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PARTIAL DIFFERENTIAL EQUATIONS

ON FRONT MOTION IN THE REACTION–DIFFUSION–ADVECTION PROBLEM WITH KPZ-NONLINEARITY

© 2025 A. O. Orlov*

Lomonosov Moscow State University, Russia
**e-mail: orlov.andrey@physics.msu.ru*

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Abstract. We obtain an asymptotic approximation to a moving inner layer (front) solution of an initial–boundary value problem for a singularly perturbed parabolic reaction–diffusion–advection equation with KPZ-nonlinearity. An asymptotic approximation for the velocity of the front is found. To prove the existence and uniqueness of a solution the asymptotic method of differential inequalities is used.

Keywords: *reaction–diffusion–advection equation, KPZ-nonlinearity, contrast structures, front motion, small parameter*

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1. INTRODUCTION. PROBLEM STATEMENT

In this paper, we consider the initial boundary value problem for the singularly perturbed parabolic equation, that differs from the classical singularly perturbed reaction–diffusion–advection equation (see [1, 2]) by the presence of an additional nonlinear term containing the square of the gradient of the desired function (KPZ-nonlinearities [3, 4]):

$$\varepsilon^2\frac{\partial^2 u}{\partial x^2}-\varepsilon\frac{\partial u}{\partial t}-\varepsilon^2 A(u,x)\left(\frac{\partial u}{\partial x}\right)^2-f(u,x,\varepsilon)=0,\quad x\in(-1,1),t\in(0,T],$$

$$\frac{\partial u}{\partial x}(-1,t,\varepsilon)=0,\quad \frac{\partial u}{\partial x}(1,t,\varepsilon)=0,\quad t\in[0,T],$$

$$u(x,0,\varepsilon)=u_{\text{init}}(x,\varepsilon),\quad x\in[-1,1], \tag{1}$$

where $\varepsilon\in(0,\varepsilon_0]$ is the small parameter, $\varepsilon>0$ is a given constant.

Traveling wave type solutions for quasilinear parabolic reaction–diffusion–advection equations are the subject of intensive study (see extensive monographs [5, 6]). Attention to nonlinearities of the form $A(u,x)\left(\frac{\partial u}{\partial x}\right)^2$ is due to both theoretical interest – the square is the limit of degree at which the Bernstein conditions on the growth of the nonlinearity are satisfied (see, e.g., [7–9]), and important applications where such nonlinearities are used in mathematical models, in particular, population dynamics models [10], in modeling free surface growth in polymer theory [3, 4, 11], and many others. We note the work [12], in which exact solutions of the KPZ equation are constructed for several physically justified nonlinearities. However, it is assumed there that $(u,x)=\text{const}\ f=f(x,t)$. The principal difference of problem (1) is that we consider an equation, where the nonlinear terms depend explicitly on the coordinate and the desired function. In this paper, we propose an algorithm for constructing an asymptotic approximation of the solution of the front view, with the velocity of motion being a function of the coordinate.

Stationary solutions of problem (1) with boundary and inner layers are studied in [13, 14]. The boundary-layer solutions of the Tikhonov-type system with KPZ-nonlinearities are studied in [15].

The paper is structured as follows. In (2), we construct an asymptotic approximation of the moving front solution using the method of A. B. Vasilieva [16]. Note that since problem (1) is singularly perturbed, at $\varepsilon=0$ the equation of problem (1) changes its type from parabolic to algebraic with three roots (see condition 2), two of them describe stable equilibrium positions of the system and represent the regular part of the asymptotic approximation of zero order of accuracy. However, the regular approximation does not allow us to describe a narrow region with a large gradient, in which the solution passes from one stable level to another. To describe the solution in this region and to harmonize the stable equilibrium positions among themselves, the so-called transition layer functions are constructed. In this way, a formal asymptotic approximation of the solution in the whole region under consideration is constructed. In (3), an algorithm for finding an asymptotic approximation of the front position is given. In (4), we give a justification of the formal asymptotics and prove the existence and uniqueness theorem using the asymptotic method of differential inequalities of N. N. Nefedov, that has shown its efficiency in many singularly perturbed problems [16]. The obtained results are illustrated in Section 5 by an example, that can be used to develop and verify new numerical methods for the considered class of problems (see [17]).

The results obtained in this paper develop the studies [1, 2], in which the front motion in the reaction–diffusion–advection equation with weak advection and smooth or modular (discontinuous at some value of the desired function nonlinearities) sources was considered, and transfer them to a new class of singularly perturbed problems with KPZ-nonlinearities. At the same time, as in [1, 2], the existence and uniqueness theorem of the solution having in both cases the same form of the contrast structure of the step type [16] is proved.

In the problem discussed below, it is assumed that at the initial moment of time the front is already formed. This means that the function $u_{\text{init}}(x, \varepsilon)$ has an internal transition layer in the neighborhood of some point $x_{00} \in (-1, 1)$, i.e., it is close to some root $\varphi^{(-)}(x)$ of the degenerate equation $f(u, x, 0) = 0$ to the left of the point x_{00} and to the root $\varphi^{(+)}(x)$ to the right of this point. In the neighborhood of x_{00} there is a sharp transition from $\varphi^{(-)}(x)$ to $\varphi^{(+)}(x)$.

We will assume that the following conditions are satisfied.

Condition 1. The functions $A(u, x)$, $f(u, x, \varepsilon)$ are sufficiently smooth in their areas of definition.

Condition 2. The derived equation $f(u, x, 0) = 0$ has exactly three solutions $u = \varphi^{(\pm,0)}(x)$, with $\varphi^{(-)}(x) < \varphi^{(0)}(x) < \varphi^{(+)}(x)$, $x \in [-1, 1]$, while the following inequalities are also valid

$$f_u(\varphi^{(\pm)}(x), x, 0) > 0, \quad f_u(\varphi^{(0)}(x), x, 0) < 0, \quad x \in [-1, 1].$$

2. CONSTRUCTION OF FORMAL ASYMPTOTICS OF THE SOLUTION

The asymptotics of the solution of problem (1) is constructed by the method of boundary functions separately in each of the regions $[-1, \hat{x}] \times [0, T]$ and $[\hat{x}, 1] \times [0, T]$ with a moving boundary (see [16]) using the effective method developed in the scientific school of Professors A. B. Vasilieva, V. F. Butuzov, and N. N. Nefedov for constructing the asymptotics of localization of the inner layer in the form of

$$U(x, \varepsilon) = \begin{cases} U^{(-)}(x, t, \varepsilon), & (x, t, \varepsilon) \in [-1, \hat{x}] \times [0, T] \times (0, \varepsilon_0], \\ U^{(+)}(x, t, \varepsilon), & (x, t, \varepsilon) \in [\hat{x}, 1] \times [0, T] \times (0, \varepsilon_0]. \end{cases}$$

