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PARTIAL DIFFERENTIAL EQUATIONS

DIRICHLET PROBLEM FOR A TWO-DIMENSIONAL WAVE EQUATION IN A CYLINDRICAL DOMAIN

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Abstract. In this work, the first boundary value problem is studied for a two-dimensional wave equation in a cylindrical domain. A uniqueness criterion has been established. The solution is constructed as the sum of an orthogonal series. When justifying the convergence of a series, the problem of small denominators from two natural arguments arose for the first time. An estimate for separation from zero with the corresponding asymptotics was established, which made it possible to prove the convergence of the series in the class of regular solutions and the stability of the solution.

Keywords: *wave equation, Dirichlet problem, uniqueness criterion, existence, stability, series, small denominators*

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1. INTRODUCTION. PROBLEM STATEMENT

Consider the wave equation

$$Lu \equiv u_{tt} - a^2(u_{xx} + u_{yy}) - bu = 0 \tag{1}$$

in the cylinder $Q = \{(x, y, t) : (x, y) \in D, 0 < t < T\}$, where $D = \{(x, y) : x^2 + y^2 < l^2\}$; $a > 0$, b, $T > 0$ and $l > 0$ are given real constants, and we set the first boundary value problem.

It is required to find the function $u(x, y, t)$, satisfying the following conditions:

$$u(x, y, t) \in C^1(\overline{Q}) \cap C^2(Q); \tag{2}$$

$$Lu(x, y, t) \equiv 0, \quad (x, y, t) \in Q; \tag{3}$$

$$u(x, y, t)|_{x^2+y^2=l^2} = 0, \quad 0 \le t \le T; \tag{4}$$

$$u(x, y, 0) = \tau(x, y), \quad u(x, y, T) = \psi(x, y), \quad (x, y) \in \overline{D}, \tag{5}$$

where $\tau(x, y)$ and $\psi(x, y)$ are given sufficiently smooth functions satisfying the matching conditions with the boundary condition (4).

It is known that the Dirichlet problem for hyperbolic type equations is incorrectly posed. S L. Sobolev showed [1], that the study of unstable oscillations (resonances of oscillations in the liquid inside thin-walled rocket tanks with natural oscillations) is closely related to the Dirichlet problem for the wave equation. In a better known form, this connection is shown in the book by V. I. Arnold [2, p. 132]. A rather complete review of the works devoted to the study of the Dirichlet problem for hyperbolic equations is given in the monograph by B. I. Ptashnik [3, pp. 89–95] and in the works [4; 5, pp. 112–118] by the author.

The works of R. Denchev [6–8] are devoted to the study of the Dirichlet problem for equation (1) at $b = 0$, $a = 1$ with a non-zero right part and homogeneous conditions on the boundary of the region Ω, when Ω is an ellipsoid, a cylinder with formations parallel to the axis t, and a parallelepiped. They also establish the criterion of singularity and existence of the solution of the problem in the Sobolev space $W_2^1(\Omega)$ under certain conditions on the right part related to the convergence of numerical series, while the arising small denominators are not studied.

In [9], for a multidimensional equation with a wave operator in the cylindrical domain $D\times(0,T)$, the conditions $\sqrt{\lambda_k}T \neq m\pi$, where $k, m \in \mathbb{N}$, under which the uniqueness theorem of the solution of the Dirichlet problem takes place, were found. Here, λ_k are the eigenvalues of the corresponding spectral problem in the domain D.

In the monograph by B. I. Ptashnik [3, pp. 95–101], the Dirichlet problem in $(p+1)$-dimensional parallelepiped $Q = [0,T] \times \Pi$, where $\Pi = \{x \in \mathbb{R}^p : 0 \leq x_r \leq \pi, r = \overline{1,p}\}$, for a strictly hyperbolic equation of even order $2n$ with constant coefficients is also studied. The solution of the problem is determined by p-dimensional Fourier series. A criterion for the uniqueness of the solution in $C^{2n}(Q)$ is established. For a series of inequalities expressing the evaluation of small denominators with the corresponding asymptotics, the justification of convergence of the series in the specified class is given. It is not shown for what numbers of the form π/T these estimates take place, only it is noted that the set of numbers π/T, for which they are not fulfilled, is the set of zero Lebesgue measure.

In the paper by V. P. Bursky [10], a necessary and sufficient condition for the trivial solvability of the homogeneous Dirichlet problem in a unit ball B centered at the origin of coordinates in space $C^2(\overline{B})$ for an equation with complex is obtained:

$$u_{xx} + u_{yy} - a^2 u_{zz} = 0.$$

In the works of S. A. Aldashev [11–14], the Dirichlet problem and the problem with mixed boundary conditions in the cylindrical domain Q (where $l = 1, T = \alpha$) for multidimensional hyperbolic equations with a wave operator are studied; the solutions of the problems are constructed as a sum of Fourier series in the spherical coordinate system. But because of the arising small denominators, one cannot assume that these series converge in the space $C^1(\overline{Q}) \cap C^2(Q)$. When proving the singularity theorems, questions also arise about the uniform convergence of the series used, since they contain small denominators.

In this paper, in the class of regular solutions of equation (1), i.e., satisfying conditions (2) and (3), the criterion of uniqueness of the solution of problem (2)–(5) is established and the solution itself is constructed in explicit form – sums of Fourier series. When justifying the convergence of the series, the problem of small denominators arose, as in the well-known works of V. I. Arnold [15, 16] and V. V. Kozlov [17], but from two natural arguments. In this connection, we establish estimates of the separability from zero of small denominators, on the basis of which we prove the convergence of the series in the class of functions $C^2(\overline{Q})$ under some conditions concerning the functions $\tau(x,y)$ and $\psi(x,y)$ and also obtain estimates of the stability of the solution.

2. UNIQUENESS CRITERION FOR THE SOLUTION OF THE DIRICHLET PROBLEM

In the cylindrical coordinate system $x = r\cos\varphi$, $y = r\sin\varphi$, $t = t$, $0 \leq r < l$, $0 \leq \varphi \leq 2\pi$, equation (1) will take the following form

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\varphi\varphi} + \frac{b}{a^2}u = \frac{1}{a^2}u_{tt}. \tag{6}$$

Dividing the variables $u(r,\varphi,t) = v(r,\varphi)T(t)$ in equation (6), we obtain the following spectral problem with respect to the function $v(r,\varphi)$:

$$v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\varphi\varphi} + \lambda^2 v = 0, \tag{7}$$

$$v(l,\varphi) = 0, \tag{8}$$

$$|v(0,\varphi)| < +\infty, \quad v(r,\varphi) = v(r,\varphi + 2\pi), \tag{9}$$

where $\lambda^2 = \frac{b}{a^2} + \mu^2$, μ is the variable separation constant.

The solution of the problem (7)–(9) is similar [18, p. 215]: we will look for in the form of $v(r,\varphi) = R(r)\Phi(\varphi)$ and obtain two one-dimensional spectral problems:

$$\Phi''(\varphi) + p^2\Phi(\varphi) = 0, \quad 0 \leq \varphi \leq 2\pi, \tag{10}$$

$$\Phi(\varphi) = \Phi(\varphi + 2\pi), \quad \Phi'(\varphi) = \Phi'(\varphi + 2\pi); \tag{11}$$

$$R''(r) + \frac{1}{r}R'(r) + \left(\lambda^2 - \frac{p^2}{r^2}\right)R(r) = 0, \quad 0 < r < l, \tag{12}$$

$$|R(0)| < +\infty, \quad R(l) = 0. \tag{13}$$

Nonzero periodic solutions of the problem (10) and (11) exist only at the whole $p = n$ and are defined by the formula

$$\Phi_n(\varphi) = a_n \cos(n\varphi) + b_n \sin(n\varphi),$$

where a_n, b_n are arbitrary constants, $n = 0, 1, 2, \ldots$ At $p = n$, the general solution of equation (12) has the form

$$R_n(r) = c_n J_n(\lambda r) + d_n Y_n(\lambda r),$$

here c_n and d_n are arbitrary constants, $J_n(\lambda r)$ and $Y_n(\lambda r)$ are cylindrical functions of the first and second kind, respectively. From the first condition in (13) it follows that $d_n = 0$, and the second condition gives the equation

$$J_n(q) = 0, \quad q = \lambda l,$$

that, as it is known, has a countable set of positive roots q_{nm}, $n = 0, 1, 2, \ldots$, $m = 1, 2, \ldots$, and eigenvalues corresponding to them

$$\lambda_{nm} = \frac{q_{nm}}{l}, \quad m = 1, 2, \ldots, \quad n = 0, 1, 2, \ldots,$$

and eigenfunctions

$$\tilde{R}_{nm}(r) = J_n(\lambda_{nm} r) = J_n\left(\frac{q_{nm}}{l} r\right)$$

of the spectral problem (12), (13).

