.

3. Let us establish the relation between the solutions of equations (1) and (4) by assuming that on the segment $[0, t_1$, the solution $u = u(t,x)$ at the variable $x$ belongs to the intersection of the subset $C^{(4)}[\mathbb{R}] \subset C[\mathbb{R}]$ with the Sobolev space $W_2^4(\mathbb{R})$, and the temporary partial derivatives $u_t = u_t(t,x)$ and $u_{tt} = u_{tt}(t,x)$ belong to the intersection $C^{(2)}[\mathbb{R}] \cap W_2^2(\mathbb{R})$.

4. Let us find sufficient conditions for the existence of a single classical global ($t \ge 0$) solution and destruction on a finite time interval of the solution of the Cauchy problem (1), (3).

## 2. CAUCHY PROBLEM FOR A LINEAR HOMOGENEOUS EQUATION

Consider the linear homogeneous equation corresponding to (1):

$$(\delta I - \partial_x^2)u_{tt} - (\alpha_2 \partial_x^2 + \alpha_1 \partial_x)u_t + (\beta_2 \partial_x^4 + \beta_1 \partial_x^2 + \gamma I)u = 0. \tag{6}$$

Let's introduce in (6) a new unknown function

$$v(t,x) = \delta u(t,x) - u_{xx}(t,x), \tag{7}$$

assuming that the partial derivatives of $u_{xx}, u_{txx}$ are continuous at $t \in \overline{\mathbb{R}}_+$. From substitution (7), provided that the initial functions $\varphi(x), \psi(x)$ belong to $C^{(2)}[\mathbb{R}]$, we can uniquely determine the initial values of the function $v = v(t,x)$:

$$v|_{t=0} = v_0(x) = \delta\varphi(x) - \varphi''(x), \quad v_t|_{t=0} = v_1(x) = \delta\psi(x) - \psi''(x),$$

and, using the membership of the positive semi-axis to the resolvent set of the differential operator $\partial_x^2$, express the solution $u(t,x)$ of equation (6) through the new unknown function $v(t,x)$:

$$u(t,x) = (\delta I - \partial_x^2)^{-1} v(t,x) = \frac{1}{2\sqrt{\delta}} \int_{-\infty}^{+\infty} e^{-|s|\sqrt{\delta}} v(t, x+s)ds. \tag{8}$$

As a result of substitution (7) we obtain the equivalent (6) integro-differential equation

$$v_{tt} + A_1 v_t + A_2 v = 0, \tag{9}$$

in which the operator coefficients are

$$A_1 = \alpha_2 I - (\alpha_2\sqrt{\delta} - \alpha_1)\sqrt{\delta}(\delta I - \partial_x^2)^{-1} - \alpha_1(\sqrt{\delta} I - \partial_x)^{-1}, \quad D(A_1) = C[\mathbb{R}],$$

$$A_2 = -\beta_2 \partial_x^2 - (\beta_2\delta + \beta_1)I + (\beta_2\delta^2 + \beta_1\delta + \gamma)(\delta I - \partial_x^2)^{-1}, \quad D(A_2) = C^{(2)}[\mathbb{R}].$$

The bounded operator $A_1$ generates a uniformly continuous group $U(\tau; A_1)$, $\tau \in \mathbb{R}$, represented by a degree series

$$U(\tau; A_1) = \sum_{n=0}^{+\infty} \frac{\tau^n}{n!} A_1^n,$$

uniformly converging on $\tau$ at each finite segment from $\mathbb{R}$, and by virtue of the permutation of operators $(\sqrt{\delta} I - \partial_x)^{-1}$ and $(\delta I - \partial_x^2)^{-1}$, the representation is true

$$U(\tau; A_1) = e^{\alpha_2 \tau} U\big(-\alpha_1 \tau; (\sqrt{\delta} I - \partial_x)^{-1}\big) U\big(-(\alpha_2\sqrt{\delta} - \alpha_1)\sqrt{\delta}\tau; (\delta I - \partial_x^2)^{-1}\big) =$$
$$= e^{\alpha_2\tau}\left(\sum_{n=0}^{+\infty} \frac{(-1)^n \alpha_1^n \tau^n}{n!} (\sqrt{\delta} I - \partial_x)^{-n}\right)\left(\sum_{m=0}^{+\infty} \frac{(-1)^m (\alpha_2\sqrt{\delta} - \alpha_1)^m \delta^{m/2} \tau^m}{m!} (\delta I - \partial_x^2)^{-m}\right),$$

as well as the evaluation

$$\|U(t; A_1)\| \le e^{(\alpha_2 + \alpha_1/\sqrt{\delta} + |\alpha_2 - \alpha_1/\sqrt{\delta}|)t}, \quad t \in \overline{R}_+.$$

In equation (9) we substitute the unknown function

$$w(t,x) = U(t/2; A_1) v(t,x), \tag{10}$$

then we can uniquely determine the initial values of the function $w(t,x)$:

$$\begin{aligned} w|_{t=0} &= w_0(x) = v_0(x), \\ w_t|_{t=0} &= w_1(x) = \frac{A_1 v_0(x)}{2} + v_1(x) = \\ &= \frac{\alpha_2 v_0(x)}{2} - \frac{\alpha_2\sqrt{\delta} - \alpha_1}{4} \int_{-\infty}^{+\infty} e^{-|s|\sqrt{\delta}} v_0(x+s)\,ds - \\ &\quad - \frac{\alpha_1}{2} \int_0^{+\infty} e^{-r\sqrt{\delta}} v_0(x+r)\,dr + v_1(x) \end{aligned}$$

and express the solution $v(t,x)$ of equation (9) through a new unknown function $w(t,x)$:

$$v(t,x) = U(-t/2; A_1) w(t,x). \tag{11}$$

As a result of substitution (10), we obtain the integro-differential equation equivalent to (9)

$$w_{tt} = \left(\frac{A_1^2}{4} - A_2\right) w, \tag{12}$$

in which the operator coefficient

$$\frac{A_1^2}{4} - A_2 = B = B_0 + B_1, \quad D(B) = C^{(2)}[\mathbb{R}],$$

where $B_0 = \beta_2 \partial_x^2$ and

$$B_1 = \left(\beta_2\delta + \beta_1 + \frac{\alpha_2^2}{4}\right) I - \left(b_2\delta^2 + \beta_1\delta + \gamma + \frac{\alpha_2(\alpha_2\sqrt{\delta} - \alpha_1)}{2}\sqrt{\delta}\right)(\delta I - \partial_x^2)^{-1} -$$
$$- \frac{\alpha_2\alpha_1}{2}(\sqrt{\delta} I - \partial_x)^{-1} + \frac{1}{4}\Big(\alpha_1(\sqrt{\delta} I - \partial_x)^{-1} + (\alpha_2\sqrt{\delta} - \alpha_1)\sqrt{\delta}(\delta I - \partial_x^2)^{-1}\Big)^2.$$