We will represent each of the functions $U^{(\pm)}(x, \varepsilon)$ as a sum of three summands:

$$U^{(\pm)}(x, t, \varepsilon) = \bar{u}^{(\pm)}(x, \varepsilon) + Q^{(\pm)}(\xi, t, \varepsilon) + R^{(\pm)}(\eta^{(\pm)}, \varepsilon),$$

where $\bar{u}^{(\pm)}(x, \varepsilon) = \bar{u}_0^{(\pm)}(x) + \varepsilon \bar{u}_1^{(\pm)}(x) + \cdots$ is the regular part of the decomposition, functions $Q^{(\pm)}(\xi, t, \varepsilon) = Q_0^{(\pm)}(\xi, t, \varepsilon) + \varepsilon Q_1^{(\pm)}(\xi, t, \varepsilon) + \cdots$ describe the behavior of the solution in the vicinity of the transition point $\hat{x}(t, \varepsilon)$, $\xi = \frac{x - \hat{x}(t, \varepsilon)}{\varepsilon}$ is the transition layer variable: $\xi \le 0$ for functions with index $(-)$ and $\xi \ge 0$ for functions with index $(+)$; functions $R^{(\pm)}(\eta^{(\pm)}, \varepsilon) = R_0^{(\pm)}(\eta^{(\pm)}) + \varepsilon R_1^{(\pm)}(\eta^{(\pm)}) + \cdots$ describe the behavior of the solution in the vicinity of the boundary points of the segment $[-1, 1]$; $\eta^{(\pm)} = \frac{x \mp 1}{\varepsilon}$ are stretched variables near the points $x = \pm 1$, respectively. Since the functions $R_i^{(\pm)}(\eta^{(\pm)})$ are defined in a standard way (see, for example, [16]), we omit the procedure of their construction. Note that these functions do not depend on the variable t and thus do not participate in the description of the moving transition layer, and the functions $R_0^{(\pm)}(\eta^{(\pm)}) = 0$ by virtue of the Neumann boundary conditions.

The position of the inner transition layer is determined from the condition C^1-combining the asymptotic representations $U^{(-)}(x,t,\varepsilon)$ and $U^{(+)}(x,t,\varepsilon)$ at the transition point $\hat{x}(t,\varepsilon)$:

$$U^{(-)}(\hat{x}(t,\varepsilon),t,\varepsilon)=U^{(+)}(\hat{x}(t,\varepsilon),t,\varepsilon)=\phi^{(0)}(\hat{x}(t,\varepsilon)), \tag{2}$$

$$\varepsilon\frac{\partial}{\partial x}U^{(-)}(\hat{x}(t,\varepsilon),t,\varepsilon)=\varepsilon\frac{\partial}{\partial x}U^{(+)}(\hat{x}(t,\varepsilon),t,\varepsilon). \tag{3}$$

We will look for the transition point $x=\hat{x}(t,\varepsilon)$ in the form of expansion by powers of the small parameter ε:

$$\hat{x}(t,\varepsilon)=x_0(t)+\varepsilon x_1(t)+\cdots \tag{4}$$

The coefficients of this expansion will be determined in the process of asymptotics construction.

The regular part of the asymptotics is determined after substituting the representation for the functions $\bar{u}^{(\pm)}(x,\varepsilon)$ into the equation.

$$\varepsilon^2\frac{\partial^2\bar{u}^{(\pm)}}{\partial x^2}-\varepsilon^2A(\bar{u}^{(\pm)},x)\left(\frac{\partial\bar{u}^{(\pm)}}{\partial x}\right)^2-f(\bar{u}^{(\pm)},x,\varepsilon)=0.$$

In the standard way [16], we obtain the algebraic equations for determining the functions of the regular part $\bar{u}_k^{(\pm)}(x)$, $k=0,1,\ldots$

Taking into account condition 2, the regular zero-order functions are defined as

$$\bar{u}_0^{(\pm)}(x)=\phi^{(\pm)}(x).$$

To shorten the record, we introduce the notations

$$\bar{f}_u^{(\pm)}(x):=f_u(\phi^{(\pm)}(x),x,0).$$

Functions $\bar{u}_k^{(\pm)}(x)$ at $k=1,2,\ldots$ are defined from equations

$$\bar{f}_u^{(\pm)}(x)\bar{u}_k^{(\pm)}(x)=\bar{h}_k^{(\pm)}(x),$$

where the functions $\bar{h}_k^{(\pm)}(x)$ are known at each k-step and are expressed recurrently through the functions $\bar{u}_k^{(\pm)}(x)$ with indices $0,1,\ldots,k-1$. The solvability of these equations follows from condition 2.

In order to obtain the equations satisfied by the transition layer functions $Q_k^{(\pm)}(\xi,t,\varepsilon)$, let us rewrite the differential operator of the problem in the variables (ξ,t). Then the equations for the functions $Q_k^{(\pm)}(\xi,t,\varepsilon)$, $k=0,1,\ldots$, are determined in the standard way [16] by equating the coefficients at the same degrees ε in both parts of the equations:

$$\begin{aligned}&\frac{\partial^2Q^{(\pm)}}{\partial\xi^2}+\frac{\partial\hat{x}(t,\varepsilon)}{\partial t}\frac{\partial Q^{(\pm)}}{\partial\xi}+A\big(\bar{u}^{(\pm)}\big(\varepsilon\xi+\hat{x}(t,\varepsilon),\varepsilon\big),\varepsilon\xi+\hat{x}(t,\varepsilon)\big)\left(\frac{\partial\bar{u}^{(\pm)}}{\partial\xi}\right)^2-\\&-A\big(\bar{u}^{(\pm)}\big(\varepsilon\xi+\hat{x}(t,\varepsilon),\varepsilon\big)+Q^{(\pm)}(\xi,t,\varepsilon),\varepsilon\xi+\hat{x}(t,\varepsilon)\big)\left(\frac{\partial Q^{(\pm)}}{\partial\xi}+\frac{\partial\bar{u}^{(\pm)}}{\partial\xi}\right)^2-\varepsilon\frac{\partial Q^{(\pm)}}{\partial t}=\\&=f\big(\bar{u}^{(\pm)}\big(\varepsilon\xi+\hat{x}(t,\varepsilon),\varepsilon\big)+Q^{(\pm)}(\xi,t,\varepsilon),\varepsilon\xi+\hat{x}(t,\varepsilon),\varepsilon\big)-f\big(\bar{u}^{(\pm)}\big(\varepsilon\xi+\hat{x}(t,\varepsilon),\varepsilon\big),\varepsilon\xi+\hat{x}(t,\varepsilon),\varepsilon\big).\end{aligned} \tag{5}$$

In contrast to the approach in [2], we will not decompose by powers of ε the transition point $\hat{x}(t,\varepsilon)$. This will simplify the algorithm for constructing the asymptotics. Note that the equations from which the functions $Q_k^{(\pm)}(\xi,t,\varepsilon)$ are found contain functions depending on $\hat{x}(t,\varepsilon)$, $\frac{\partial\hat{x}(t,\varepsilon)}{\partial t}$, and that explains the presence of the argument ε at $Q_k^{(\pm)}(\xi,t,\varepsilon)$.