Thus, the spectral problem (10), (11) has a system of eigenfunctions

$$\Phi_n(\varphi) = \left\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos(n\varphi), \frac{1}{\sqrt{\pi}}\sin(n\varphi)\right\}, \tag{14}$$

orthonormalized, complete and forming a basis in the space $L_2(0, 2\pi)$, and the spectral problem (12), (13) – a system of eigenfunctions

$$R_{nm}(r) = \frac{J_n(\lambda_{nm} r)}{\|J_n(\lambda_{nm} r)\|_{L_2(0,l)}} = \frac{\sqrt{2}}{l} \frac{J_n(\lambda_{nm} r)}{|J_{n+1}(q_{nm})|}, \tag{15}$$

complete and an orthonormalized basis in $L_2(0, l)$ with weight r.

Then, the spectral problem (7)–(9) has eigenvalues $\lambda^2_{nm} = \frac{b}{a^2} + \mu^2_{nm} = \left(\frac{q_{nm}}{l}\right)^2$, and the system of eigenfunctions corresponds to them, taking into account (14) and (15)

$$v_{nm}(r, \varphi) = \left\{\frac{1}{\sqrt{2\pi}} R_{0m}(r), \frac{1}{\sqrt{\pi}} R_{nm}(r)\cos(n\varphi), \frac{1}{\sqrt{\pi}} R_{nm}(r)\sin(n\varphi)\right\}, \tag{16}$$

that is complete and forms an orthonormalized basis in the space $L_2(D)$ with weight r.

Further, we will assume that $b \geq 0$, because if $b < 0$, then, starting from some numbers $n > n_0$ or $m > m_0$, the right part of $\lambda^2_{nm} = \frac{b}{a^2} + \mu^2_{nm}$, takes only positive values, i.e., the sign of the coefficient b, essentially does not affect the obtained results.

Let $u(r, \varphi, t)$ be the solution of problem (2)–(5). Based on the system (16) we introduce the functions

$$A_{0m}(t) = \frac{1}{\sqrt{2\pi}} \iint_D u(r, \varphi, t) R_{0m}(r) r\, dr\, d\varphi, \tag{17}$$

$$A_{nm}(t) = \frac{1}{\sqrt{\pi}} \iint_D u(r, \varphi, t) R_{nm}(r) \cos(n\varphi) r\, dr\, d\varphi, \tag{18}$$

$$B_{nm}(t) = \frac{1}{\sqrt{\pi}} \iint_D u(r, \varphi, t) R_{nm}(r) \sin(n\varphi) r\, dr\, d\varphi. \tag{19}$$

Differentiating equality (18) by t twice and considering equation (6), we obtain

$$A''_{nm}(t) = \frac{1}{\sqrt{\pi}} \iint_D u_{tt}(r,\varphi,t) R_{nm}(r)\cos(n\varphi) r\,dr\,d\varphi =$$

$$= \frac{a^2}{\sqrt{\pi}} \iint_D \left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\varphi\varphi}\right) R_{nm}(r)\cos(n\varphi) r\,dr\,d\varphi + bA_{nm}(t) = J_1 + J_2 + bA_{nm}(t), \quad (20)$$

where

$$J_1 = \frac{a^2}{\sqrt{\pi}} \iint_D \left(u_{rr} + \frac{1}{r}u_r\right) R_{nm}(r)\cos(n\varphi) r\,dr\,d\varphi = \frac{a^2}{\sqrt{\pi}} \int_0^{2\pi} \cos(n\varphi) \int_0^l (ru_r)'_r R_{nm}(r)\,dr\,d\varphi, \quad (21)$$

$$J_2 = \frac{a^2}{\sqrt{\pi}} \iint_D \frac{1}{r} u_{\varphi\varphi} R_{nm}(r)\cos(n\varphi)\,dr\,d\varphi = \frac{a^2}{\sqrt{\pi}} \int_0^l \frac{1}{r} R_{nm}(r) \int_0^{2\pi} u_{\varphi\varphi}\cos(n\varphi)\,d\varphi\,dr. \quad (22)$$

Let us calculate the internal integrals in the right-hand sides of the equalities (21) and (22):

$$\int_0^l (ru_r)'_r R_{nm}(r)\,dr = ru_r R_{nm}(r)\Big|_0^l - \int_0^l u_r r R'_{nm}(r)\,dr = -\int_0^l u_r r R'_{nm}(r)\,dr =$$

$$= ruR'_{nm}(r)\Big|_0^l + \int_0^l u(rR'_{nm}(r))'\,dr = -\lambda_{nm}^2 \int_0^l urR_{nm}(r)\,dr + n^2 \int_0^l u\frac{R_{nm}(r)}{r}\,dr,$$

$$\int_0^{2\pi} u_{\varphi\varphi}\cos(n\varphi)\,d\varphi = -n^2 \int_0^{2\pi} u\cos(n\varphi)\,d\varphi.$$

Substituting these values in (21) and (22), and then (21) and (22) into equality (20), we obtain

$$A''_{nm}(t) + a^2\mu_{nm}^2 A_{nm}(t) = 0. \quad (23)$$

The general solution of equation (23) is determined by the formula

$$A_{nm}(t) = a_{nm}\cos(a\mu_{nm}t) + b_{nm}\sin(a\mu_{nm}t), \quad (24)$$

where a_{nm} and b_{nm} are arbitrary constants. For their determination we will use the boundary conditions (5):

$$A_{nm}(0) = \frac{1}{\sqrt{\pi}} \iint_D u(r,\varphi,0) R_{nm}(r)\cos(n\varphi) r\,dr\,d\varphi = \frac{1}{\sqrt{\pi}} \iint_D \tau(r,\varphi) R_{nm}(r)\cos(n\varphi) r\,dr\,d\varphi =: \tau_{nm}, \quad (25)$$

$$A_{nm}(T) = \frac{1}{\sqrt{\pi}} \iint_D u(r,\varphi,T) R_{nm}(r)\cos(n\varphi) r\,dr\,d\varphi = \frac{1}{\sqrt{\pi}} \iint_D \psi(r,\varphi) R_{nm}(r)\cos(n\varphi) r\,dr\,d\varphi =: \psi_{nm}. \quad (26)$$

Subordinating the general solution (24) to the boundary conditions (25) and (26), we find

$$a_{nm} = \tau_{nm}, \quad b_{nm} = \frac{1}{\sin(a\mu_{nm}T)}(\psi_{nm} - \tau_{nm}\cos(a\mu_{nm}T))$$

provided that

$$\Delta_{nm}(T) = \sin(a\mu_{nm}T) \neq 0 \quad \text{at all} \quad n, m \in \mathbb{N}. \quad (27)$$

Then

$$A_{nm}(t) = \tau_{nm}\frac{\sin(a\mu_{nm}(T-t))}{\sin(a\mu_{nm}T)} + \psi_{nm}\frac{\sin(a\mu_{nm}t)}{\sin(a\mu_{nm}T)}. \quad (28)$$

