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PARTIAL DIFFERENTIAL EQUATIONS

EXISTENCE OF SOLUTIONS OF THE BOUNDARY VALUE PROBLEM FOR THE DIFFUSION EQUATION WITH PIECEWISE CONSTANT ARGUMENTS

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Abstract. In this paper the boundary value problem (BVP) for diffusion equation with piecewise constant arguments is studied. By using the separation of variables method, the considered BVP is reduced to the investigation of the existence conditions of solutions of initial value problems for differential equation with piecewise constant arguments. Existence conditions of infinitely many solutions or emptiness for considered differential equation are established, and explicit formulas for these solutions are obtained. Several examples are given to illustrate the obtained results.

Keywords: *diffusion equation, piecewise constant argument, periodic solution*

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1. INTRODUCTION. PROBLEM STATEMENT

Differential equations with piecewise constant arguments are encountered in the study of hybrid systems and can model certain harmonic oscillators with almost periodic effects [1, 2]. A wide review of studies devoted to ordinary equations and partial differential equations with piecewise constant arguments is given in [3, 4].

In articles [5, 6], differential equations of special kind with piecewise constant argument are studied. Periodic (solvable) problems are reduced to a system of linear algebraic equations, all conditions for the existence of its n-periodic solutions are described, by means of which explicit formulas for solutions of differential equations are found.

Partial derivative equations with piecewise constant temporal argument arise naturally in the process of approximation [7].

In [8], the existence, oscillation and asymptotic bounds of solutions of initial problems with piecewise constant lags are studied for a partial derivative equation with piecewise constant argument.

Boundary and initial problems for the diffusion equation with piecewise constant arguments were studied in [9] and [10], respectively. The equation with piecewise constant mixed arguments of the form

$$u_t(x,t) = a^2 u_{xx}(x,t) + b u_{xx}(x,[t-1]) + c u(x,[t]) + d u(x,[t+1])$$

was considered in [11], where the questions of existence of solutions, convergence of solutions to zero, unboundedness of solutions and their oscillations were investigated.

In the paper [12], the asymptotic behavior of the solution of the diffusion equation with piecewise constant argument of generalized form is found.

In this paper, we consider a boundary value problem for the diffusion equation with piecewise constant arguments of the form [10, 13]

$$u_t(x,t) = a^2 u_{xx}(x,t) - b u_{xx}(x,[t]) - c u_{xx}(x,[t+1]), \quad 0 < x < 1,\ t > 0, \tag{1}$$

$$u(0,t) = u(1,t) = 0, \tag{2}$$

$$u(x,0) = v(x). \tag{3}$$

Adapting the method of [10, 14], we first obtain the formal solution of the problem (1)–(3) in the form of a series. For this purpose, after the separation of variables, we study the first order differential equation with piecewise constant time argument, obtain the existence condition and the explicit formula for its solution. Then, applying the method of [5, 6, 15, 16], we will find N-periodic solutions and their explicit formulas of this differential equation. In a special case, we prove the existence of an infinite number of solutions of the differential equation with piecewise constant argument, which shows the incorrectness of the result about the uniqueness given in [13].

2. DIFFERENTIAL EQUATION WITH PIECEWISE CONSTANT ARGUMENT

Let v_j be the coefficients of the sinusoidal Fourier series for the function $v(x)$, i.e.,

$$v(x) = \sum_{j=1}^{+\infty} v_j \sin(j\pi x), \quad v_j = 2\int_0^1 v(x)\sin(j\pi x)dx.$$

The solution of the problem (1)–(3) is found in the form

$$u(x,t) = \sum_{j=1}^{+\infty} T_j(t)\sin(j\pi x). \tag{4}$$

Substituting the function (4) into equation (1) and initial conditions from (3), we obtain

$$\sum_{j=1}^{\infty} \big(T_j'(t) + a^2\pi^2 j^2 T_j(t) + b\pi^2 j^2 T_j([t]) + c\pi^2 j^2 T_j([t+1])\big)\sin(j\pi x) = 0,$$

$$u(x,0) = \sum_{j=1}^{\infty} T_j(0)\sin(j\pi x) = v(x), \quad T_j(0) = v_j.$$

Hence, taking into account orthogonality of functions $\sin(n\pi x)$, we have an infinite sequence of ordinary differential equations with piecewise constant argument

$$T_j'(t) + a^2\pi^2 j^2 T_j(t) + b\pi^2 j^2 T_j([t]) + c\pi^2 j^2 T_j([t+1]) = 0, \quad t > 0, j \in \mathbb{N}, \tag{5}$$

with the initial condition

$$T_j(0) = v_j. \tag{6}$$

Definition 1. The function $T(t)$ is called a solution to the problem (5), (6), if it satisfies the following conditions:
(i) $T(t)$ is continuous with $\mathbb{R}_+$;
(ii) the derivative of $T'(t)$ exists and is continuous with $\mathbb{R}_+$, except for points $[t] \in \mathbb{R}_+$ where one-sided derivatives exist;
(iii) $T(t)$ satisfies (5) and (6) at $\mathbb{R}_+$ with a possible exception at $[t] \in \mathbb{R}_+$.