Equation (12) can be written as an abstract ordinary differential equation

$$W_{tt} = BW, \quad t \in \mathbb{R}_+, \tag{13}$$

where $W = W(t) : t \to w(t,x)$ is the sought vector-function defined for $t \in \overline{\mathbb{R}}_+$ with values in the space $C[\mathbb{R}]$.

For equation (13), we consider an abstract Cauchy problem with initial conditions

$$W|_{t=0} = W_0, \quad W'|_{t=0} = W_1, \tag{14}$$

where $W_0 = w_0(x)$, $W_1 = w_1(x)$ are elements of the space $C[\mathbb{R}]$.

The Cauchy problem (13), (14) is uniformly correct [6, § 1.4], only when the operator $B$ is the producing operator of a strongly continuous cosine operator-function $C(\tau; B)$, $\tau \in \mathbb{R}$.

In the space $C[\mathbb{R}]$, the operator $B_0$ is the derivative operator of the strongly continuous cosine operator-function $C(\tau; B_0)$, $\tau \in \mathbb{R}$ [6, § 1.5]:

$$C(\tau; B_0)g(x) = 2^{-1}[U(\tau\sqrt{\beta_2}; \partial_x) + U(-\tau\sqrt{\beta_2}; \partial_x)]g(x) = 2^{-1}[g(x + \tau\sqrt{\beta_2}) + g(x - \tau\sqrt{\beta_2})],$$

for which the estimate of the norm is fair

$$\|C(t; B_0)\| \le 1, \quad t \in \overline{R}_+.$$

The corresponding sine operator-function $S(\tau; B_0)$, $\tau \in \mathbb{R}$, has the form

$$S(\tau; B_0)g(x) = \int_0^\tau C(s; B_0)g(x)ds = \frac{1}{2\sqrt{\beta_2}} \int_{x-\tau\sqrt{\beta_2}}^{x+\tau\sqrt{\beta_2}} g(\xi)d\xi$$

and the norm estimation is valid for it

$$\|S(t; B_0)\| \le t, \quad t \in \overline{R}_+.$$

The bounded operator $B_1$ generates a strongly continuous cosine operator-function $C(\tau; B_1)$, for which the representation [6, §§ 1.4, 4.2] is valid on an arbitrary element $g(x) \in C[\mathbb{R}]$

$$C(\tau; B_1)g(x) = \sum_{n=0}^{+\infty} \frac{\tau^{2n}}{(2n)!} B_1^n g(x), \quad \tau \in \mathbb{R},$$

and the power series converges uniformly on $\tau$ on each finite segment from $\mathbb{R}$. Note that the operator-valued function $C(\tau; B_1)$ is continuous in the uniform operator topology, and the norm estimate is valid for it

$$\|C(t; B_1)\| \leqslant \sum_{n=0}^{+\infty} \frac{t^{2n}}{(2n)!} \|B_1\|^n \leqslant \mathrm{ch}(c_1 t), \quad t \in \overline{R}_+,$$

where $c_1^2 = 2\beta_2\delta + 2\beta_1 + \gamma/\delta + (\alpha_2\sqrt{\delta} + \alpha_1 + |\alpha_2\sqrt{\delta} - \alpha_1|)^2/(4\delta)$.

The operator $B$ is obtained by perturbing the unbounded operator $B_0$ by the bounded operator $B_1$, but the perturbation by the bounded operator preserves [6, § 8.2] the ability of the operator $B_0$ to generate the cosine operator-function, so $B = B_0 + B_1$ is the derivative operator of the strongly continuous cosine operator-function $C(\tau; B)$, $\tau \in \mathbb{R}$, and hence the abstract Cauchy problem (13), (14) is uniformly correct.

The solution of the Cauchy problem (13), (14) for any initial data $W_0 \in D(B)$ and $W_1 \in C_1[\mathbb{R}]$ is defined by the formula

$$W(t) = C(t; B)W_0 + S(t; B)W_1,$$

where $S(t; B)$ is the sine operator-function associated with $C(t; B)$:

$$S(t; B)g = \int_0^t C(\tau; B)g d\tau, \quad g \in C[\mathbb{R}],$$

$C_1[\mathbb{R}] = \{g \in C[\mathbb{R}] : C(t; B)g \in C^{(1)}(\mathbb{R}, C[\mathbb{R}])\}$ is a linear manifold. It is obvious that $D(B) = C^{(2)}[\mathbb{R}] \subset C_1[\mathbb{R}]$.

In order to derive an estimate of the norm of the solution of equation (13) that is the abstract function $W(t)$, we find estimates of the norms of the cosine and sine of the operator functions generated by the operator $B$, for which we obtain a representation of the operator-valued function $C(t; B)$ via $C(t; B_0)$ and $C(t; B_1)$.

Considering the derivative operator $B$ as the result of perturbing the derivative operator $B_0$ by the operator $B_1$, that in turn gives rise to the cosine operator-function, for $g(x) \in D(B_0) \cap D(B_1) = C^{(2)}[\mathbb{R}]$, we obtain [6, § 8.2] the representation of

$$C(t; B)g(x) = C(t; B_0)g(x) + \frac{t^2}{2} \int_0^1 j_1(t\sqrt{1 - s^2}, B_0)C(ts; B_1)g(x)ds,$$

where $j_1(t, B_0)g(x) = \frac{4}{\pi}\int_0^1 \sqrt{1-r^2}C(tr; B_0)g(x)dr$.