Having differentiated equality (19) twice by t taking into account equation (6), we obtain

$$B''_{nm}(t) + a^2\mu_{nm}^2 B_{nm}(t) = 0.$$

From here (by analogy with the function $A_{nm}(t)$), we will find under condition (27)

$$B_{nm}(t) = \tilde{\tau}_{nm}\frac{\sin(a\mu_{nm}(T-t))}{\sin(a\mu_{nm}T)} + \tilde{\psi}_{nm}\frac{\sin(a\mu_{nm}t)}{\sin(a\mu_{nm}T)}, \quad (29)$$

where

$$\tilde{\tau}_{nm} = \frac{1}{\sqrt{\pi}} \iint_D \tau(r,\varphi) R_{nm}(r) \sin(n\varphi) r\, dr\, d\varphi, \tag{30}$$

$$\tilde{\psi}_{nm} = \frac{1}{\sqrt{\pi}} \iint_D \psi(r,\varphi) R_{nm}(r) \sin(n\varphi) r\, dr\, d\varphi. \tag{31}$$

Now let us differentiate equality (17) twice by t and, similarly, on the basis of equation (6) we obtain that the function $A_{0m}(t)$ is a solution of the differential equation

$$A''_{0m}(t) + a^2\mu_{0m}^2 A_{0m}(t) = 0.$$

From here (by analogy with the function $A_{nm}(t)$), we find

$$A_{0m}(t) = \tau_{0m}\frac{\sin(a\mu_{0m}(T-t))}{\sin(a\mu_{0m}T)} + \psi_{0m}\frac{\sin(a\mu_{0m}t)}{\sin(a\mu_{0m}T)} \tag{32}$$

provided $\sin(\mu_{0m}T) \neq 0$ for all $m \in \mathbb{N}$, where

$$\tau_{0m} = \frac{1}{\sqrt{2\pi}} \iint_D \tau(r,\varphi) R_{0m}(r) r\, dr\, d\varphi, \tag{33}$$

$$\psi_{nm} = \frac{1}{\sqrt{2\pi}} \iint_D \psi(r,\varphi) R_{0m}(r) r\, dr\, d\varphi. \tag{34}$$

Now let us prove the uniqueness of the solution of problem (2)–(5). Let $\tau(x,y) = \psi(x,y) \equiv 0$ and conditions (27) be satisfied for all $m \in \mathbb{N}$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Then, by virtue of equations (25), (26), (30), (31), (33) and (34) all $\tau_{nm} = 0$, $\tilde{\tau}_{nm} = 0$, $\psi_{nm} = 0$, $\tilde{\psi}_{nm} = 0$, at $n = 0,1,2,\ldots$, $m = 1,2,\ldots$ Hence and on the basis of formulas (32), (29), (28) and (17)–(19) we have the following equations

$$\iint_D u(r,\varphi,t) R_{nm}(r)\cos(n\varphi) r\, dr\, d\varphi = 0, \quad \iint_D u(r,\varphi,t) R_{nm}(r)\sin(n\varphi) r\, dr\, d\varphi = 0$$

at all $n = 0,1,2,\ldots$, $m = 1,2,\ldots$, $t \in [0,T]$. From these equalities, based on the completeness of the system of functions (16) in the space $L_2(D)$ with weight r, it follows that $u(r,\varphi,t) = 0$ is almost everywhere in $\overline{D}$ at any $t \in [0,T]$. Since by virtue of (2) the function $u(r,\varphi,t)$ is continuous in $\overline{Q}$, then $u(r,\varphi,t) \equiv 0$ in $\overline{Q}$.

Suppose for some $n = n_0$ or $m = m_0$ the expression $\Delta_{n_0 m}(T) = 0$ or $\Delta_{n m_0}(T) = 0$. For definiteness, suppose that $\Delta_{n_0 m}(T) = 0$. Then the homogeneous problem (2)–(5) ($\tau(x,y) = \psi(x,y) \equiv 0$) has a nonzero solution

$$u_{n_0 m}(r,\varphi,t) = \sin(a\mu_{n_0 m}t)\left(a_{0m}R_{0m}(r) + a_{n_0 m}R_{n_0 m}(r)\cos(n_0\varphi) + b_{n_0 m}R_{n_0 m}(r)\sin(n_0\varphi)\right), \tag{35}$$

where a_{0m}, $a_{n_0 m}$ and $b_{n_0 m}$ are arbitrary constants.

Consider the zeros of the expression $\Delta_{nm}(T)$. Equality

$$\Delta_{nm}(T) = \sin(a\mu_{nm}T) = 0$$

only takes place when

$$T = \frac{\pi k}{a\mu_{nm}}, \quad k \in \mathbb{N}. \tag{36}$$

So, $\Delta_{nm}(T)$ goes to zero when T is determined by formula (36).

Thus, the criterion of uniqueness of the solution of problem (2)–(5) is established.

Theorem 1. *If there exists a solution of problem* (2)–(5), *then it is singular if and only if conditions* (27) *are satisfied at all* n *and* m.

3. EXISTENCE OF A SOLUTION TO THE PROBLEM

If the conditions (27) are satisfied, the solution of the problem (2)–(5) is defined by the sum of the series

$$u(r,\varphi,t)=\frac{1}{\sqrt{2\pi}}\sum_{m=1}^{\infty}A_{0m}(t)R_{0m}(r)+\frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(A_{nm}(t)\cos(n\varphi)+B_{nm}(t)\sin(n\varphi)\right)R_{nm}(r), \tag{37}$$

where the coefficients $A_{0m}(t)$, $A_{nm}(t)$, and $B_{nm}(t)$ are found by formulas (32), (28) and (29), respectively. Since $\Delta_{nm}(T)$ is the denominator of the coefficients of the series (37) and, as shown above, equation $\sin(a\mu_{nm}T)=0$ has a countable set of zeros (36), the problem of small denominators arises. In this regard, estimates about separability from zero should be established. For simplicity, in what follows we assume that $b=0$. The expression $\Delta_{nm}(T)$ at $b=0$ is represented in the following form:

$$\Delta_{nm}(\nu)=\sin(\nu q_{nm}),\quad \nu=\frac{aT}{l}. \tag{38}$$

Lemma 1. *If one of the following conditions is met:*

1) *the number $\nu/2=p$ is natural and odd;*

2) *the number $\nu/2=p/q$ is fractional-rational and the relation $(2r-p)/(2q)$ is not an integer where $r\in\mathbb{N}_0$ and $0\le r<q$,*

then there exist positive constants C_0 and m_0 ($m_0\in\mathbb{N}$) such that for all $m>m_0$ the evaluation is valid.

$$|\Delta_{nm}(\nu)|\ge C_0>0. \tag{39}$$

Proof. For zeros q_{nm} of the Bessel function $J_n(q)$ at large values $m>m_0$, where m_0 is a sufficiently large natural number, the asymptotic formula [19, p. 241] is valid.

$$q_{nm}=\frac{\pi}{2}(2m+n-1/2)+O\left((4m+2n-1)^{-1}\right). \tag{40}$$

Substitution (40) into (38) gives

$$\Delta_{nm}(\nu)=\sin\left(\frac{\nu\pi}{2}(2m+n-1/2)\right)+O\left((4m+2n-1)^{-1}\right), \tag{41}$$

since

$$\sin O\left((4m+2n-1)^{-1}\right)\approx O\left((4m+2n-1)^{-1}\right),\quad \cos O\left((4m+2n-1)^{-1}\right)\approx 1+O\left((4m+2n-1)^{-1}\right)$$

at large $m>m_0$.