We require that the transition layer functions $Q_k^{(\pm)}(\xi,t,\varepsilon)$, $k=0,1,\ldots$, satisfy the conditions of equality to zero at infinity: $Q_k^{(-)}(\xi,t,\varepsilon)\to0$ at $\xi\to-\infty$, $Q_k^{(+)}(\xi,t,\varepsilon)\to0$ at $\xi\to+\infty$, $k=0,1,\ldots$, $t\in[0,T]$.

Equating the coefficients at ε^0 in the right and left parts of equations (5), we obtain equations for the function $Q_0^{(-)}(\xi,t,\varepsilon)$ at $\xi\le0$ and the function $Q_0^{(+)}(\xi,t,\varepsilon)$ at $\xi\ge0$:

$$\begin{aligned}&\frac{\partial^2Q_0^{(\pm)}}{\partial\xi^2}+\frac{\partial\hat{x}(t,\varepsilon)}{\partial t}\frac{\partial Q_0^{(\pm)}}{\partial\xi}-A\big(\varphi^{(\pm)}(\hat{x}(t,\varepsilon))+Q_0^{(\pm)}(\xi,t,\varepsilon),\hat{x}(t,\varepsilon)\big)\left(\frac{\partial Q_0^{(\pm)}}{\partial\xi}\right)^2=\\&=f\big(\varphi^{(\pm)}(\hat{x}(t,\varepsilon))+Q_0^{(\pm)}(\xi,t,\varepsilon),\hat{x}(t,\varepsilon),0\big).\end{aligned} \tag{6}$$

We obtain the additional conditions at $\xi = 0$ from the continuous cross-linking condition (2) written in zero order at ε:

$$Q_0^{(-)}(0,t,\varepsilon) + \phi^{(-)}(\hat{x}(t,\varepsilon)) = Q_0^{(+)}(0,t,\varepsilon) + \phi^{(+)}(\hat{x}(t,\varepsilon)) = \phi^{(0)}(\hat{x}(t,\varepsilon)).$$

We also add conditions at infinity: $Q_0^{(-)}(\xi,t,\varepsilon) \to 0$ at $\xi \to -\infty$, $Q_0^{(+)}(\xi,t,\varepsilon) \to 0$ at $\xi \to +\infty$, $t \in [0,T]$.

Let's introduce the operator D, acting by the rule

$$D\hat{x} := \frac{\partial \hat{x}(t,\varepsilon)}{\partial t}, \tag{7}$$

and functions

$$\begin{gathered}
\tilde{u}^{(\pm)}(\xi,t,\varepsilon) = \phi^{(\pm)}(\hat{x}(t,\varepsilon)) + Q_0^{(\pm)}(\xi,t,\varepsilon), \\
\tilde{u}(\xi,t,\varepsilon) = \begin{cases} \phi^{(-)}(\hat{x}(t,\varepsilon)) + Q_0^{(-)}(\xi,t,\varepsilon), & \text{if } \xi \le 0, \\ \phi^{(+)}(\hat{x}(t,\varepsilon)) + Q_0^{(+)}(\xi,t,\varepsilon), & \text{if } \xi \ge 0, \end{cases} \\
\tilde{v}^{(-)}(\xi,t,\varepsilon) = \frac{\partial \tilde{u}}{\partial \xi}(\xi,t,\varepsilon), \quad \xi \le 0, \\
\tilde{v}^{(+)}(\xi,t,\varepsilon) = \frac{\partial \tilde{u}}{\partial \xi}(\xi,t,\varepsilon), \quad \xi \ge 0.
\end{gathered} \tag{8}$$

Remark. It follows from the form of equations (6), that in the functions $Q_0^{(\pm)}(\xi,t,\varepsilon)$, $\tilde{u}(\xi,t,\varepsilon)$, $\tilde{u}^{(\pm)}(\xi,t,\varepsilon)$, $\tilde{v}^{(\pm)}(\xi,t,\varepsilon)$, we can switch to another set of arguments – $(\xi,\hat{x})$. In the future, we will use both sets, choosing the most convenient for each particular case.

Let us rewrite equations (6), as well as the additional conditions, using (8):

$$\begin{gathered}
\frac{\partial^2 \tilde{u}^{(\pm)}}{\partial \xi^2} + D\hat{x}\frac{\partial \tilde{u}^{(\pm)}}{\partial \xi} - A(\tilde{u}^{(\pm)},\hat{x})\left(\frac{\partial \tilde{u}^{(\pm)}}{\partial \xi}\right)^2 = f(\tilde{u}^{(\pm)},\hat{x},0), \\
\tilde{u}^{(\pm)}(0,\hat{x}) = \phi^{(0)}(\hat{x}), \quad \tilde{u}^{(\pm)}(\pm\infty,\hat{x}) = \phi^{(\pm)}(\hat{x}).
\end{gathered} \tag{9}$$

Along with the problems (9), let us consider the problem

$$\frac{\partial^2 \hat{u}}{\partial \xi^2} + W\frac{\partial \hat{u}}{\partial \xi} - A(\hat{u},\hat{x})\left(\frac{\partial \hat{u}}{\partial \xi}\right)^2 = f(\hat{u},\hat{x},0), \quad \hat{u}(0,\hat{x}) = \phi^{(0)}(\hat{x}), \quad \hat{u}(\pm\infty,\hat{x}) = \phi^{(\pm)}(\hat{x}). \tag{10}$$

Let us formulate and prove the existence result of the solution of problem (10) in the form of a lemma.

Lemma. *For each $\hat{x} \in (-1,1)$, there exists a single value W such that the problem* (10) *has a single smooth monotone solution $\hat{u}(\xi,\hat{x})$, satisfying the estimation*

$$|\hat{u}(\xi,\hat{x}) - \phi^{(\pm)}(\hat{x})| < C\exp\{-\kappa|\xi|\},$$

where C and κ are some positive constants. In this case, the dependence $W(\hat{x})$ is defined as

$$W(\hat{x}) = \int_{\phi^{(-)}(\hat{x})}^{\phi^{(+)}(\hat{x})} f(u,\hat{x},0)\exp\left\{-2\int_{\phi^{(-)}(\hat{x})}^{u} A(y,\hat{x})dy\right\}du \times \\ \times \left[\int_{-\infty}^{+\infty}\left(\frac{\partial \hat{u}}{\partial \xi}(\xi,\hat{x})\right)^2 \exp\left\{-2\int_{\phi^{(-)}(\hat{x})}^{\tilde{u}(\xi,\hat{x})} A(y,\hat{x})dy\right\}d\xi\right]^{-1}.$$

The smoothness of the function $W(\hat{x})$ coincides with the smoothness of the functions $f(u,\hat{x},0)$ and $A(u,\hat{x})$.