For $t \in \mathbb{R}_+$ we obtain estimates of the norms: $\|j_1(t, B_0)\| \le \frac{4}{\pi}\int_0^1 \sqrt{1-r^2}dr = 1$ and

$$\|C(t; B)\| \le 1 + \frac{t^2}{2}\int_0^1 \mathrm{ch}\,(c_1 ts)\,ds = 1 + \frac{t}{2c_1}\,\mathrm{sh}(c_1 t) = \sigma_1(t), \tag{15}$$

$$\|S(t; B)\| \le t + \frac{1}{2c_1}\int_0^t \tau\,\mathrm{sh}(c_1\tau)\,d\tau \le t\left(1 + \frac{\mathrm{ch}\,(c_1 t)}{2c_1^2}\right) = \sigma_2(t). \tag{16}$$

Using formulas (11) and (8) of inverse substitutions we have

$$u(t, x) = (\delta I - \partial_x^2)^{-1}v(t, x) = (\delta I - \partial_x^2)^{-1}U(-t/2; A_1)w(t, x). \tag{17}$$

Then, using the permutability of the resolvent $(\delta I - \partial_x^2)^{-1}$ and the semigroup $U(-t/2; A_1)$ both among themselves and with the cosine operator-function generated by the operator $B$, we find a solution of the Cauchy problem for equation (6):

$$u(t, x) = U(-t/2; A_1)\big[C(t; B)\varphi(x) + S(t; B)(A_1\varphi(x)/2 + \psi(x))\big]. \tag{18}$$

Thus, there is

**Theorem 1.** *Let the initial functions $\varphi(x)$ and $\psi(x)$ belong to the subset $C^{(4)}[\mathbb{R}]$ of the space $C[\mathbb{R}]$, then the Cauchy problem for the linear homogeneous equation* (6) *is uniformly correct, the classical solution is given by the formula* (18) *and the evaluation is valid for it*

$$\sup_{x\in\mathbb{R}} |u(t, x)| \le e^{-(\alpha_2 - \alpha_1/\sqrt{\delta} - |\alpha_2 - \alpha_1/\sqrt{\delta}|)t/2}\times$$

$$\times\left[\sigma_1(t)\sup_{x\in\mathbb{R}}|\varphi(x)| + \sigma_2(t)\left(\sup_{x\in\mathbb{R}}|\psi(x)| + \frac{\alpha_2\sqrt{\delta} + \alpha_1 + |\alpha_2\sqrt{\delta} - \alpha_1|}{2\sqrt{\delta}}\sup_{x\in\mathbb{R}}|\varphi(x)|\right)\right], \quad t \in \overline{R}_+.$$

**Remark 1.** The classical solution $W(t)$ of the abstract Cauchy problem (13), (14) belongs to $C^{(2)}(\overline{R}_+, C[\mathbb{R}])$ and for it $BW(t) \in C(\overline{R}_+, C[\mathbb{R}])$, hence $w(t, x) = U(t/2; A_1) \times (\delta I - \partial_x^2)u(t, x) \in C^{2,2}(\overline{\mathbb{R}}_+, \mathbb{R})$. By virtue of (17), the solution of the Cauchy problem (6), (3) is $u(t, x) \in C^{2,4}(\overline{R}_+, \mathbb{R})$.

## 3. LOCAL SOLUTION OF THE CAUCHY PROBLEM FOR THE NONLINEAR EQUATION (4)

Equation (4) is obtained from equation (1) through differentiating both parts by the variable $x$ and then substituting $u_x = v$ (the left parts of these equations coincide).

Let's act on both parts of equation (4) by the operator $(\delta I - \partial_x^2)^{-1}$ and obtain the equivalent equation

$$v_{tt} + A_1 v_t + A_2 v = f_1(v), \tag{19}$$

in which the nonlinearity $f_1(u) = [\delta(\delta I - \partial_x^2)^{-1} - I]f(u)$, and the operators $A_1$ and $A_2$ are the same as in equation (9).

Equation (19) is reduced to an abstract semi-linear equation by substituting $v(t, x) = U(-t/2; A_1)w(t, x)$

$$W_{tt} = BW + f_2(t, U(-t/2; A_1)W), \tag{20}$$

where the operator $B$ is the same as in (13) and the nonlinear operator $f_2$ is defined by the formula

$$f_2(t, \cdot) = U(t/2; A_1)[\delta(\delta I - \partial_x^2)^{-1} - I]f(\cdot),$$

here $f(\cdot)$ is the superposition operator: $f(g) = f(g(x))$, $g(x) \in C[\mathbb{R}]$.

Given $t \in \overline{\mathbb{R}}_+$, it is fair to estimate the norm of the operator $f_2(t,\cdot)$ in the space $C[\mathbb{R}]$:

$$\|F(t,g)\|_C \le 2e^{(\alpha_2+\alpha_1/\sqrt{\delta}+|\alpha_2-\alpha_1/\sqrt{\delta}|)t/2} f(\|g\|_C). \tag{21}$$

For equation (20) we consider an abstract Cauchy problem with initial conditions

$$W|_{t=0} = W_0', \quad W'|_{t=0} = W_1', \tag{22}$$

where $W_0' = (w_0(x))'$ and $W_1' = (w_1(x))'$ are elements of the space $C[\mathbb{R}]$.

From the continuous differentiability of the superposition operator in the space of continuous functions and boundedness of the operators $U(t/2; A_1)$ and $(\delta I - \partial_x^2)^{-1}$, the continuous Fréchet differentiability of the operator $f_2(t,\cdot)$ in the space $C[\mathbb{R}]$ follows and, consequently, there exists an interval $[0, t_0)$, within which the abstract Cauchy problem (20), (22) has [7, § 3] the only classical solution $W = W(t)$ (provided that the initial data $W_0'$, $W_1'$ belong to the domain of definition of the operator $B$) that satisfies the integral equation

$$W(t) = C(t;B)W_0' + S(t;B)W_1' + \int_0^t S(t-\tau;B) f_2(\tau, U(-\tau/2;A_1)W)d\tau. \tag{23}$$