Let the number $\nu/2=p\in\mathbb{N}$ odd. Then, from equality (41) for all $m>m_0$ and $n\in\mathbb{N}_0$ we obtain

$$|\Delta_{nm}(\nu)|\ge\left|\sin\left(\pi p(2m+n)-\frac{p\pi}{2}\right)\right|-\left|O\left((4m+2n-1)^{-1}\right)\right|=$$
$$=\left|\sin\frac{p\pi}{2}\right|-\left|O\left((4m+2n-1)^{-1}\right)\right|=1-\left|O\left((4m+2n-1)^{-1}\right)\right|>\frac{1}{2} \tag{42}$$

by virtue of

$$\left|O\left((4m+2n-1)^{-1}\right)\right|<C_1<\frac{1}{2}$$

at large m.

Let $\nu/2=p/q$, $p,q\in\mathbb{N}$, $(p,q)=1$, $p/q\notin\mathbb{N}$. In this case, let us divide $p(2m+n)$ by q with remainder: $p(2m+n)=qs+r$, $s,r\in\mathbb{N}_0$, $0\le r<q$. Then the relation (41) will take the form

$$\Delta_{nm}(\nu)=\sin\left(s\pi+\frac{r\pi}{q}-\frac{p\pi}{2q}\right)+O\left((4m+2n-1)^{-1}\right)=(-1)^s\sin\left(\pi\frac{2r-p}{2q}\right)+O\left((4m+2n-1)^{-1}\right).$$

If $r = 0$, then we have case 1) of the lemma. Then $1 \leq r \leq q - 1$. Hence (since the relation $(2r - p)/(2q)$ is not an integer) it follows that

$$|\Delta_{nm}(\nu)| \geq \left|\sin\left(\pi\frac{2r-p}{2q}\right)\right| - \left|O\left((4m+2n-1)^{-1}\right)\right| \geq \left|\sin\left(\pi\frac{2r-p}{2q}\right)\right| - C_1 \geq C_2 - C_1 > 0, \tag{43}$$

where

$$C_2 = \min_{1\leq r\leq q-1} |\sin(\pi(2r-p)/2q)|.$$

Then, from (42) and (43) under the condition $C_1 < C_2$, follows the validity of the estimate (39).

Lemma 2. *Let one of the conditions of Lemma 1 be satisfied, then for all $m > m_0$, $n \in \mathbb{N}_0$ and any $t \in [0, T]$ the following estimates are valid*

$$|A_{nm}(t)| \leq M_1\big(|\tau_{nm}| + |\psi_{nm}|\big), \tag{44}$$

$$|B_{nm}(t)| \leq M_1\big(|\widetilde{\tau}_{nm}| + |\widetilde{\psi}_{nm}|\big), \tag{45}$$

$$|A'_{nm}(t)| \leq M_2\mu_{nm}\big(|\tau_{nm}| + |\psi_{nm}|\big), \quad |B'_{nm}(t)| \leq M_2\mu_{nm}\big(|\widetilde{\tau}_{nm}| + |\widetilde{\psi}_{nm}|\big),$$

$$|A''_{nm}(t)| \leq M_3\mu_{nm}^2\big(|\tau_{nm}| + |\psi_{nm}|\big), \quad |B''_{nm}(t)| \leq M_3\mu_{nm}^2\big(|\widetilde{\tau}_{nm}| + |\widetilde{\psi}_{nm}|\big),$$

hereafter M_i are positive constants depending on T, a and l.

The fairness of these estimates follows directly from formulas (28) and (29) on the basis of inequalities (39).

Now formally from the series (37) at $b = 0$ by postal differentiation, we obtain the series

$$u_{tt} = \frac{1}{\sqrt{2\pi}}\sum_{m=1}^{\infty} A''_{0m}(t)R_{0m}(r) + \frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\big(A''_{nm}(t)\cos(n\varphi) + B''_{nm}(t)\sin(n\varphi)\big)R_{nm}(r),$$

$$u_{\varphi\varphi} = -\frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} n^2\big(A_{nm}(t)\cos(n\varphi) + B_{nm}(t)\sin(n\varphi)\big)R_{nm}(r),$$

$$u_{rr} = \frac{1}{\sqrt{2\pi}}\sum_{m=1}^{\infty} A_{0m}(t)R''_{0m}(r) + \frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\big(A_{nm}(t)\cos(n\varphi) + B_{nm}(t)\sin(n\varphi)\big)R''_{nm}(r),$$

which at any $(r, \varphi, t) \in \overline{Q}$ are majorized respectively by numerical series

$$\frac{4M_3}{\sqrt{2\pi}}\sum_{m>m_0}^{\infty} \mu_{0m}^2\big(|\tau_{0m}| + |\psi_{0m}|\big)|R_{0m}(r)| +$$

$$+ \frac{M_3}{\sqrt{\pi}}\sum_{n=1}^{\infty}\sum_{m>m_0}^{\infty} \mu_{nm}^2\big(|\tau_{nm}| + |\psi_{nm}| + |\widetilde{\tau}_{nm}| + |\widetilde{\psi}_{nm}|\big)|R_{nm}(r)|, \tag{46}$$

$$\frac{M_1}{\sqrt{\pi}}\sum_{n=1}^{\infty}\sum_{m>m_0}^{\infty} n^2\big(|\tau_{nm}| + |\psi_{nm}| + |\widetilde{\tau}_{nm}| + |\widetilde{\psi}_{nm}|\big)|R_{nm}(r)|, \tag{47}$$

$$\frac{M_1}{\sqrt{2\pi}}\sum_{m>m_0}^{\infty} \big(|\tau_{0m}| + |\psi_{0m}|\big)|R''_{0m}(r)| +$$

$$+ \frac{M_1}{\sqrt{\pi}}\sum_{n=1}^{\infty}\sum_{m>m_0}^{\infty} \big(|\tau_{nm}| + |\psi_{nm}| + |\widetilde{\tau}_{nm}| + |\widetilde{\psi}_{nm}|\big)|R''_{nm}(r)|. \tag{48}$$

Lemma 3. *Let $0 < r_0 \leq r \leq l$, where r_0 is a small positive fixed constant. Then at $m > m_0$ and any fixed $n \in \mathbb{N}_0$ there are the following estimates*

$$|R_{nm}(r)| \leq M_4, \tag{49}$$

$$|R'_{nm}(r)| \leq M_5\mu_{nm}, \tag{50}$$

$$|R''_{nm}(r)| \leq M_6\mu_{nm}^2. \tag{51}$$

Proof. Based on the asymptotic formula for the Bessel function of the first kind $J_\nu(z)$ at large values of the argument z [20, p. 98]

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}}\left[\cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) - \frac{1}{2z}\sin\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\right] + O(z^{-5/2}) \tag{52}$$

we have

$$|J_n(\mu_{nm}r)| \le \sqrt{\frac{2}{\pi r_0\mu_{nm}}}\left(1 + \frac{1}{2r_0\mu_{nm}}\right) \le 2\sqrt{\frac{2}{\pi r_0\mu_{nm}}}, \tag{53}$$

as $1(2r_0\mu_{nm}) < 1$ at large m.

Similarly, we obtain the estimates

$$|J_{n+1}(q_{nm})| = |J_{n+1}(l\mu_{nm})| \le 2\sqrt{\frac{2}{\pi l\mu_{nm}}}, \tag{54}$$

from which follows the estimation (49).

Now find the derivative

$$R'_{nm}(r) = \frac{\sqrt{2}}{l|J_{n+1}(q_{nm})|}\mu_{nm}J'_n(z), \quad z = \mu_{nm}r. \tag{55}$$

Using the equality

$$J'_\nu(z) = \frac{1}{2}[J_{\nu-1}(z) - J_{\nu+1}(z)] \tag{56}$$

and formula (52), we obtain the asymptotic formula for $J'_n(z)$ at large z

$$J'_n(z) = \frac{1}{2}\sqrt{\frac{2}{\pi z}}\left[\cos\left(z - \frac{(n-1)}{2}\pi - \frac{\pi}{4}\right) - \cos\left(z - \frac{(n+1)}{2}\pi - \frac{\pi}{4}\right)\right] + O(z^{-3/2}) =$$
$$= \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{n\pi}{2} + \frac{\pi}{4}\right) + O(z^{-3/2}),$$

on the basis of which, similarly to estimates (53) and (54), we find

$$|J'_n(\mu_{nm}r)| \le 2\sqrt{\frac{2}{\pi r_0\mu_{nm}}}. \tag{57}$$

Then from equality (55) by virtue of estimates (57) and (54), follows estimate (50).