Proof. In order to use the known result from [18], we make a monotonic transformation proposed by A. V. Bitsadze in [19]:

$$z(\xi,\hat{x}) := z(\hat{u}(\xi,\hat{x}),\hat{x}) = \int_{\phi^{(-)}(\hat{x})}^{\hat{u}(\xi,\hat{x})} \exp\left\{-\int_{\phi^{(-)}(\hat{x})}^{y} A(r,\hat{x})dr\right\}dy, \quad (\hat{u},\hat{x}) \in [\phi^{(-)}(\hat{x}),\phi^{(+)}(\hat{x})] \times [-1,1].$$

Let's introduce the notations

$$z^{(\pm,0)}(\hat{x}) = \int_{\phi^{(-)}(\hat{x})}^{\phi^{(\pm,0)}(\hat{x})} \exp\left\{-\int_{\phi^{(-)}(\hat{x})}^{y} A(r,\hat{x})dr\right\} dy.$$

Due to the monotonicity of the transformation $z(\hat{u},\hat{x})$ by $\hat{u}$ we can define the inverse function

$$\hat{u}(\xi,\hat{x}) = h(z(\xi,\hat{x}),\hat{x}), \quad (z,\hat{x}) \in [0, z^{(+)}(\hat{x})] \times [-1,1].$$

Thus, the problem (10) transforms into the problem

$$\frac{\partial^2 z}{\partial \xi^2} + W\frac{\partial z}{\partial \xi} - f(h(z,\hat{x}),\hat{x},0)\exp\left\{-\int_{\phi^{(-)}(\hat{x})}^{h(z,\hat{x})} A(r,\hat{x})dr\right\} = 0,$$

$$z(-\infty,\hat{x}) = 0, \quad z(0,\hat{x}) = z^{(0)}(\hat{x}), \quad z(+\infty,\hat{x}) = z^{(+)}(\hat{x}), \tag{11}$$

for which, by virtue of conditions 1 and 2, the following statements are true [18].

1. For each $\hat{x} \in (-1,1)$, there exists a single value W, such that the problem (11) has a single smooth monotone solution $\hat{z}(\xi,\hat{x})$, satisfying the estimation

$$|z(\xi,\hat{x}) - z^{(\pm)}(\hat{x})| < C\exp\{-\kappa|\xi|\},$$

where C and κ are some positive constants.

2. The dependence $W(\hat{x})$ is defined as

$$W(\hat{x}) = \int_0^{z^{(+)}(\hat{x})} f(h(z,\hat{x}),\hat{x},0)\exp\left\{-\int_{\phi^{(-)}(\hat{x})}^{h(z,\hat{x})} A(r,\hat{x})dr\right\} dz \left[\int_{-\infty}^{+\infty}\left(\frac{\partial \hat{z}}{\partial \xi}(\xi,\hat{x})\right)^2 d\xi\right]^{-1}. \tag{12}$$

The smoothness of the function $W(\hat{x})$ coincides with the smoothness of the functions $f(u,\hat{x},0)$ and $A(u,\hat{x})$.

Finally, returning to the function $\hat{u}(\xi,\hat{x})$ using the transformation $\hat{u}(\xi,\hat{x}) = h(z(\xi,\hat{x}),\hat{x})$ and recalculating the integrals in expression (13), we have the statement of the lemma. The lemma is proved.

Let's condition.

Condition 3. Task

$$\frac{dx}{dt} = W(x), \quad x(0) = x_{00} \tag{13}$$

has a solution $x = x_0(t)$, such that $x_0(t) \in (-1,1)$ at $t \in [0,T]$; $W(x) > 0$ for all $x \in [-1,1]$.

The inequality $W(x) > 0$ in condition 3 guarantees the absence of stationary solutions for problem (13). Let us denote by (9a) the problems (9) in which we replace $\hat{x}$ by $x_0(t)$, or, otherwise, in which we put $\varepsilon = 0$.

It follows from the lemma and condition 3, that problems (9a) are singularly solvable, since the condition $D\hat{x}_0 = W(x_0)$ is satisfied. Thus

$$\frac{\partial \tilde{u}^{(+)}}{\partial \xi}(0,x_0(t)) - \frac{\partial \tilde{u}^{(-)}}{\partial \xi}(0,x_0(t)) = 0.$$

By virtue of the assumed smoothness of the functions $f(u,\hat{x},0)$, $A(u,\hat{x})$ (see condition 1), problems (9) are regular perturbations of problems (9a), so they are also uniquely solvable. Note that by virtue of the representation (4)

$$\frac{\partial \tilde{u}^{(+)}}{\partial \xi}(0,\hat{x}(t,\varepsilon)) - \frac{\partial \tilde{u}^{(-)}}{\partial \xi}(0,\hat{x}(t,\varepsilon)) = O(\varepsilon).$$

Thus, the construction of the zero-order transition layer functions is completed.

The first-order transition layer functions are found from the following problems:

$$\frac{\partial^2 Q_1^{(\pm)}}{\partial \xi^2} + D\hat{x}\frac{\partial Q_1^{(\pm)}}{\partial \xi} - 2\tilde{A}(\xi,t)\tilde{v}^{(\pm)}(\xi,\hat{x})\frac{\partial Q_1^{(\pm)}}{\partial \xi} - \left(\tilde{A}_u(\xi,t)(\tilde{v}^{(\pm)}(\xi,\hat{x}))^2 + \tilde{f}_u(\xi,t)\right)Q_1^{(\pm)} = r_1^{(\pm)}(\xi,t,\varepsilon),$$

$$Q_1^{(\pm)}(0,t,\varepsilon) + \bar{u}_1^{(\pm)}(\hat{x}) = 0, \quad Q_1^{(\pm)}(\pm\infty,t,\varepsilon) = 0, \tag{14}$$

where the notations are defined

$$\tilde f_u(\xi,t)=f_u(\tilde u(\xi,\hat x),\hat x,0),\quad \tilde A(\xi,t)=A_u(\tilde u(\xi,\hat x),\hat x),\quad \tilde A_u(\xi,t)=A_u(\tilde u(\xi,\hat x),\hat x) \tag{15}$$

and

$$r_1^{(\pm)}(\xi,t,\varepsilon)=\frac{\partial Q_0^{(\pm)}}{\partial t}(\xi,t,\varepsilon)+2\tilde A(\xi,t)\tilde v^{(\pm)}(\xi,\hat x)\frac{d\varphi^{(\pm)}}{dx}(\hat x)+$$
$$+\left(\bar u_1^{(\pm)}(\hat x)+\xi\frac{d\varphi^{(\pm)}}{dx}(\hat x)\right)\left(\tilde f_u(\xi,t)+\tilde A_u(\xi,t)\left(\tilde v^{(\pm)}(\xi,\hat x)\right)^2\right)+\xi\left(\tilde f_x(\xi,t)+\tilde A_x(\xi,t)\left(\tilde v^{(\pm)}(\xi,\hat x)\right)^2\right)+\tilde f_\varepsilon(\xi,t).$$