From equation (23), using estimates (15), (16), (21), and (2), we derive the integral inequality

$$\|W(t)\|_C \le \sigma_1(t)\|W_0'\|_C + \sigma_2(t)\|W_1'\|_C +$$
$$+\,2\int_0^t \sigma_2(t-\tau)e^{(\alpha_2+\alpha_1/\sqrt{\delta}+|\alpha_2-\alpha_1/\sqrt{\delta}|)\tau/2}\chi\big(e^{-(\alpha_2-\alpha_1/\sqrt{\delta}-|\alpha_2-\alpha_1/\sqrt{\delta}|)\tau/2}\big) f(\|W(\tau)\|_C)\,d\tau, \tag{24}$$

where

$$\|W_0'\|_C = \|(w_0(x))'\|_C = \|(v_0(x))'\|_C = \sup_{x\in\mathbb{R}} |\delta\varphi'(x) - \varphi'''(x)|,$$
$$\|W_1'\|_C = \|(w_1(x))'\|_C = \|(v_1(x))'\|_C = \|(A_1 v_0(x)/2 + v_1(x))'\|_C \le$$
$$\le \frac{\alpha_2\sqrt{\delta}+\alpha_1+|\alpha_2\sqrt{\delta}-\alpha_1|}{2\sqrt{\delta}} \sup_{x\in\mathbb{R}}|\delta\varphi'(x)-\varphi'''(x)| + \sup_{x\in\mathbb{R}}|\delta\psi'(x)-\psi'''(x)|.$$

Denoting

$$\sigma_3(t) = \sigma_1(t)\|W_0'\|_C + \sigma_2(t)\|W_1'\|_C,$$
$$\sigma_4(\tau) = e^{(\alpha_2+\alpha_1/\sqrt{\delta}+|\alpha_2-\alpha_1/\sqrt{\delta}|)\tau/2}\chi(e^{-(\alpha_2-\alpha_1/\sqrt{\delta}-|\alpha_2-\alpha_1/\sqrt{\delta}|)\tau/2})$$

and using the inequality

$$\sigma_5(t) = t(1+\operatorname{ch}(c_1 t)/(2c_1^2) \geqslant (t-\tau)(1+\operatorname{ch}(c_1(t-\tau))/(2c_1^2) = \sigma_2(t-\tau), \quad t \geqslant \tau \geqslant 0,$$

let us write the integral inequality (24) in the form

$$\|W(t)\|_C \le \sigma_3(t) + 2\sigma_5(t)\int_0^t \sigma_4(\tau) f(\|W(\tau)\|_C)d\tau. \tag{25}$$

From inequality (25), we derive [3] an estimate of the norm in the space $C[\mathbb{R}]$ of the solution of equation (20) on the segment $[0, t_1]$:

$$\|W(t)\|_C \le \sigma_3(t)\Phi^{-1}(\Psi(t)) = \sigma_6(t),$$

where

$$\Psi(t) = \Phi(1) + 2\sigma_5(t)\int_0^t \sigma_4(\tau)\frac{\chi(\sigma_3(\tau))}{\sigma_3(\tau)}d\tau,$$

$\Phi(\xi) = \int_{\xi_0}^{\xi} |f(s)|^{-1} ds$ for $\xi_0, \xi > 0$; $\Phi^{-1}$ is the inverse function to $\Phi$, the segment $[0, t_1] \subset [0, t_0)$ is defined by those values $t$ for which the values of the function $\Psi(t)$ belong to the region of existence $\operatorname{Dom}(\Phi^{-1})$ of the inverse function $\Phi^{-1}$.

Thus, there is

**Theorem 2.** *Let the function $f$ satisfy the conditions* (2), *and the initial functions $\varphi(x)$, $\psi(x)$ of the Cauchy problem* (4), (5) *belong to the space $C[\mathbb{R}]$ together with their derivatives up to the fifth order inclusive, then on the segment $[0, t_1]$ there exists a single classical solution $u = u(t, x)$ of this problem in the space $C[\mathbb{R}]$, for which the estimation is valid*

$$\sup_{x\in\mathbb{R}} |v(t, x)| = \sup_{x\in\mathbb{R}} |u_x(t, x)| \le e^{-(\alpha_2-\alpha_1/\sqrt{\delta}-|\alpha_2-\alpha_1/\sqrt{\delta}|)t/2}\sigma_6(t) = \sigma_7(t), \quad t \in [0, t_1].$$

## 4. RELATIONSHIP BETWEEN SOLUTIONS OF EQUATIONS (1) AND (4)

Further, we will assume that the solution of equation (1) belongs to the intersection of the space $C[\mathbb{R}]$ with the space $L_2(\mathbb{R})$ of functions with integrable square.

Recall that the scalar product and norm in $L_2(\mathbb{R})$ are defined by the formulas $(\varphi, \psi) = \int_{-\infty}^{+\infty} \varphi(x)\psi(x)dx$ and $\|\varphi\|_2 = \left(\int_{-\infty}^{+\infty} |\varphi(x)|^2 dx\right)^{1/2}$, respectively, and that for functions $g(x)$ belonging to the intersection of the space of continuous bounded functions $C(\mathbb{R})$ with the Sobolev space $W_2^1(\mathbb{R})$, the following estimate is valid

$$\|g\|_C \le \|g\|_{W_2^1} = \left(\int_{-\infty}^{+\infty} [(g(x))^2 + (g'(x))^2]dx\right)^{1/2}, \tag{26}$$

and if $g(x) \in C^{(2)}(\mathbb{R})$, then [8] the limits of the functions $g(x), g'(x)$ at $x \to \pm\infty$ are zero.

**Lemma.** *From the existence of a local classical solution $v = v(t, x)$, $t \in [0, t_1]$, of equation* (4) *follows the existence of a corresponding solution of*

$$u = u(t, x) = \lim_{x_0\to-\infty} \int_{x_0}^{x} v(t, s)ds = \int_{-\infty}^{x} v(t, s)ds \tag{27}$$

*of equation* (1) *on the same time interval $[0, t_1]$ if the conditions are fulfilled*

$$u(t, x) \in C^{(4)}[\mathbb{R}] \cap W_2^4(\mathbb{R}), \quad u_t(t, x), u_{tt}(t, x) \in C^{(2)}[\mathbb{R}] \cap W_2^2(\mathbb{R}), \quad t \in [0, t_1]. \tag{28}$$

**Proof.** First of all, we note that from conditions (28), the limit equalities follow

$$\lim_{x\to\pm\infty} \partial_x^k u(t, x) = 0, \quad k = \overline{0, 4};$$
$$\lim_{x\to\pm\infty} \partial_t^n \partial_x^m u(t, x) = 0, \quad n = 1, 2, \quad m = \overline{0, 2}; \quad t \in [0, t_1]. \tag{29}$$