From (12) we calculate the second derivative

$$J''_n(\mu_{nm}r) = -\frac{1}{r}J'_n(\mu_{nm}r) + \left(\frac{n^2}{r^2} - \mu^2_{nm}\right)J_n(\mu_{nm}r). \tag{58}$$

Hence, taking into account estimates (53) and (57), we have

$$|J''_n(\mu_{nm}r)| \le \frac{1}{r_0}2\sqrt{\frac{2}{\pi r_0\mu_{nm}}} + \frac{n^2}{r_0^2}2\sqrt{\frac{2}{\pi r_0\mu_{nm}}} + \mu^2_{nm}2\sqrt{\frac{2}{\pi r_0\mu_{nm}}}.$$

From this inequality, by virtue of (54), we verify the validity of the estimate (51).

Lemma 4. *Let* $0 < r_0 \le r \le l$. *Then for large* n *and any fixed* $m \in \mathbb{N}$*, the following estimates are valid*

$$|R_{nm}(r)| \le M_7, \tag{59}$$
$$|R'_{nm}(r)| \le M_8 n, \tag{60}$$
$$|R''_{nm}(r)| \le M_9 n^2. \tag{61}$$

Proof. To obtain these estimates, let us use Langer's asymptotic formula at large values of order p of the Bessel function [20, p. 103]

$$J_p(t) = \frac{1}{\pi}\sqrt{1 - \frac{\operatorname{arctg}\omega}{\omega}}K_{1/3}(z) + O(p^{-4/3}), \tag{62}$$

where

$$\omega = \sqrt{1 - \left(\frac{t}{p}\right)^2}, \quad t < p, \quad z = p(\text{Arth}\,\omega - \omega),$$

$K_{1/3}(z)$ – McDonald's function.

Using a power series expansion of the function

$$arctg\,\omega = \omega - \frac{\omega^3}{3} + \frac{\omega^5}{5} - \frac{\omega^7}{7} + \ldots,$$

evaluate the expression

$$\frac{\omega^2}{3}\left(1 - \frac{3}{5}\omega^2\right) < 1 - \frac{arctg\,\omega}{\omega} < \frac{\omega^2}{3}.$$

Hence at $0 < \omega < 1$ we have

$$\sqrt{\frac{2}{15}}\omega < \left(1 - \frac{arctg\,\omega}{\omega}\right)^{1/2} < \frac{\omega}{\sqrt{3}}. \tag{63}$$

Then from the formula (62), taking into account the estimation (63), we obtain

$$|J_p(t)| \le \frac{\omega}{\pi\sqrt{3}} K_{1/3}(z), \tag{64}$$

$$|J_p(t)| > \sqrt{\frac{2}{15}}\frac{\omega}{\pi} K_{1/3}(z). \tag{65}$$

Now on the basis of estimates (64) and (65) we have

$$|J_n(\mu_{nm} r)| \le \frac{\omega_1}{\pi\sqrt{3}} K_{1/3}(z_1), \tag{66}$$

$$|J_n(q_{nm})| \ge \sqrt{\frac{2}{15}}\frac{\omega_2}{\pi} K_{1/3}(z_2), \tag{67}$$

where

$$\omega_1 = \sqrt{1 - \left(\frac{q_{nm} r}{nl}\right)^2}, \quad z_1 = n(\text{Arth}\,\omega_1 - \omega_1),$$

$$\omega_2 = \sqrt{1 - \left(\frac{q_{nm}}{n+1}\right)^2}, \quad z_2 = (n+1)(\text{Arth}\,\omega_2 - \omega_2).$$

From inequalities (66) and (67), estimate (59) follows, since $\omega_1 \approx \omega_2$ at large n.

Based on formulas (55) and (56), we estimate the derivative $R'_{nm}(r)$:

$$|R'_{nm}(r)| \le \frac{q_{nm}}{\sqrt{2} l^2 |J_{n+1}(q_{nm})|}(|J_{n-1}(\mu_{nm} r)| + |J_{n+1}(\mu_{nm} r)|).$$

Hence, taking into account estimates (66) and (67), we obtain (60).

By virtue of equality (58) on the basis of (59) and (60), we are convinced of the fairness of the estimate (61).

Remark. Note that the function $R_{nm}(r)$ and its derivatives $R'_{nm}(r)$, $R''_{nm}(r)$, starting from some number n, tend to zero at $r \to 0$. Therefore, in Lemmas 3 and 4 the estimates (49)–(51) and (59)–(61) are obtained at $r \ge r_0 > 0$.

By virtue of lemmas 3 and 4, rows (46)–(48) are majorized by the combination of rows

$$M_{10} \sum_{m > m_0}^{\infty} m^2(|\tau_{0m}| + |\psi_{0m}|), \quad M_{11} \sum_{n=1}^{\infty}\sum_{m>m_0}^{\infty} n^2(|\tau_{0m}| + |\psi_{0m}|),$$

$$M_{12} \sum_{n=1}^{\infty}\sum_{m>m_0}^{\infty} \mu_{nm}^2(|\tau_{nm}| + |\psi_{nm}| + |\tilde{\tau}_{nm}| + |\tilde{\psi}_{nm}|). \tag{68}$$

Let us denote by $C^{4,4}(\overline{D})$ the set of functions $f(r,\varphi)$, that have continuous mixed derivatives on r and φ up to and including fourth order in the closed region $\overline{D}$.

Lemma 5. *Let* $\tau(r,\varphi)$, $\psi(r,\varphi)\in C^{4,4}(\overline{D})$, and $\tau^{(0,i)}(r,0)=\tau^{(0,i)}(r,2\pi)$, $i=\overline{0,3}$, $\tau^{(k,4)}(0,\varphi)=0$, $k=\overline{0,3}$, $\psi^{(0,i)}(r,0)=\psi^{(0,i)}(r,2\pi)$, $i=\overline{0,3}$, $\psi^{(k,4)}(0,\varphi)=0$, $k=\overline{0,3}$. *Then the coefficients of* τ_{nm}, $\widetilde{\tau}_{nm}$, ψ_{nm}, $\widetilde{\psi}_{nm}$ *at* $\mu_{nm}\to+\infty$ *have estimates of*

$$\tau_{nm}=O\left(\frac{1}{n\mu_{nm}^4}\right),\quad \widetilde{\tau}_{nm}=O\left(\frac{1}{n\mu_{nm}^4}\right),\quad \psi_{nm}=O\left(\frac{1}{n\mu_{nm}^4}\right),\quad \widetilde{\psi}_{nm}=O\left(\frac{1}{n\mu_{nm}^4}\right).$$

Proof. Consider the coefficients $\tau_{nm},\psi_{nm},\tilde{\tau}_{nm}$, and $\tilde{\psi}_{nm}$ defined by formulas (25), (26), (30), and (31), respectively. Let us represent τ_{nm} in the following form:

$$\tau_{nm}=\frac{1}{\sqrt{\pi}}\int_0^l R_{nm}(\mu_{nm}r)I(r)r\,dr, \tag{69}$$

where

$$I(r)=\int_0^{2\pi}\tau(r,\varphi)\cos(n\varphi)\,d\varphi.$$

By the condition $\tau'_\varphi(r,0)=\tau'_\varphi(r,2\pi)$ and $\tau'''_\varphi(r,0)=\tau'''_\varphi(r,2\pi)$, then the integral $I(r)$ can be transformed by fourfold integration by parts into the form

$$I(r)=\frac{1}{n^4}\int_0^{2\pi}\tau_\varphi^{(4)}(r,\varphi)\cos(n\varphi)\,d\varphi. \tag{70}$$