Here, the derivatives of $\tilde f_x(\xi,t)$, $\tilde f_\varepsilon(\xi,t)$ are computed at the same point as the derivative of $\tilde f_u(\xi,t)$ in (15). Similarly, $\tilde A_x(\xi,t)$ is computed at the same point as $\tilde A_u(\xi,t)$. In all the notations introduced here, the argument ε is implied, but we omit it for brevity. The problem for the function $Q_1^{(-)}(\xi,t,\varepsilon)$ will be solved on the semi-straight $\xi\le 0$, and for the function $Q_1^{(+)}(\xi,t,\varepsilon)$ – on the semi-straight $\xi\ge 0$. The solutions of problems (14) are written in explicit form:

$$Q_1^{(\pm)}(\xi,t,\varepsilon)=-\bar u_1^{(\pm)}(\hat x)\frac{\tilde v^{(\pm)}(\xi,\hat x)}{\tilde v^{(\pm)}(0,\hat x)}+$$
$$+\tilde v^{(\pm)}(\xi,\hat x)\int_0^\xi\frac{e^{-(D\hat x)\eta}}{(\tilde v^{(\pm)}(\eta,\hat x))^2p^{(\pm)}(\eta,\hat x)}\int_{\pm\infty}^{\eta}\tilde v^{(\pm)}(\sigma,\hat x)p^{(\pm)}(\sigma,\hat x)e^{(D\hat x)\sigma}r_1^{(\pm)}(\sigma,t,\varepsilon)d\sigma d\eta, \tag{16}$$

where

$$p^{(\pm)}(\xi,\hat x)=\exp\left\{-2\int_0^\xi A(\tilde u^{(\pm)}(y,\hat x),\hat x)\tilde v^{(\pm)}(y,\hat x)dy\right\}.$$

It follows from the expression for the functions $r_1^{(\pm)}(\xi,t,\varepsilon)$, that they have exponential valuations [16], and from (16) we deduce in the standard way that similar valuations are true for functions $Q_1^{(\pm)}(\xi,t,\varepsilon)$.

Similarly to the first approximation, one can find for any $k=2,3,\ldots$ transition layer functions $Q_k^{(\pm)}(\xi,t,\varepsilon)$: they are determined from boundary value problems with the same differential operator as in problems (14).

3. ASYMPTOTIC APPROXIMATION OF FRONT POSITION

Let us describe the algorithm for finding an asymptotic approximation of the front position. The unknown coefficients $x_i(t)$, $i\in\mathbb N$, of the expansion are determined from the crossing conditions (3) of the derivatives of the asymptotic approximations. Let us introduce the function

$$H(\varepsilon,t):=\varepsilon\left(\frac{dU^{(+)}}{dx}(\hat x,t,\varepsilon)-\frac{dU^{(-)}}{dx}(\hat x,t,\varepsilon)\right)=H_0(\varepsilon,t)+\varepsilon H_1(\varepsilon,t)+\varepsilon^2H_2(\varepsilon,t)+\cdots, \tag{17}$$

where

$$H_0(\varepsilon,t)=\frac{\partial Q_0^{(+)}}{\partial\xi}(0,\hat x)-\frac{\partial Q_0^{(-)}}{\partial\xi}(0,\hat x),$$
$$H_1(\varepsilon,t)=\frac{d\phi^{(+)}}{dx}(\hat x)-\frac{d\phi^{(-)}}{dx}(\hat x)+\left(\frac{\partial Q_1^{(+)}}{\partial\xi}(0,t,\varepsilon)-\frac{\partial Q_1^{(-)}}{\partial\xi}(0,t,\varepsilon)\right)$$

etc.

The C^1-linking condition (3) is expressed by the equality $H(\varepsilon,t)=0$. By virtue of the lemma and condition 3, taking into account the decomposition of the transition point (4), this equality is satisfied in the order ε^0.

The analysis of problems (9), (10) shows that the function $H_0(\varepsilon,t)$ can be represented as

$$H_0(\varepsilon,t)=(D\hat x-W(\hat x))\left[\frac{1}{\tilde v^{(\pm)}(0,\hat x)}\int_0^{\pm\infty}(\tilde v^{(\pm)}(\xi,\hat x))^2e^{(D\hat x)\xi}p^{(\pm)}(\xi,\hat x)d\xi\right]_-^+ +O(\varepsilon^2). \tag{18}$$

Hereinafter, $[\,]_-^+$ means the difference between the expressions labeled $+$ and $-$.

As follows from the decomposition (17) and the representation (18), the higher order terms $x_i(t)$, $i \geq 1$, in (4) can be found from the following Cauchy problems:

$$\frac{dx_i}{dt} - W'(x_0(t))x_i(t) = G_i(t), \quad x_i(0) = 0,$$

where $G_i(t)$ are known functions.

4. JUSTIFICATION OF FORMAL ASYMPTOTICS

Let's say

$$X_n(t,\varepsilon) = \sum_{i=0}^{n+1} \varepsilon^i x_i(t), \quad \xi = \frac{x - X_n(t,\varepsilon)}{\varepsilon}.$$

The curve $X_n(t,\varepsilon)$ divides the area $\overline{D} : (x,t) \in [-1,1] \times [0,T]$ into two sub-areas:

$$\bar{D}_n^{(-)} : (x,t) \in [-1, X_n(t,\varepsilon)] \times [0,T] \quad \text{and} \quad \bar{D}_n^{(+)} : (x,t) \in [X_n(t,\varepsilon), 1] \times [0,T].$$

Let's define the functions

$$U_n^{(-)}(x,t,\varepsilon) = \sum_{i=0}^{n} \varepsilon^i \left(\bar{u}_i^{(-)}(x) + Q_i^{(-)}(\xi,t,\varepsilon) + R_i^{(-)}(\eta^{(-)}) \right), \quad (x,t) \in \bar{D}_n^{(-)},$$
$$U_n^{(+)}(x,t,\varepsilon) = \sum_{i=0}^{n} \varepsilon^i \left(\bar{u}_i^{(+)}(x) + Q_i^{(+)}(\xi,t,\varepsilon) + R_i^{(+)}(\eta^{(+)}) \right), \quad (x,t) \in \bar{D}_n^{(+)},$$

where $\hat{x}(t,\varepsilon)$, included in the expressions for the transition layer functions, are replaced by $X_n(t,\varepsilon)$, and denoted by

$$U_n(x,t,\varepsilon) = \begin{cases} U_n^{(-)}(x,t,\varepsilon), & (x,t) \in \bar{D}_n^{(-)}, \\ U_n^{(+)}(x,t,\varepsilon), & (x,t) \in \bar{D}_n^{(+)}. \end{cases} \tag{19}$$

To prove the existence and uniqueness of the moving front solution, we use the asymptotic method of differential inequalities [16]. Let us construct continuous functions $\alpha(x,t,\varepsilon)$, $\beta(x,t,\varepsilon)$ in such a way that they satisfy the following conditions.