Let $v = v(t, x)$ be the classical solution of equation (4) on the time segment $[0, t_1]$. Then, using relations (29), we obtain the equations

$$\int_{-\infty}^{x} \partial_t^i \partial_s^j v(t, s)ds = \int_{-\infty}^{x} (\partial_t^i \partial_s^j u(t, s))_s ds = \partial_t^i \partial_x^j u(t, x) - \lim_{s\to-\infty} \partial_t^i \partial_s^j u(t, s) = \partial_t^i \partial_x^j u(t, x).$$

Further, by virtue of continuity of the function $f'$, we have

$$\int_{-\infty}^{x} \partial_s^2 f(v(t, s))ds = (f(u_x(t, x)))_x - f'\left(\lim_{x_0\to-\infty} u_x(t, x_0)\right) \lim_{x_0\to-\infty} u_{xx}(t, x_0) = u_{xx}(t, x)f'(u_x(t, x)).$$

Now, using the obtained representations and substituting function (27) into equation (1), we obtain the identity equality on the segment $[0, t_1]$, whence it follows that function (27) is a solution of equation (1). The lemma is proved.

**Remark 2.** From the conditions (28) for the solution of the Cauchy problem (1), (3) $u = u(t, x)$, the conditions that the initial functions must satisfy are required to follow:

$$\varphi(x) \in C^{(4)}[\mathbb{R}] \cap W_2^4(\mathbb{R}), \quad \psi(x) \in C^{(2)}[\mathbb{R}] \cap W_2^2(\mathbb{R}). \tag{30}$$

## 5. EXISTENCE OF A GLOBAL SOLUTION OF THE CAUCHY PROBLEM FOR EQ. (1)

Consider the so-called energy integral for equation (1):

$$y(t) = \delta(u, u) + (u_x, u_x) = \int_{-\infty}^{+\infty} (\delta u^2 + u_x^2)dx, \quad t \in [0, t_1]. \tag{31}$$

Applying the Cauchy-Bunyakovsky inequality $|(\varphi, \psi)| \leq \|\varphi\|_2\|\psi\|_2$ to the derivative of the energy integral $y'(t) = 2(\delta(u_t, u) + (u_{tx}, u_x))$, we derive an auxiliary estimate on the segment $t \in [0, t_1]$:

$$y'(t) \leq y(t) + z(t), \tag{32}$$

where

$$z(t) = \delta(u_t, u_t) + (u_{tx}, u_{tx}) = \int_{-\infty}^{+\infty} (\delta u_t^2 + u_{tx}^2)dx, \quad t \in [0, t_1], \tag{33}$$

is the second integral of energy for equation (1).

**Theorem 3.** *Let the conditions of lemma and theorem* 2 *be satisfied and let the parameters* $\alpha_i, \beta_i, i = 1, 2, \gamma, \delta$ *of equation* (1), *the nonlinearity* $f$ *and the initial functions* $\varphi(x), \psi(x)$ *satisfy conditions* (30) *and*

$$E_0 = \delta\|\psi\|_2^2 + \|\psi'\|_2^2 + \beta_2\|\varphi''\|_2^2 + \gamma\|\varphi\|_2^2 + 2\int_{-\infty}^{+\infty} F(\varphi'(x))dx - \beta_1\|\varphi'\|_2^2 \geq 0;$$

$$F(\eta) = \int_0^\eta f(s)ds \geq 0, \quad \eta \in \mathbb{R}; \quad F(\varphi'(x)) \in L(\mathbb{R}).$$

*Then, there exists a single global solution of the Cauchy problem* (1), (3) *and for it the estimation is valid*

$$\sup_{x\in\mathbb{R}} |u(t, x)| \leq \begin{cases} \sqrt{c_2/\delta}e^{(1+\beta_1)t/2}, & 0 < \delta < 1, \\ \sqrt{c_2}e^{(1+\beta_1)t/2}, & \delta \geq 1, \end{cases} \quad t \geq 0,$$

*where*

$$c_2 = \big(E_0 + (1 + \beta_1)(\delta\|\varphi\|_2^2 + \|\varphi'\|_2^2)\big)/(1 + \beta_1).$$

**Proof.** Multiply both parts of equation (1) by the partial time derivative $u_t = u_t(t, x)$ and integrate from $-\infty$ to $+\infty$. Then, integrating by parts and taking into account, by virtue of (29), the equality to zero outside the integral summands, we obtain

$$\begin{aligned} &\frac{\delta}{2}\frac{d}{dt}\|u_t\|_2^2 + (u_{ttx}, u_{tx}) + \alpha_2(u_{tx}, u_{tx}) - \frac{\alpha_1}{2}\int_{-\infty}^{+\infty} (u_t^2)_x dx + \\ &+ \beta_2(u_{xx}, u_{txx}) - \beta_1(u_x, u_{tx}) + \frac{\gamma}{2}\frac{d}{dt}\|u\|_2^2 + (f(u_x), u_{tx}) = 0. \end{aligned} \tag{34}$$

Let us introduce the potential $F(\eta) = \int_0^\eta f(s)ds$, generated by the nonlinearity $f$ of equation (1), and, taking into account that $\int_{-\infty}^{+\infty}(u_t^2)_x dx = u_t^2|_{-\infty}^{+\infty} = 0$, we rewrite the equality (34) as

$$\frac{1}{2}\frac{d}{dt}E(t) = 0, \tag{35}$$

where

$$\begin{aligned} E(t) = \delta\|u_t\|_2^2 + \|u_{tx}\|_2^2 + \beta_2\|u_{xx}\|_2^2 - \beta_1\|u_x\|_2^2 + \gamma\|u\|_2^2 + \\ + 2\int_{-\infty}^{+\infty} F(u_x)dx + 2\alpha_2\int_0^t \|u_{sx}\|_2^2 d\tau \end{aligned}$$

is the energy functional of equation (1).

From relation (35) it follows that the energy functional $E(t)$ does not depend on time, then, integrating both parts of (35), we obtain the conservation law

$$E(t) = E(0) \equiv E_0, \tag{36}$$

where

$$E_0 = \delta\|\psi\|_2^2 + \|\psi'\|_2^2 + \beta_2\|\varphi''\|_2^2 - \beta_1\|\varphi'\|_2^2 + \gamma\|\varphi\|_2^2 + 2\int_{-\infty}^{+\infty} F(\varphi'(x))dx$$

is the initial energy.