Now let us write the integral (69), taking into account the representation (70), as

$$\tau_{nm}=\frac{\sqrt{2}}{l\sqrt{\pi}|J_{n+1}(q_{nm})|n^4}\int_0^{2\pi}J(\varphi)\cos(n\varphi)\,d\varphi, \tag{71}$$

where

$$J(\varphi)=\int_0^l\tau_\varphi^{(4)}(r,\varphi)J_n(\mu_{nm}r)r\,dr. \tag{72}$$

Note that the function $X_n(r)=r^{-n}J_n(\xi)$, $\xi=\mu_{nm}r$ is a solution of the differential equation

$$X''_n(r)+\frac{2n+1}{r}X'_n(r)+\mu_{nm}^2X_n(r)=0. \tag{73}$$

Then the integral (72), taking into account equation (73), is transformed as follows:

$$\begin{aligned}
J(\varphi)&=\int_0^l\tau_\varphi^{(4)}(r,\varphi)X_n(r)r^{n+1}\,dr=\\
&=-\frac{1}{\mu_{nm}^2}\int_0^l\tau_\varphi^{(4)}(r,\varphi)\left[X''_n(r)+\frac{2n+1}{r}X'_n(r)\right]r^{n+1}\,dr=\\
&=-\frac{1}{\mu_{nm}^2}\int_0^l\tau_\varphi^{(4)}(r,\varphi)\left[(r^{n+1}X'_n(r))'+nr^nX'_n(r)\right]dr=\\
&=-\frac{1}{\mu_{nm}^2}\int_0^l\tau_{r,\varphi}^{(2,4)}(r,\varphi)r^{n+1}X_n(r)\,dr-\\
&\quad-\frac{1}{\mu_{nm}^2}\int_0^l\tau_{r,\varphi}^{(1,4)}(r,\varphi)r^nX_n(r)\,dr+\\
&\quad+\frac{n^2}{\mu_{nm}^2}\int_0^l\tau_{r,\varphi}^{(0,4)}(r,\varphi)r^{n-1}X_n(r)\,dr=\\
&=-\frac{1}{\mu_{nm}^2}J_1-\frac{1}{\mu_{nm}^2}J_2+\frac{n^2}{\mu_{nm}^2}J_3,
\end{aligned} \tag{74}$$

where

$$J_1 = \int_0^l \tau_{r,\varphi}^{(2,4)}(r,\varphi) r^{n+1} X_n(r)\,dr,$$
$$J_2 = \int_0^l \tau_1(r,\varphi) r^{n+1} X_n(r)\,dr,$$
$$J_3 = \int_0^l \tau_2(r,\varphi) r^{n+1} X_n(r)\,dr,$$
$$\tau_1(r,\varphi) = \frac{\tau_{r,\varphi}^{(1,4)}(r,\varphi)}{r}, \quad \tau_2(r,\varphi) = \frac{\tau_{r,\varphi}^{(0,4)}(r,\varphi)}{r^2}.$$

Similarly to the integral $J(\varphi)$ by formula (74), we transform the integrals J_i, $i = 1, 2$:

$$J_i = -\frac{1}{\mu_{nm}^2} J_{i1} - \frac{1}{\mu_{nm}^2} J_{i2} + \frac{n^2}{\mu_{nm}^2} J_{i3}, \tag{75}$$

where

$$J_{11} = \int_0^l \tau_{r,\varphi}^{(4,4)}(r,\varphi) r^{n+1} X_n(r)\,dr = \int_0^l \tau_{r,\varphi}^{(4,4)}(r,\varphi) J_n(\mu_{nm} r) r\,dr,$$
$$J_{12} = \int_0^l \tau_{r,\varphi}^{(3,4)}(r,\varphi) r^{n} X_n(r)\,dr = \int_0^l \frac{\tau_{r,\varphi}^{(3,4)}(r,\varphi)}{r} J_n(\mu_{nm} r) r\,dr,$$
$$J_{13} = \int_0^l \tau_{r,\varphi}^{(2,4)}(r,\varphi) r^{n-1} X_n(r)\,dr = \int_0^l \frac{\tau_{r,\varphi}^{(2,4)}(r,\varphi)}{r^2} J_n(\mu_{nm} r) r\,dr,$$
$$J_{21} = \int_0^l \tau_{1r}''(r,\varphi) r^{n+1} X_n(r)\,dr = \int_0^l \tau_{1r}''(r,\varphi) J_n(\mu_{nm} r) r\,dr,$$
$$J_{22} = \int_0^l \tau_{1r}'(r,\varphi) r^{n} X_n(r)\,dr = \int_0^l \frac{\tau_{1r}'(r,\varphi)}{r} J_n(\mu_{nm} r) r\,dr,$$
$$J_{23} = \int_0^l \tau_1(r,\varphi) r^{n-1} X_n(r)\,dr = \int_0^l \frac{\tau_1(r,\varphi)}{r^2} J_n(\mu_{nm} r) r\,dr.$$

We transform the integral J_3 as follows:

$$J_3 = \int_0^l \tau_{r,\varphi}^{(0,4)}(r,\varphi) r^{-1} J_n(\mu_{nm} r)\,dr =$$
$$= \int_0^l \tau_{r,\varphi}^{(0,4)}(r,\varphi) r^{-n-2} r^{n+1} J_n(\mu_{nm} r)\,dr =$$
$$= \frac{\tau_{r,\varphi}^{(0,4)}(r,\varphi)}{r} J_{n+1}(\mu_{nm} r)\bigg|_0^l - \frac{1}{\mu_{nm}} \int_0^l d\left[r^{-n-2}\tau_{r,\varphi}^{(0,4)}(r,\varphi)\right] r^{n+1} J_{n+1}(\mu_{nm} r)\,dr =$$
$$= -\frac{1}{\mu_{nm}} \int_0^l \tau_{r,\varphi}^{(1,4)}(r,\varphi) r^{-1} J_{n+1}(\mu_{nm} r)\,dr + \frac{n+2}{\mu_{nm}} \int_0^l \tau_{r,\varphi}^{(0,4)}(r,\varphi) r^{-2} J_{n+1}(\mu_{nm} r)\,dr =$$
$$= -\frac{1}{\mu_{nm}} J_{31} + \frac{n+2}{\mu_{nm}} J_{32}. \tag{76}$$

After substituting (75) and (76) into equality (74), we obtain

$$J(\varphi) = \frac{1}{\mu_{nm}^4}(J_{11} + J_{12} + J_{21} + J_{22}) - \frac{n^2}{\mu_{nm}^4}(J_{13} + J_{23}) - \frac{n^2}{\mu_{nm}^3} J_{31} + \frac{n^2(n+2)}{\mu_{nm}^3} J_{32}. \tag{77}$$

If $\tau_{r,\varphi}^{(0,4)}(r,\varphi) \in C^4[0,l]$ and $\tau_{r,\varphi}^{(k,4)}(0,\varphi) = 0$, $k = \overline{0,3}$, then the representations are fair

$$\begin{aligned}
\tau_{r,\varphi}^{(0,4)}(r,\varphi) &= \frac{\tau_{r,\varphi}^{(4,4)}(\theta,\varphi)r^4}{4!}, \quad 0 < \theta < r,\\
\tau_{r,\varphi}^{(1,4)}(r,\varphi) &= \frac{\tau_{r,\varphi}^{(4,4)}(\theta,\varphi)r^3}{3!},\\
\tau_{r,\varphi}^{(2,4)}(r,\varphi) &= \frac{\tau_{r,\varphi}^{(4,4)}(\theta,\varphi)r^2}{2!},\\
\tau_{r,\varphi}^{(3,4)}(r,\varphi) &= \tau_{r,\varphi}^{(4,4)}(\theta,\varphi)r.
\end{aligned}$$