Let's denote

$$E_j(t) = e^{-a^2\pi^2 j^2 t} - \frac{b}{a^2}(1 - e^{-a^2\pi^2 j^2 t}), \quad D_j(t) = \frac{c}{a^2}(1 - e^{-a^2\pi^2 j^2 t}), \quad j \in \mathbb{N}.$$

Theorem 1. *Let a, b, c be real numbers. If $D_j(1) \neq -1$, then the equation (5) has a single solution represented at the intervals $t \in [n, n+1)$, $n = 0, 1, 2, \ldots$, in the form of*

$$T_j(t) = \left(E_j(t-n) - D_j(t-n)\frac{E_j(1)}{1 + D_j(1)}\right)\frac{E_j^n(1)}{(1 + D_j(1))^n}v_j. \tag{7}$$

Theorem 2. 1. *If* $D_j(1) = -1$ *and* $E_j(1) = 0$ *for* $j > 0$*, then the problem* (5), (6) *has infinitely many solutions. In particular, this problem has a single one-periodic and infinitely many* N*-periodic solutions,* $N = 2, 3, \ldots$

2. *Let* $D_j(1) = -1$ *and* $E_j(1) \neq 0$*. Then if* $v_j \neq 0$*, then problems* (5), (6) *have no solution. If* $v_j = 0$*, then this problem has a trivial solution.*

Example 1. Let $j = 1$, $a \in \mathbb{R}$, $c = a^2/(e^{-a^2\pi^2 j^2} - 1)$, $b = -a^2 e^{-a^2\pi^2 j^2}/(e^{-a^2\pi^2 j^2} - 1)$, $v_1 = 1$. In this case, $D_j(1) = -1$, $E_j(1) = 0$. Functions

$$F_2(t) = \begin{cases} \left(\frac{1}{1-e^{a^2\pi^2}} + \frac{e^{a^2\pi^2}}{e^{a^2\pi^2}-1} e^{-a^2\pi^2 t}\right) v_1 - \frac{1-e^{-a^2\pi^2 t}}{e^{-a^2\pi^2}-1} T_{11}(1), & t \in [0,1), \\ \left(\frac{1}{1-e^{a^2\pi^2}} + \frac{e^{a^2\pi^2}}{e^{a^2\pi^2}-1} e^{-a^2\pi^2 (t-1)}\right) T_{11}(1) - \frac{1-e^{-a^2\pi^2(t-1)}}{e^{-a^2\pi^2}-1} v_1, & t \in [1,2], \end{cases}$$

and

$$F_3(t) = \begin{cases} \left(-\frac{b}{a^2}(1 - e^{-a^2\pi^2 t}) + e^{-a^2\pi^2 t}\right) v_1 - \frac{c}{a^2}(1 - e^{-a^2\pi^2 t}) T_{11}(1), & t \in [0,1), \\ \left(-\frac{b}{a^2}(1 - e^{-a^2\pi^2 (t-1)}) + e^{-a^2\pi^2 (t-1)}\right) T_{11}(1) - \frac{c}{a^2}(1 - e^{-a^2\pi^2 (t-1)}) T_{21}(2), & t \in [1,2), \\ \left(-\frac{b}{a^2}(1 - e^{-a^2\pi^2 (t-2)}) + e^{-a^2\pi^2 (t-2)}\right) T_{21}(2) - \frac{c}{a^2}(1 - e^{-a^2\pi^2 (t-2)}) v_1, & t \in [2,3), \end{cases}$$

are two- and three-periodic solutions of the problem (5), (6) at $j = 1$, respectively, where $T_{11}(1)$, $T_{21}(2)$ are arbitrary numbers. Having chosen these constants, we give the solutions and their graphs.

The function $F_2(t)$ at $T_{11}(1) = 3$ and $a = \frac{1}{\pi}$ has the following form (Fig. 1, a)

$$F_2(t) = \begin{cases} \frac{1}{1-e} + \frac{e^{1-t}}{e-1} - \frac{3(1-e^{-t})}{e^{-1}-1}, & t \in [0,1), \\ \frac{1-e^{1-t}}{1-e^{-1}} + 3\left(\frac{1}{1-e} + \frac{e^{2-t}}{e-1}\right), & t \in [1,2]. \end{cases} \tag{8}$$