Let us require that the initial energy is non-negative: $E_0 \geq 0$, i.e., the inequality

$$\delta\|\psi\|_2^2 + \|\psi'\|_2^2 + \beta_2\|\varphi''\|_2^2 + \gamma\|\varphi\|_2^2 + 2\int_{-\infty}^{+\infty} F(\varphi'(x))dx \geq \beta_1\|\varphi'\|_2^2,$$

where the function $F(\varphi'(x))$ belongs to the space $L(\mathbb{R})$ of functions absolutely integrable on $\mathbb{R}$.

From the conservation law (36) we deduce

$$\begin{gathered}\delta\|u_t\|_2^2 + \|u_{tx}\|_2^2 + \beta_2\|u_{xx}\|_2^2 + \gamma\|u\|_2^2 + \\ + 2\int_{-\infty}^{+\infty} F(u_x)dx + 2\alpha_2\int_0^t \|u_{sx}\|_2^2 ds = E_0 + \beta_1\|u_x\|_2^2.\end{gathered} \tag{37}$$

Suppose that

$$F(\eta) \geq 0, \quad \eta \in \mathbb{R}, \tag{38}$$

then from equality (37), reducing the left part, we obtain

$$z(t) \leq E_0 + \beta_1(\delta\|u\|_2^2 + \|u_x\|_2^2) = E_0 + \beta_1 y(t), \quad t \in [0, t_1]. \tag{39}$$

From inequalities (32) and (39), the integral inequality follows

$$y(t) \leq E_0 t + y(0) + (1+\beta_1)\int_0^t y(s)ds, \quad t \in [0, t_1]. \tag{40}$$

Applying to (40) Gronwall's lemma [9, § 1, formula (1.10)], we obtain an estimate of the first energy integral

$$y(t) \leq \left(\frac{E_0}{1+\beta_1} + y(0)\right) e^{(1+\beta_1)t} = \sigma_8(t), \tag{41}$$

true on the entire positive semi-axis of $t \in \overline{\mathbb{R}}_+$, and hence the classical solution of $u = u(t,x)$ at $t \in \overline{\mathbb{R}}_+$ belongs to the Sobolev space $W_2^1(\mathbb{R})$:

$$\|u\|_{W_2^1}^2 = \|u\|_2^2 + \|u_x\|_2^2 \leq \begin{cases} \left(1 + \frac{1-\delta}{\delta}\right) y(t) \leq \frac{1}{\delta}\sigma_8(t), & 0 < \delta < 1, \\ \delta\|u\|_2^2 + \|u_x\|_2^2 = y(t) \leq \sigma_8(t), & \delta \geq 1. \end{cases}$$

Now, using inequalities (26) and (41), we obtain an estimate of the solution $u = u(t,x)$, $t \in \overline{\mathbb{R}}_+$ of the Cauchy problem (1), (3) in the space $C[\mathbb{R}]$:

$$\|u\|_C = \sup_{x\in\mathbb{R}} |u(t,x)| \leq \|u\|_{W_2^1} \leq \begin{cases} \sqrt{\frac{\sigma_8(t)}{\delta}}, & 0 < \delta < 1, \\ \sqrt{\sigma_8(t)}, & \delta \geq 1, \end{cases}$$

ensuring the existence of a global solution. The theorem is proved.

## 6. DECOMPOSITION OF THE SOLUTION OF THE CAUCHY PROBLEM FOR EQ. (1)

Let us find sufficient conditions for the occurrence of a gap of the second kind for the energy integral (31) on the segment $[0, t_2] \subseteq [0, t_1]$, that we choose so that the inequality $y(t) > 0$, following from the initial condition $y(0) = \delta\|\varphi\|_2^2 + \|\varphi'\|_2^2 > 0$, holds.

Applying the Cauchy-Bunyakovsky inequality to the square of the derivative of the energy integral $y(t)$ on the segment $t \in [0, t_2]$, we have

$$[y'(t)]^2 \leq 4y(t)z(t).$$

Let us derive an estimate of the square of the norm of the partial derivative $u_{tt}$, using the representation of equation (1) in an equivalent form

$$u_{tt} = -A_1u_t - A_2u + (\delta I - \partial_x^2)^{-1}u_{xx}f'(u_x),$$

obtained by acting on both parts of equation (1) by a linear bounded operator $(\delta I - \partial_x^2)^{-1}$. For this purpose, we obtain auxiliary estimates

$$\|u_{xx}f'(u_x)\|_2^2 \leq \sup_{x\in}(f'(u_x))^2 \int\limits_{-\infty}^{+\infty} u_{xx}^2\,dx \leq \Big(f'\Big(\sup_{x\in}|u_x|\Big)\Big)^2\|u_{xx}\|_2^2 \leq \sigma_9(t)\|u_{xx}\|_2^2,$$

where $\sigma_9(t) = (f'(\sigma_7(t)))^2$ — is a continuous function on the segment $[0, t_1]$;

$$\|A_1u_t\|_2^2 \leq \big\|\alpha_2u_t - (\alpha_2\sqrt{\delta} - \alpha_1)\sqrt{\delta}(\delta I - \partial_x^2)^{-1}u_t - \alpha_1(\sqrt{\delta}I - \partial_x)^{-1}u_t\big\|_2^2 \leq$$

$$\leq \left(\alpha_2\|u_t\|_2 + \left|\alpha_2 - \frac{\alpha_1}{\sqrt{\delta}}\right|\|u_t\|_2 + \frac{\alpha_1}{\sqrt{\delta}}\|u_t\|_2\right)^2 \leq c_3\|u_t\|_2^2 \leq c_3z(t),$$

where $c_3 = (\alpha_2 + \alpha_1/\sqrt{\delta} + |\alpha_2 - c\alpha_1/\sqrt{\delta}|)^2$;

$$\|A_2u\|_2^2 \leq \big\|-\beta_2\partial_x^2u - (\beta_2\delta + \beta_1)u + (\beta_2\delta^2 + \beta_1\delta + \gamma)(\delta I - \partial_x^2)^{-1}u\big\|_2^2 \leq$$

$$\leq \big(\beta_2\|u_{xx}\|_2 + (\beta_2\delta + \beta_1)\|u\|_2 + \big(\beta_2\delta + \beta_1 + \gamma/\delta\big)\|u\|_2\big)^2 \leq$$

$$\leq 2\big(\beta_2^2\|u_{xx}\|_2^2 + \big(2(\beta_2\delta + \beta_1) + \gamma/\delta\big)^2\|u\|_2^2\big) \leq 2\beta_2^2\|u_{xx}\|_2^2 + c_4y(t),$$

where $c_4 = 2(2(\beta_2\delta + \beta_1) + \gamma/\delta)^2$.