1. Ordering condition:

$$\alpha(x,t,\varepsilon) \leq \beta(x,t,\varepsilon), \quad x \in [-1,1], t \in [0,T], \varepsilon \in (0,\varepsilon_0]. \tag{20}$$

2. Action of the differential operator on upper and lower solutions:

$$L[\beta] := \varepsilon^2 \frac{\partial^2 \beta}{\partial x^2} - \varepsilon \frac{\partial \beta}{\partial t} - \varepsilon^2 A(\beta,x) \left(\frac{\partial \beta}{\partial x} \right)^2 - f(\beta,x,\varepsilon) \leq 0 \leq$$
$$\leq L[\alpha] := \varepsilon^2 \frac{\partial^2 \alpha}{\partial x^2} - \varepsilon \frac{\partial \alpha}{\partial t} - \varepsilon^2 A(\alpha,x) \left(\frac{\partial \alpha}{\partial x} \right)^2 - f(\alpha,x,\varepsilon) \tag{21}$$

for all $x \in (-1,1)$ and $t \in [0,T]$, except those $x(t)$, in which the functions $\alpha(x,t,\varepsilon)$ and $\beta(x,t,\varepsilon)$ are nonsmooth.

3. Boundary conditions:

$$\frac{\partial \alpha}{\partial x}(-1,t,\varepsilon) \geq 0 \geq \frac{\partial \beta}{\partial x}(-1,t,\varepsilon), \quad \frac{\partial \alpha}{\partial x}(+1,t,\varepsilon) \leq 0 \leq \frac{\partial \beta}{\partial x}(+1,t,\varepsilon), \quad t \in [0,T], \varepsilon \in (0,\varepsilon_0]. \tag{22}$$

4. Conditions on the initial function:

$$\alpha(x,0,\varepsilon) \leq u_{\text{init}}(x,\varepsilon) \leq \beta(x,0,\varepsilon), \quad x \in [-1,1], \varepsilon \in (0,\varepsilon_0]. \tag{23}$$

5. Conditions on the jump of derivatives:

$$\frac{\partial\beta}{\partial x}(\overline{x}(t)-0,t,\varepsilon)\geq\frac{\partial\beta}{\partial x}(\overline{x}(t)+0,t,\varepsilon), \tag{24}$$

where $\overline{x}(t)$ is the point at which the upper solution is nonsmooth;

$$\frac{\partial\alpha}{\partial x}(\underline{x}(t)-0,t,\varepsilon)\leq\frac{\partial\alpha}{\partial x}(\underline{x}(t)+0,t,\varepsilon), \tag{25}$$

where $\underline{x}(t)$ is the point at which the lower solution is nonsmooth.

It is known (see [20]) that if the conditions (20)–(25) are satisfied, there exists a single solution of problem (1) for which the inequalities are satisfied

$$\alpha(x,t,\varepsilon)\leq u(x,t,\varepsilon)\leq\beta(x,t,\varepsilon),\quad (x,t)\in[-1,1]\times[0,T].$$

Let us prove the following existence and uniqueness theorem.

Theorem. *When conditions* 1–3 *are satisfied for any sufficiently smooth initial function* $u_{init}(x)$*, lying between upper and lower solutions*

$$\alpha(x,0,\varepsilon)\leq u_{\text{init}}(x,\varepsilon)\leq\beta(x,0,\varepsilon),$$

there exists a single solution $u(x,t,\varepsilon)$ *of problem* (1)*, that at any* $t\in[0,T]$ *is enclosed between these upper and lower solutions, and for which the function* $U_n(x,t,\varepsilon)$ *is a uniform in the domain* $[-1,1]\times[0,T]$ *asymptotic approximation with accuracy* $O(\varepsilon^{n+1})$.

Proof. The upper and lower solutions of the problem will be constructed as a modification of the asymptotic series (19). Set the function

$$x_\beta(t,\varepsilon)=X_{n+1}(t)-\varepsilon^{n+1}\delta(t),$$

and the positive function $\delta(t)>0$ will be defined below. Let us construct the upper solution of the problem in each of the regions $\overline{D}_\beta^{(-)}:(x,t)\in[-1,x_\beta(t,\varepsilon)]\times[0,T]$ and $\overline{D}_\beta^{(+)}:(x,t)\in[x_\beta(t,\varepsilon),1]\times[0,T]$:

$$\beta(x,t,\varepsilon)=\begin{cases}\beta^{(-)}(x,t,\varepsilon), & (x,t)\in\overline{D}_\beta^{(-)},\\ \beta^{(+)}(x,t,\varepsilon), & (x,t)\in\overline{D}_\beta^{(+)}.\end{cases}$$

We will connect the functions $\beta^{(-)}(x,t,\varepsilon)$ and $\beta^{(+)}(x,t,\varepsilon)$ at the point $x_\beta(t,\varepsilon)$ in such a way, that the following equality is satisfied

$$\beta^{(-)}(x_\beta(t,\varepsilon),t,\varepsilon)=\beta^{(+)}(x_\beta(t,\varepsilon),t,\varepsilon)=\phi^{(0)}(x_\beta(t,\varepsilon)).$$

Note that the function $\beta(x,t,\varepsilon)$ is not smooth. Let us introduce a stretched variable

$$\xi_\beta=\frac{x-x_\beta(t,\varepsilon)}{\varepsilon}.$$

Let us construct the functions $\beta^{(\pm)}(x,t,\varepsilon)$ as modifications of the formal asymptotics (19):

$$\begin{gathered}\beta^{(-)}(x,t,\varepsilon)=U_{n+1}^{(-)}|_{\xi_\beta}+\varepsilon^{n+1}(\mu+q_\beta^{(-)}(\xi_\beta,t,\varepsilon))+\varepsilon^{n+1}R_\beta^{(-)}(\eta^{(-)}),\\ (x,t)\in D_\beta^{(-)},\xi_\beta\leq 0,\eta^{(-)}\geq 0;\\ \beta^{(+)}(x,t,\varepsilon)=U_{n+1}^{(+)}|_{\xi_\beta}+\varepsilon^{n+1}(\mu+q_\beta^{(+)}(\xi_\beta,t,\varepsilon))+\varepsilon^{n+1}R_\beta^{(+)}(\eta^{(+)}),\\ (x,t)\in D_\beta^{(+)},\xi_\beta\geq 0,\eta^{(+)}\leq 0.\end{gathered}$$

Here under the notation $U_{n+1}^{(\pm)}|_{\xi_\beta}$ we understand the functions from (19), where the argument ξ of the transition layer functions is replaced by ξ_β, and X_{n+1} – by x_β.