By virtue of this in the integrals J_{31} and J_{32}, the functions $\tau_{r,\varphi}^{(0,4)}(r,\varphi)r^{-5/2}$, $\tau_{r,\varphi}^{(1,4)}(r,\varphi)r^{-3/2}$ are continuously differentiable on $[0,l]$, so on this interval they have complete bounded variation, i.e., finite variation. Taking into account the theorem from [21, p. 653], the integrals J_{31} and J_{32} at $\mu_{nm} \to \infty$ have the following evaluation

$$J_{31} = O(\mu_{nm}^{-3/2}), \quad J_{32} = O(\mu_{nm}^{-3/2}). \tag{78}$$

In the integrals J_{1i}, $i = 1,2,3$, the integrand functions $\tau_{r,\varphi}^{(4,4)}(r,\varphi)$, $\tau_{r,\varphi}^{(3,4)}(r,\varphi)r^{-1}$, and $\tau_{r,\varphi}^{(2,4)}(r,\varphi)r^{-2}$ are continuous on the segment $[0,l]$. Then by virtue of Young's theorem [21, p. 654], these integrals at $\mu_{nm} \to \infty$ have the following evaluation

$$J_{1i} = O(\mu_{nm}^{-1/2}). \tag{79}$$

Now consider the integrals J_{2i}, $i = 1,2,3$. In them, the functions $\tau_{1r}''(r,\varphi)$, $\tau_{1r}'(r,\varphi)r^{-1}$ and $\tau_{1r}(r,\varphi)r^{-2}$ are also continuous on the segment $[0,l]$, so the estimates are valid

$$J_{2i} = O(\mu_{nm}^{-1/2}), \quad \mu_{nm} \to \infty. \tag{80}$$

Then from the representation (71), taking into account equality (77) and estimates (78)–(80), we obtain

$$\tau_{nm} = O\left(\frac{1}{n\mu_{nm}^4}\right).$$

Similarly, from formulas (26), (30), and (31), the rest of the estimates follow. The lemma is proved.

Numerical series (68), by virtue of formula (40), are majorized by convergent series, respectively

$$M_{13}\sum_{m>m_0}^{\infty}\frac{1}{m^2}, \quad M_{14}\sum_{n=1}^{\infty}\sum_{m>m_0}^{\infty}\frac{n}{(4m+2n-1)^4}, \quad M_{15}\sum_{n=1}^{\infty}\sum_{m>m_0}^{\infty}\frac{1}{n(4m+2n-1)^2}.$$

If for the numbers ν from lemma 1, for some $m = m_1, m_2, \ldots, m_s \le m_0$, where $1 \le m_1 < m_2 < \cdots < m_s$, $\Delta_{nm_i}(\nu) = 0$, then it is necessary and sufficient for the solvability of problem (2)–(5) that the conditions are satisfied

$$\tau_{nm_i} = \psi_{nm_i} = 0, \quad \tilde{\tau}_{nm_i} = \tilde{\psi}_{nm_i} = 0, \quad i = \overline{1,s}. \tag{81}$$

In this case, the solution of the problem (2)–(5) is defined as a sum of series:

$$\begin{aligned}
u(r,\varphi,t) = {}& \frac{1}{\sqrt{2\pi}}\left(\sum_{m=1}^{m_1-1} + \sum_{m=m_1+1}^{m_2-1} + \cdots + \sum_{m=m_{s-1}+1}^{m_s-1} + \sum_{m=m_s+1}^{\infty}\right)A_{0m}(t)R_{0m}(r) +\\
&+ \frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty}\left(\sum_{m=1}^{m_1-1} + \sum_{m=m_1+1}^{m_2-1} + \cdots + \sum_{m=m_{s-1}+1}^{m_s-1} + \sum_{m=m_s+1}^{\infty}\right)\times\\
&\times (A_{nm}(t)\cos(n\varphi) + B_{nm}(t)\sin(n\varphi))R_{nm}(r) +\\
&+ \sum_{i=1}^{s} C_{nm_i}u_{nm_i}(r,\varphi,t),
\end{aligned} \tag{82}$$

here $u_{nm_i}(r,\varphi,t)$ are determined by formula (35), where m_0 should be replaced by m_i, C_{nm_i} are arbitrary constants; if in the finite sums in the right-hand side of (82), the upper limit is less than the lower limit, they should be considered as zeros.

Thus, the following has been proved

Theorem 2. *Let the conditions of lemmas* 1 *and* 5 *be satisfied. Then if* $\Delta_{nm}(\nu) \neq 0$ *at all* $m = \overline{1, m_0}$*, then problem* (2)–(5) *is uniquely solvable, and this solution is defined by row* (37); *if* $\Delta_{nm}(\nu) = 0$ *at some* $m = m_1, m_2, \dots, m_s \leq m_0$*, then problem* (2)–(5) *is solvable only when conditions* (81) *are satisfied, and the solution is defined by row* (82).

Note that the fulfillment of the condition $\Delta_{nm}(\nu) \neq 0$ at $m = \overline{1, m_0}$ can be achieved if $\nu \neq \pi k/q_{nm}$ (by virtue of formula (36) at $b = 0$).

4. STABILITY OF THE PROBLEM SOLUTION

Consider the following norms:

$$\|u(r,\varphi,t)\|_{L_2(D)} = \iint_D u^2(r,\varphi,t) r\,dr\,d\varphi,$$

$$\|u(r,\varphi,t)\|_{C(\overline{Q})} = \max_{r,\varphi,t\in\overline{Q}} |u(r,\varphi,t)|,$$

$$\|f^{(2,2)}_{r,\varphi}(r,\varphi)\|_{L_2(D)} = \iint_D (f^{(2,2)}_{r,\varphi}(r,\varphi))^2 r\,dr\,d\varphi,$$

$$\|g^{(2,2)}_{r,\varphi}(r,\varphi)\|^2_{C(\overline{D})} = \max_{r,\varphi\in\overline{D}} |g^{(2,2)}_{r,\varphi}(r,\varphi)|.$$

Theorem 3. *Let the conditions of Theorem* 2 *and* $\Delta_{nm}(\nu) \neq 0$ *be satisfied at* $m = \overline{1, m_0}$. *Then for the solution* (37) *of the problem* (2)–(5), *the following estimates are valid*

$$\|u(r,\varphi,t)\|_{L_2(D)} \leq M_{16}(\|\tau(r,\varphi)\|_{L_2(D)} + \|\psi(r,\varphi)\|_{L_2(D)}), \tag{83}$$

$$\|u(r,\varphi,t)\|_{C(\overline{Q})} \leq M_{17}(\|\tau^{(2,2)}_{r,\varphi}(r,\varphi)\|_{C(\overline{D})} + \|\psi^{(2,2)}_{r,\varphi}(r,\varphi)\|_{C(\overline{D})}). \tag{84}$$

Proof. The constructed system of eigenfunctions (16) is orthonormalized in the space $L_2(D)$ with weight r. Then from formula (37) on the basis of estimates (44), (45), and (49), we will have

$$\|u(r,\varphi,t)\|^2_{L_2(D)} = \sum_{m=1}^{\infty} A^2_{0m}(t) + \sum_{n,m=1}^{\infty} A^2_{nm}(t) + B^2_{nm}(t) \leq$$

$$\leq 2M_1^2 M_4^2 \left[\sum_{m=1}^{\infty} \left(|\tau_{0m}|^2 + |\psi_{0m}|^2\right) + \sum_{n,m=1}^{\infty} \left(|\tau_{nm}|^2 + |\widetilde{\tau}_{nm}|^2 + |\psi_{nm}|^2 + |\widetilde{\psi}_{nm}|^2\right)\right] =$$

$$= 2M_1^2 M_4^2 \left(\|\tau(r,\varphi)\|^2_{L_2(D)} + \|\psi(r,\varphi)\|^2_{L_2(D)}\right).$$

Hence we obtain the estimate (83).