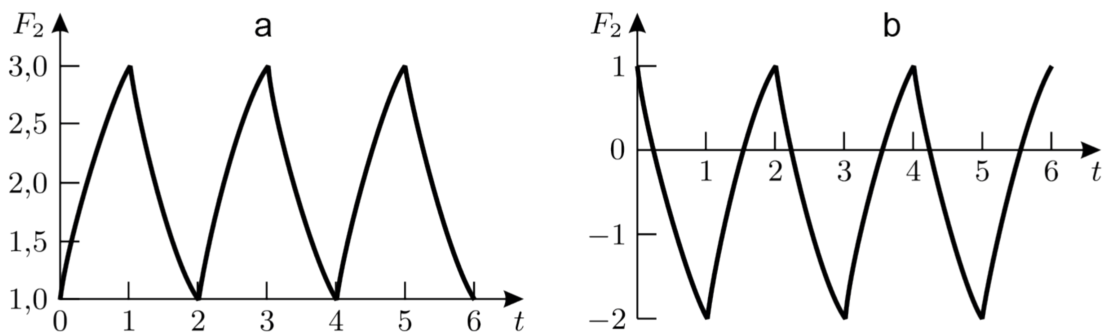

Fig. 1. Graphs of the function $F_2(t)$

and at $T_{11}(1) = -2$ and $a = \frac{1}{\pi}$ (Fig. 1, *b*)

$$F_2(t) = \begin{cases} \frac{1}{1-e} + \frac{e^{1-t}}{e-1} + \frac{2(1-e^{-t})}{e^{-1}-1}, & t \in [0,1), \\ \frac{e^{1-t}-1}{e^{-1}-1} - 2\left(\frac{1}{1-e} + \frac{e^{2-t}}{e-1}\right), & t \in [1,2]. \end{cases} \tag{9}$$

The function $F_3(t)$ at $T_{11}(1) = 2$, $T_{21}(2) = \frac{3}{2}$ and $a = \frac{1}{\pi}$ is represented as (Fig. 2, *a*)

$$F_3(t) = \begin{cases} \frac{1}{1-e} + \frac{e(2-e^{-t})}{e-1}, & t \in [0,1), \\ \frac{2}{1-e} + \frac{e(3+e^{1-t})}{2(e-1)}, & t \in [1,2), \\ \frac{3}{2(1-e)} + \frac{e(2+e^{2-t})}{2(e-1)}, & t \in [2,3], \end{cases} \tag{10}$$

at $T_{11}(1) = -2$, $T_{21}(2) = -\frac{3}{2}$ and $a = \frac{1}{\pi}$ (Fig. 2, *b*)

$$F_3(t) = \begin{cases} \frac{1}{1-e} + \frac{e(3e^{-t}-2)}{e-1}, & t \in [0,1), \\ \frac{2}{e-1} - \frac{e(3+e^{1-t})}{2(e-1)}, & t \in [1,2), \\ \frac{3}{2(e-1)} - \frac{e(5e^{2-t}-2)}{2(e-1)}, & t \in [2,3], \end{cases} \tag{11}$$

and at $T_{11}(1) = 3$, $T_{21}(2) = -4$ and $a = \frac{1}{\pi}$ (Fig. 2, *c*)

$$F_3(t) = \begin{cases} \frac{1}{1-e} + \frac{e(3-2e^{-t})}{e-1}, & t \in [0,1), \\ \frac{3}{1-e} + \frac{e(7e^{1-t}-4)}{e-1}, & t \in [1,2), \\ \frac{4}{e-1} - \frac{e(5e^{2-t}-1)}{e-1}, & t \in [2,3]. \end{cases} \tag{12}$$

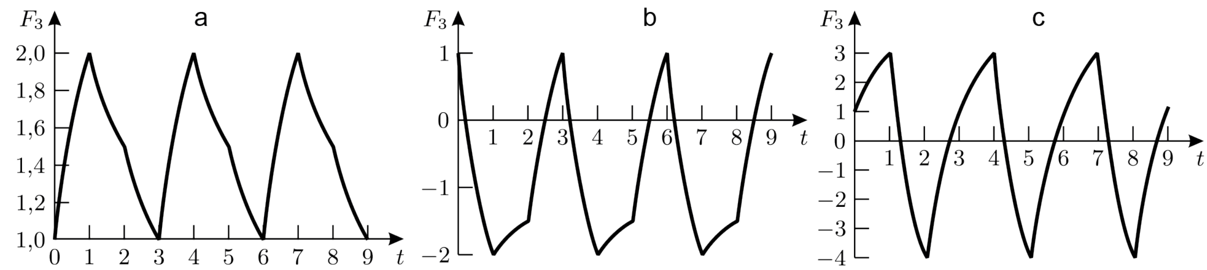

Fig. 2. Graphs of the function $F_3(t)$

Remark 1. In Example 1, the parameters of the equation satisfy the conditions of the singularity theorem from [13]. It shows the incorrectness of the results of Theorem 2 of [11], which asserts the uniqueness of the solution of the problem (5), (6).