Taking them into account, we have

$$\|u_{tt}\|_2^2 \leq \big(\|A_1u_t\|_2 + \|A_2u\|_2 + \|(\delta I - \partial_x^2)^{-1}u_{xx}f'(u_x)\|_2\big)^2 \leq$$

$$\leq 3\left(\|A_1u_t\|_2^2 + \|A_2u\|_2^2 + \frac{1}{\delta^2}\|u_{xx}f'(u_x)\|_2^2\right) \leq 3\left(c_3z(t) + 2\beta_2^2\|u_{xx}\|_2^2 + c_4y(t) + \frac{\sigma_9(t)}{\delta^2}\|u_{xx}\|_2^2\right),$$

whence follows the inequality

$$\|u_{tt}\|_2^2 \leq 3c_3z(t) + 3c_4y(t) + c_5\|u_{xx}\|_2^2, \quad t \in [0, t_2], \tag{42}$$

where $c_5 = 6\beta_2^2 + 3c_6\delta^2$, $c_6 = \max_{t\in[0,t_1]}\sigma_9(t)$.

Let's return to the conservation law (37) and obtain the relation from it

$$z(t) + \beta_2\|u_{xx}\|_2^2 + \gamma\|u\|_2^2 + 2\alpha_2\int_0^t \|u_{sx}\|_2^2ds \leq E_0 + \beta_1\|u_x\|_2^2 + 2\left|\int_{-\infty}^{+\infty} F(u_x)dx\right|. \tag{43}$$

Earlier, when proving the existence of a global solution, we assumed the fulfillment of condition (38) — non-negativity of the potential $F(\eta)$ on the whole numerical axis $\eta \in \mathbb{R}$. Now, when considering the destruction of the solution, we require to fulfill the inequality for the nonlinearity $f$

$$\left|\int_{-\infty}^{+\infty} dx\int_0^{w(x)} f(s)ds\right| \leq \left|\int_{-\infty}^{+\infty} w(x)f(w(x))dx\right|, \tag{44}$$

where $w(x)$ is an arbitrary function from $C[\mathbb{R}]$, for which the functions $F(w(x))$ and $w(x)f(w(x))$ belong to the space $L_1(\mathbb{R})$.

Using inequality (44), we evaluate the integral in the right-hand side of (43). Integrating by parts, applying the limit equality (29) and the Cauchy-Bunyakovsky inequality, we have

$$2\left|\int\limits_{-\infty}^{+\infty} F(u_x)\,dx\right| \le 2\left|\int\limits_{-\infty}^{+\infty} f(u_x)\,du(x)\right| = \left|u(x)f(u_x)|_{-\infty}^{+\infty} - \int\limits_{-\infty}^{+\infty} uf'(u_x)u_{xx}\,dx\right| \le$$

$$\le 2|(uf'(u_x), u_{xx})| \le 2\|uf'(u_x)\|_2\|u_{xx}\|_2 \le \|uf'(u_x)\|_2^2 + \|u_{xx}\|_2^2 \le$$

$$\le \sup_{x\in\mathbb{R}}(f'(u_x))^2 \int\limits_{-\infty}^{+\infty} u_2^2\,dx + \|u_{xx}\|_2^2 \le (f'(\sigma_7(t)))^2\|u\|_2^2 + \|u_{xx}\|_2^2 = \sigma_9(t)\|u\|_2^2 + \|u_{xx}\|_2^2,$$

whence follows the inequality

$$2\left|\int_{-\infty}^{+\infty} F(u_x)dx\right| \le c_6\|u\|_2^2 + \|u_{xx}\|_2^2, \quad t \in [0, t_2]. \tag{45}$$

Applying the estimation (45) to the relation (43) under the condition

$$\beta_2 > 1, \tag{46}$$

we obtain the inequality

$$\|u_{xx}\|_2^2 \le \frac{E_0}{\beta_2 - 1} + \frac{\beta_1 + c_6}{\beta_2 - 1}y(t) - \frac{1}{\beta_2 - 1}z(t), \quad t \in [0, t_2],$$

using which we increase the right part of the estimate (42):

$$\|u_{tt}\|_2^2 \le \frac{c_5}{\beta_2 - 1}E_0 + \left(3c_4 + c_5\frac{(\beta_1 + c_6)}{\beta_2 - 1}\right)y(t) + \left(3c_3 - \frac{c_5}{\beta_2 - 1}\right)z(t), \quad t \in [0, t_2].$$

Let us calculate the second order derivative of the functional (31) and express its value through the second integral of energy (33):

$$y''(t) + 2(u_{tt}, u_{xx}) - 2\delta(u_{tt}, u) = 2z(t).$$

Using the estimates

$$2(u_{tt}, u_{xx}) \le 2|(u_{tt}, u_{xx})| \le \|u_{tt}\|_2^2 + \|u_{xx}\|_2^2 \le 3c_3z(t) + 3c_4y(t) + (c_5 + 1)\|u_{xx}\|_2^2 \le$$

$$\le \frac{c_5 + 1}{\beta_2 - 1}E_0 + \left(3c_4 + (c_5 + 1)\frac{\beta_1 + c_6}{\beta_2 - 1}\right)y(t) + \left(3c_3 - \frac{c_5 + 1}{\beta_2 - 1}\right)z(t),$$

$$-2\delta(u_{tt}, u) \le 2\delta|(u_{tt}, u)| \le \delta\|u_{tt}\|_2^2 + \delta\|u\|_2^2 \le 3\delta c_3z(t) + \delta(3c_4 + 1)y(t) + \delta c_5\|u_{xx}\|_2^2 \le$$

$$\le \frac{\delta c_5}{\beta_2 - 1}E_0 + \delta\left(3c_4 + 1 + c_5\frac{\beta_1 + c_6}{\beta_2 - 1}\right)y(t) + \delta\left(3c_3 - \frac{c_5}{\beta_2 - 1}\right)z(t),$$

increase the left side of it:

$$y''(t) + c_7 + c_8y(t) \ge c_9z(t), \quad t \in [0, t_2], \tag{47}$$

where

$$c_7 = \frac{(\delta + 1)c_5 + 1}{\beta_2 - 1}E_0, \quad c_8 = 3(\delta + 1)c_4 + \delta + ((\delta + 1)c_5 + 1)\frac{(\beta_1 + c_6)}{\beta_2 - 1}, \quad c_9 = 2 + \frac{(\delta + 1)c_5 + 1}{\beta_2 - 1} - 3(\delta + 1)c_3.$$