Let (r,φ,t) be an arbitrary point $\overline{Q}$. Then from formula (37), taking into account estimates (44), (45) and (49), we have

$$|u(r,\varphi,t)| \leq M_1 M_4 \left[\sum_{m=1}^{\infty} (|\tau_{0m}| + |\psi_{0m}|) + \sum_{n,m=1}^{\infty} (|\tau_{nm}| + |\psi_{nm}| + |\tilde{\tau}_{nm}| + |\tilde{\psi}_{nm}|)\right]. \tag{85}$$

Further, based on the reasoning given in the proof of Lemma 5, we will represent the coefficient τ_{nm} as

$$\tau_{nm} = -\frac{\sqrt{2}}{l\sqrt{\pi}|J_{n+1}(q_{nm})|n^2} \int_0^{2\pi} J(\varphi)\cos(n\varphi)\,d\varphi,$$

where

$$J(\varphi) = \int_0^l \tau_{r,\varphi}^{(0,2)}(r,\varphi) J_n(\mu_{nm} r) r\, dr = -\frac{1}{\mu_{nm}^2}(J_1' + J_2' - n^2 J_3'),$$
$$J_1' = \int_0^l \tau_{r,\varphi}^{(2,2)}(r,\varphi) J_n(\mu_{nm} r) r\, dr,$$
$$J_2' = \int_0^l \frac{\tau_{r,\varphi}^{(1,2)}(r,\varphi)}{r} J_n(\mu_{nm} r) r\, dr,$$
$$J_3' = \int_0^l \frac{\tau_{r,\varphi}^{(0,2)}(r,\varphi)}{r^2} J_n(\mu_{nm} r) r\, dr.$$

If $\tau_{r,\varphi}^{(0,2)}(r,\varphi) \in C^2[0,l]$ and $\tau_{r,\varphi}^{(0,2)}(0,\varphi) = \tau^{(1,2)}(0,\varphi) = 0$, then the functions $\tau_{r,\varphi}^{(1,2)}(r,\varphi) r^{-1} = \tau_{r,\varphi}^{(2,2)}(\theta,\varphi)$, $\tau_{r,\varphi}^{(0,2)}(r,\varphi) = \tau_{r,\varphi}^{(2,2)}(\theta,\varphi)/2$, $0 < \theta < r$ are continuous on the segment $[0,l]$, then

$$|\tau_{nm}| \le \frac{M_{18}}{\mu_{nm}^2} |\tau_{nm}^{(2,2)}|,$$

where

$$\tau_{nm}^{(2,2)} = \frac{1}{\sqrt{\pi}} \iint_D \tau_{r,\varphi}^{(2,2)}(r,\varphi) \cos(n\varphi) R_{nm}(r) r\, dr\, d\varphi. \tag{86}$$

Similarly, we obtain the estimates

$$|\tilde{\tau}_{nm}| \le \frac{M_{18}}{\mu_{nm}^2} |\tilde{\tau}_{nm}^{(2,2)}|,$$
$$\tilde{\tau}_{nm}^{(2,2)} = \frac{1}{\sqrt{\pi}} \iint_D \tau_{r,\varphi}^{(2,2)}(r,\varphi) \sin(n\varphi) R_{nm}(r) r\, dr\, d\varphi, \tag{87}$$
$$|\psi_{nm}| \le \frac{M_{18}}{\mu_{nm}^2} |\psi_{nm}^{(2,2)}|,$$
$$|\tilde{\psi}_{nm}| \le \frac{M_{18}}{\mu_{nm}^2} |\tilde{\psi}_{nm}^{(2,2)}|,$$

where $\psi_{nm}^{(2,2)}$ and $\tilde{\psi}_{nm}^{(2,2)}$ are defined according to formulas (86) and (87), but with the replacement of $\tau(r,\varphi)$ with $\psi(r,\varphi)$.

Now, continuing the estimation (85), we have

$$|u(r,\varphi,t)| \le M_{19} \left[\sum_{m=1}^{\infty} \frac{1}{\mu_{0m}^2} (|\tau_{0m}^{(2,2)}| + |\psi_{0m}^{(2,2)}|) + \sum_{n,m=1}^{\infty} \frac{1}{\mu_{nm}^2} (|\tau_{nm}^{(2,2)}| + |\tilde{\tau}_{nm}^{(2,2)}| + |\psi_{nm}^{(2,2)}| + |\tilde{\psi}_{nm}^{(2,2)}|) \right].$$

Hence, using Bunyakovsky's inequality, we obtain

$$|u(r,\varphi,t)| \le M_{20} \Biggl\{ \left(\sum_{m=1}^{\infty} \frac{1}{\mu_{0m}^4} \right)^{1/2} \left[\left(\sum_{m=1}^{\infty} |\tau_{0m}^{(2,2)}|^2 \right)^{1/2} + \left(\sum_{m=1}^{\infty} |\psi_{0m}^{(2,2)}|^2 \right)^{1/2} \right] +$$
$$+ \left(\sum_{n,m=1}^{\infty} \frac{1}{\mu_{nm}^4} \right)^{1/2} \left[\left(2 \sum_{n,m=1}^{\infty} \left(|\tau_{nm}^{(2,2)}|^2 + |\widetilde{\tau}_{nm}^{(2,2)}|^2 \right) \right)^{1/2} + \left(2 \sum_{n,m=1}^{\infty} \left(|\psi_{nm}^{(2,2)}|^2 + |\widetilde{\psi}_{nm}^{(2,2)}|^2 \right) \right)^{1/2} \right] \Biggr\} \le$$
$$\le M_{21} \left[\left(\sum_{m=1}^{\infty} |\tau_{0m}^{(2,2)}|^2 \right)^{1/2} + \left(\sum_{n,m=1}^{\infty} \left(|\tau_{nm}^{(2,2)}|^2 + |\widetilde{\tau}_{nm}^{(2,2)}|^2 \right) \right)^{1/2} + \right.$$

$$+\left(\sum_{m=1}^{\infty}|\psi_{0m}^{(2,2)}|^2\right)^{1/2}+\left(\sum_{n,m=1}^{\infty}\left(|\psi_{nm}^{(2,2)}|^2+|\widetilde{\psi}_{nm}^{(2,2)}|^2\right)\right)^{1/2}\Bigg]\le$$

$$\le M_{21}\sqrt{2}\Bigg[\left(\sum_{m=1}^{\infty}|\tau_{0m}^{(2,2)}|^2+\sum_{n,m=1}^{\infty}\left(|\tau_{nm}^{(2,2)}|^2+|\widetilde{\tau}_{nm}^{(2,2)}|^2\right)\right)^{1/2}+$$

$$+\left(\sum_{m=1}^{\infty}|\psi_{0m}^{(2,2)}|^2+\sum_{n,m=1}^{\infty}\left(|\psi_{nm}^{(2,2)}|^2+|\widetilde{\psi}_{nm}^{(2,2)}|^2\right)\right)^{1/2}\Bigg]=$$

$$=\sqrt{2}M_{21}\left(\|\tau^{(2,2)}(r,\varphi)\|_{L_2(D)}+\|\psi^{(2,2)}(r,\varphi)\|_{L_2(D)}\right)\le M_{22}\left(\|\tau^{(2,2)}(r,\varphi)\|_{C(\overline{D})}+\|\psi^{(2,2)}(r,\varphi)\|_{C(\overline{D})}\right).$$

From the last inequality, the estimate (84) follows directly.

CONFLICT OF INTERESTS

The author of this paper declares that he has no conflict of interests.

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