The positive value μ is chosen, so that conditions (20) and (21) are satisfied. The functions $R_\beta^{(\pm)}(\eta^{(\pm)})$ are chosen, so that condition (22) is satisfied (their construction is not considered in this paper). The functions

$q_\beta^{(\pm)}(\xi_\beta, t, \varepsilon)$ are needed to eliminate the inconsistencies that arise, when the operator acts on the upper solution. Let us define them from the following problems:

$$\frac{\partial^2 q_\beta^{(\pm)}}{\partial \xi_\beta^2} + Dx_\beta \frac{\partial q_\beta^{(\pm)}}{\partial \xi_\beta} - 2\tilde{A}(\xi_\beta, t)\tilde{v}^{(\pm)}(\xi_\beta, x_\beta)\frac{\partial q_\beta^{(\pm)}}{\partial \xi_\beta} -$$

$$- \big(\tilde{A}_u(\xi_\beta, t)\big(\tilde{v}^{(\pm)}(\xi_\beta, x_\beta)\big)^2 + \tilde{f}_u(\xi_\beta, t)\big)q_\beta^{(\pm)} - qf^{(\pm)}(\xi_\beta, t, \varepsilon) = 0,$$

$$q_\beta^{(\pm)}(0, t, \varepsilon) + \mu = 0, \quad q_\beta^{(\pm)}(\pm\infty, t, \varepsilon) = 0, \tag{26}$$

where $qf^{(\pm)}(\xi_\beta, t, \varepsilon) = \mu\big(\tilde{A}_u(\xi_\beta, t)\big(\tilde{v}^{(\pm)}(\xi_\beta, x_\beta)\big)^2 + \tilde{f}_u(\xi_\beta, t) - \bar{f}_u^{(\pm)}(x_\beta)\big)$.

Explicit expressions for these functions can be obtained

$$q_\beta^{(\pm)}(\xi_\beta, t, \varepsilon) = -\mu\frac{\tilde{v}^{(\pm)}(\xi, x_\beta)}{\tilde{v}^{(\pm)}(0, x_\beta)} +$$

$$+ \tilde{v}^{(\pm)}(\xi_\beta, x_\beta)\int\limits_0^{\xi_\beta} \frac{e^{-(Dx_\beta)\eta}}{(\tilde{v}^{(\pm)}(\eta, x_\beta))^2 p^{(\pm)}(\eta, x_\beta)} \int\limits_{\pm\infty}^{\eta} \tilde{v}^{(\pm)}(\sigma, x_\beta)e^{(Dx_\beta)\sigma}p^{(\pm)}(\sigma, x_\beta)qf^{(\pm)}(\sigma, t, \varepsilon)\, d\sigma\, d\eta. \tag{27}$$

The functions $q^{(\pm)}(\xi_\beta, t, \varepsilon)$ have exponential estimates [16].

We can simplify expressions (27) as follows:

$$q_\beta^{(\pm)}(\xi_\beta, t, \varepsilon) =$$

$$= -\mu - \mu\bar{f}_u^{(\pm)}(x_\beta)\tilde{v}^{(\pm)}(\xi_\beta, x_\beta)\int\limits_0^{\xi_\beta} \frac{e^{-(Dx_\beta)\eta}}{(\tilde{v}^{(\pm)}(\eta, x_\beta))^2 p^{(\pm)}(\eta, x_\beta)} \int\limits_{\pm\infty}^{\eta} \tilde{v}^{(\pm)}(\sigma, x_\beta)e^{(Dx_\beta)\sigma}p^{(\pm)}(\sigma, x_\beta)\, d\sigma\, d\eta.$$

Using a similar algorithm, we construct the lower solution. Set the function

$$x_\alpha(t, \varepsilon) = X_{n+1}(t) + \varepsilon^{n+1}\delta(t),$$

where $\delta(t)$ is the same function as in the construction of the upper solution.

Let's construct the lower solution of the problem in each of the regions $\overline{D}_\alpha^{(-)} : (x, t) \in [-1, x_\alpha(t, \varepsilon)] \times [0, T]$ and $\overline{D}_\alpha^{(+)} : (x, t) \in [x_\alpha(t, \varepsilon), 1] \times [0, T]$:

$$\alpha(x, t, \varepsilon) = \begin{cases} \alpha^{(-)}(x, t, \varepsilon), & (x, t) \in \overline{D}_\alpha^{(-)}, \\ \alpha^{(+)}(x, t, \varepsilon), & (x, t) \in \overline{D}_\alpha^{(+)}. \end{cases}$$

We will merge the functions $\alpha^{(-)}(x, t, \varepsilon)$ and $\alpha^{(+)}(x, t, \varepsilon)$ at the point $x_\alpha(t, \varepsilon)$ in such a way that the equality is satisfied

$$\alpha^{(-)}(x_\alpha(t, \varepsilon), t, \varepsilon) = \alpha^{(+)}(x_\alpha(t, \varepsilon), t, \varepsilon) = \phi^{(0)}(x_\alpha(t, \varepsilon)).$$

Note that the function $\alpha(x, t, \varepsilon)$ is not smooth. Let us introduce a stretched variable

$$\xi_\alpha = \frac{x - x_\alpha(t, \varepsilon)}{\varepsilon}.$$

Let us construct the functions $\alpha^{(\pm)}(x, t, \varepsilon)$ as modifications of the formal asymptotics (19):

$$\alpha^{(-)}(x, t, \varepsilon) = U_{n+1}^{(-)}|_{\xi_\alpha} - \varepsilon^{n+1}(\mu + q_\alpha^{(-)}(\xi_\alpha, t, \varepsilon)) + \varepsilon^{n+1}R_\alpha^{(-)}(\eta^{(-)}),$$

$$(x, t) \in D_\alpha^{(-)}, \xi_\alpha \le 0, \eta^{(-)} \ge 0;$$

$$\alpha^{(+)}(x, t, \varepsilon) = U_{n+1}^{(+)}|_{\xi_\alpha} - \varepsilon^{n+1}(\mu + q_\alpha^{(+)}(\xi_\alpha, t, \varepsilon)) + \varepsilon^{n+1}R_\alpha^{(+)}(\eta^{(+)}),$$

$$(x, t) \in D_\alpha^{(+)}, \xi_\alpha \ge 0, \eta^{(+)} \le 0.$$

Here $\mu > 0$ is the same value as in the expression for the upper solution, and $q_\alpha^{(\pm)}(\xi_\alpha, t, \varepsilon)$ are determined from problems (26), in which the stretched variable ξ_β is replaced by ξ_α, and x_β is replaced by x_α.