Let us now reduce the right-hand side of inequality (47):

$$y(t)y''(t) - \frac{c_9}{4}(y'(t))^2 + c_7y(t) + c_8y^2(t) \ge 0, \quad t \in [0, t_2]. \tag{48}$$

We require that the coefficient at the square of the derivative in inequality (48) be greater than one, i.e., we require the inequality $c_9/4 > 1$ or (in the detailed notation)

$$6(\delta+1)\beta_2^2 - (2+3(\delta+1)c_3)\beta_2 + 3(\delta+1)(c_6/\delta^2 + c_3) + 3 > 0. \tag{49}$$

Two cases arise here: if the discriminant of the quadratic trinomial

$$D_1 = D_1(\delta, c_3, c_6) = (2+3(\delta+1)c_3)^2 - 72(\delta+1)((\delta+1)(c_6/\delta^2 + c_3) + 1) < 0, \tag{50}$$

then inequality (49) is valid for all values of $\beta_2 > 1$; If $D_1 \geq 0$, then inequality (49) holds at

$$1 < \beta_2 < \frac{2+3(\delta+1)c_3 - \sqrt{D_1(\delta, c_3, c_6)}}{12(\delta+1)} \quad \text{or} \quad \beta_2 > \frac{2+3(\delta+1)c_3 + \sqrt{D_1(\delta, c_3, c_6)}}{12(\delta+1)}. \tag{51}$$

From condition (50), follows the inequality

$$9c_3^2 - 12\left(6 - \frac{1}{(\delta+1)^2}\right)c_3 - 72\frac{c_6}{\delta^2} - \frac{72(\delta+1)-4}{(\delta+1)^2} < 0, \tag{52}$$

and the discriminant of the quadratic trinomial

$$D_2 = D_2(\delta, c_6) = 36\left(6 - (\delta+1)^{-2}\right)^2 + 648(\delta^{-2}c_6 + (\delta + 17/18)(\delta+1)^{-2}) \geq 0,$$

therefore inequality (52), and hence (50), is satisfied at

$$0 < c_3 = \left(\alpha_2 + \frac{\alpha_1}{\sqrt{\delta}} + \left|\alpha_2 - \frac{\alpha_1}{\sqrt{\delta}}\right|\right)^2 < \frac{6(6-(\delta+1)^{-2}) + \sqrt{D_2(\delta, c_6)}}{9}, \tag{53}$$

i.e., if condition (53) is satisfied, the inequality $c_9/4 > 1$ is valid for any value of the parameter $\beta_2 > 1$.

In the case of $D_1 \geq 0$, inequality (49) is satisfied for parameter values satisfying conditions (51), in which the values $\delta$, $c_3$ and $c_6$ are related by the relation

$$(\alpha_2 + \alpha_1/\sqrt{\delta} + |\alpha_2 - \alpha_1/\sqrt{\delta}|)^2 \geq \frac{6(6-(\delta+1)^{-2}) + \sqrt{D_2(\delta, c_6)}}{9}.$$

Comparing inequality (48) with one of the basic ordinary differential inequalities for the energy integral [10, Appendix A, § 5], we conclude that if the initial conditions are fulfilled

$$(\delta(\varphi, \psi) + (\varphi', \psi'))^2 > \left(\frac{c_8}{c_9-4}(\delta\|\varphi\|_2^2 + \|\varphi'\|_2^2) + \frac{c_7}{c_9-2}\right)(\delta\|\varphi\|_2^2 + \|\varphi'\|_2^2), \tag{54}$$

then the time $t_2$ of existence of the solution of the Cauchy problem (1), (3) cannot be arbitrarily large, namely, there is an estimate from above

$$t_2 \leq T_\infty \leq \frac{1}{c_{10}(\delta\|\varphi\|_2^2 + \|\varphi'\|_2^2)^{(c_9-4)/4}}, \tag{55}$$

where

$$c_{10}^2 = \frac{(c_9-4)^2}{4(\delta\|\varphi\|_2^2 + \|\varphi'\|_2^2)^{c_9/2}}\left((\delta(\varphi, \psi) + (\varphi', \psi'))^2 - \left(\frac{c_8(\delta\|\varphi\|_2^2 + \|\varphi'\|_2^2)}{c_9-4} + \frac{c_7}{c_9-2}\right)(\delta\|\varphi\|_2^2 + \|\varphi'\|_2^2)\right) > 0,$$

and for the functionality of $y(t)$, it is fair to estimate from below

$$y(t) = \int_{-\infty}^{+\infty}(\delta u^2 + u_x^2)dx \geq \frac{1}{((\delta\|\varphi\|_2^2 + \|\varphi'\|_2^2)^{(c_9-4)/4} - c_{10}t)^{4/(c_9-4)}}, \tag{56}$$

and, hence, there is no time-global solution of the Cauchy problem (1), (3).

Thus, the following theorem is proven

**Theorem 4.** *Let the conditions of lemma and theorem* 2 *be satisfied and let the parameters* $\alpha_i, \beta_i, i = 1, 2, \gamma, \delta$ *of equation* (1), *the nonlinearity* $f$ *and the initial functions* $\varphi(x), \psi(x)$ *satisfy conditions* (30), (44), (46), (49), (54), *respectively, then the time* $t_2$ *of existence of the solution* $u(t, x)$ *of the Cauchy problem* (1), (3) *cannot be arbitrarily large, namely it is bounded from above and the estimation* (55) *takes place, and for the energy integral* $y(t)$ *the estimation from below* (56) *is valid.*

## CONFLICT OF INTERESTS

The author of this paper declares that he has no conflict of interests.

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