Let us make sure that the constructed functions $\alpha(x,t,\varepsilon)$ and $\beta(x,t,\varepsilon)$ satisfy the differential inequalities (20)–(25). The ordering condition (20) can be checked similarly as it was done in [2].

Let us show that inequality (21) holds. From the way of constructing the upper and lower solutions the following equations follow

$$L[\alpha^{(\pm)}] = \varepsilon^{n+1}\bar{f}_u^{(\pm)}(x_\alpha)\mu + O(\varepsilon^{n+2}), \quad L[\beta^{(\pm)}] = -\varepsilon^{n+1}\bar{f}_u^{(\pm)}(x_\beta)\mu + O(\varepsilon^{n+2}).$$

The inequalities near the boundary (22) are fulfilled due to a standard modification of the boundary-layer functions [16] (their verification is not intended for this paper).

Let's check the jump condition of the derivative (24)

$$\varepsilon\left(\frac{\partial\beta^{(+)}}{\partial x}\bigg|_{x=x_\beta} - \frac{\partial\beta^{(-)}}{\partial x}\bigg|_{x=x_\beta}\right) = -\varepsilon^{n+1}\frac{1}{\tilde{v}(0,x_0)}\left(L(x_0)\frac{d\delta}{dt} - L(x_0)W'(x_0(t))\delta(t) + F(x_0)\right) + O(\varepsilon^{n+2}),$$

where

$$F(x_0) = \mu\left[\bar{f}_u^{(\pm)}(x_0)\int_{\pm\infty}^{0} p(\sigma,x_0)\tilde{v}(\sigma,x_0)e^{(Dx_0)\sigma}d\sigma\right]_-^+,$$

$$L(x_0) = \int_{-\infty}^{+\infty}\tilde{v}^2(\xi,x_0)e^{(Dx_0)\xi}p(\xi,x_0)d\xi > 0.$$

Here, the index at the functions $\tilde{v}(\xi,x_0)$, $p(\xi,x_0)$ is omitted due to their smoothness at $\xi = 0$.

Let's define the function $\delta(t)$ as a solution to the problem

$$L(x_0)\frac{d\delta}{dt} - L(x_0)W'(x_0(t))\delta(t) + F(x_0) = \sigma, \quad \delta(0) = \delta_0,$$

where σ is a sufficiently large positive value and $\delta_0 > 0$. In this case, the solution to the problem $\delta(t)$ is a positive function. Thus,

$$\varepsilon\left(\frac{\partial\beta^{(+)}}{\partial x}\bigg|_{x=x_\beta} - \frac{\partial\beta^{(-)}}{\partial x}\bigg|_{x=x_\beta}\right) = -\varepsilon^{n+1}\frac{\sigma}{\tilde{v}(0,x_0)} + O(\varepsilon^{n+2}).$$

The expression in the right-hand side is negative due to $\sigma > 0$. With the same choice of function $\delta(t)$, the derivative jump inequality will be satisfied for the lower solution $\alpha(x,t,\varepsilon)$. The theorem is proved.

5. EXAMPLE

Consider the initial boundary value problem

$$\varepsilon^2\frac{\partial^2 u}{\partial x^2} - \varepsilon\frac{\partial u}{\partial t} - \varepsilon^2\left(\frac{\partial u}{\partial x}\right)^2 = e^u(1-e^{-u})\left(\frac{1}{2} - e^{-u}\right)(1 - \phi^{(0)}(x) - e^{-u}), \quad x\in(-1,1), t\in(0,T],$$

$$\frac{\partial u}{\partial x}(-1,t,\varepsilon) = 0, \quad \frac{\partial u}{\partial x}(1,t,\varepsilon) = 0, \quad t\in[0,T],$$

$$u(x,0,\varepsilon) = u_{\text{init}}(x,\varepsilon), \quad x\in[-1,1].$$

We will assume that for all $x\in[-1,1]$, the inequality $1/4 < \phi^{(0)}(x) < 1/2$ is satisfied. The members of the regular part of zero order are easily determined:

$$\bar{u}_0^{(-)}(x) = 0, \quad \bar{u}_0^{(+)}(x) = \ln 2.$$

The problem for the function $\tilde{u}(\xi,x_0)$ has the following form:

$$\frac{\partial^2\tilde{u}}{\partial\xi^2} + W\frac{\partial\tilde{u}}{\partial\xi} - \left(\frac{\partial\tilde{u}}{\partial\xi}\right)^2 = e^{\tilde{u}}(1-e^{-\tilde{u}})\left(\frac{1}{2} - e^{-\tilde{u}}\right)(1 - \phi^{(0)}(x_0) - e^{-\tilde{u}}),$$

$$\tilde{u}(0,x_0) = -\ln(1-\phi^{(0)}(x_0)), \quad \tilde{u}(-\infty,x_0) = 0, \quad \tilde{u}(\infty,x_0) = \ln 2. \tag{28}$$

By replacing $z(\xi,x_0) := z(\tilde{u}(\xi,x_0)) = 1 - e^{-\tilde{u}(\xi,x_0)}$ the problem (28) is transformed to the form

$$\frac{\partial^2 z}{\partial \xi^2} + W\frac{\partial z}{\partial \xi} = z\left(z-\frac{1}{2}\right)(z-\phi^0(x_0)), \quad z(-\infty,x_0)=0, \quad z(\infty,x_0)=\frac{1}{2}. \tag{29}$$

The solution of problem (29) is determined by the formula

$$z = \left(2 + \left(\frac{1}{\phi(x_0)} - 2\right)\exp\left\{-\frac{\xi}{2\sqrt{2}}\right\}\right)^{-1}.$$

Making the inverse substitution, we obtain the expression for the solution of the original problem (28):

$$\tilde{u}(\xi,x_0) = -\ln\left(1 - \left(2 + \left(\frac{1}{\phi(x_0)} - 2\right)\exp\left\{-\frac{\xi}{2\sqrt{2}}\right\}\right)^{-1}\right).$$

The initial problem for determining the front position in the zero approximation has the form

$$\frac{dx_0}{dt} = \sqrt{2}\left(\phi^{(0)}(x_0) - \frac{1}{4}\right), \quad x_0(0) = x_{00}. \tag{30}$$

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