3. PROBLEM SOLVING

Definition 2. The function $u(x,t)$ is called a solution of the problem (1)–(3), if the following conditions are satisfied:

(i) $u(x,t)$ is continuous on the set $\Omega = [0,1] \times \mathbb{R}_+$, $\mathbb{R}_+ = [0,\infty)$;

(ii) the partial derivatives of u_t and u_{xx} exist and are continuous at Ω with a possible exception at points $(x,[t]) \in \Omega$, where one-sided derivatives exist on the second argument;

(iii) $u(x,t)$ satisfies (1)–(3) at Ω with a possible exception at $(x,[t]) \in \Omega$.

Assumption. *Let the function $v(\cdot)$ have continuous derivatives up to and including third order at the segment $[0,1]$ and satisfy the conditions $v(0) = v(1) = v''(0) = v''(1) = 0$.*

Theorem 3. *Let the assumption $c \neq -a^2$ and $D_j(1) \neq -1$ at $j \in \mathbb{N}$ be satisfied. Then the problem* (1)–(3) *has a single solution represented as a series*

$$u(x,t) = \sum_{j=1}^{+\infty} \left(E_j(t-n) - D_j(t-n) \frac{E_j(1)}{1+D_j(1)} \right) \frac{E_j^n(1)}{(1+D_j(1))^n} v_j \sin(j\pi x), \quad t \in [n, n+1), n = 0,1,2,\ldots$$

Theorem 4. 1. *Let the assumption be satisfied, $D_{j_0}(1) = -1$ and $E_{j_0}(1) = 0$. Then the problem* (1)–(3) *has an infinite number of solutions represented by $t \in [n, n+1)$, $n = 0,1,2,\ldots$, as*

$$u(x,t) = \sum_{\substack{j=1 \\ j \neq j_0}}^{+\infty} \left(E_j(t-n) - D_j(t-n) \frac{E_j(1)}{1+D_j(1)} \right) \frac{E_j^n(1)}{(1+D_j(1))^n} v_j \sin(j\pi x) + T_{j_0}(t) \sin(j\pi x), \tag{12}$$

where $T_{j_0}(t)$ is an arbitrary solution of the problem (5), (6) (*see point 2 in Theorem 2*).

2. *If $D_{j_0}(1) = -1$, $E_{j_0}(1) \neq 0$ and $v_{j_0} \neq 0$ at $j = j_0$, then the problem* (1)–(3) *has no solution.*

Example 2. Let $a = 1/\pi$, $c = 2$, $b = 3$ in equation (1) and $u(x,0) = \sum_{j=1}^{5} \frac{\sin(j\pi x)}{j}$ in condition (3). Then the solution of the problem (1)–(3) has the following form (Fig. 3)

$$u(x,t) = \sum_{j=1}^{5} \left[\left(E_j(t-n) - D_j(t-n) \frac{E_j(1)}{1+D_j(1)} \right) \frac{E_j^n(1)}{(1+D_j(1))^n} v_j \right] \sin(j\pi x), \quad t \in [n, n+1), n = 0, 1, 2, \ldots$$

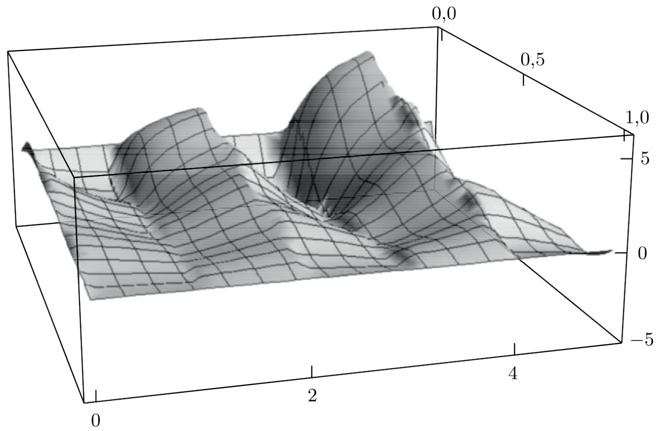

Fig. 3. Graph of the function $u(x,t)$

Example 3. Let $a \in \mathbb{R}$, $c = \frac{a^2}{e^{-a^2\pi^2 j^2} - 1}$, $b = -a^2 e^{-a^2\pi^2 j^2}/(e^{-a^2\pi^2 j^2} - 1)$, $v(x) = \sin(\pi x) + 2\sin(2\pi x)$. Then the solution of the problem (1)–(3) is defined by the formula

$$u(x,t) = T_1(t)\sin(\pi x) + 2T_2(t)\sin(2\pi x).$$

Note that $D_1(1) = -1$, $E_1(1) = 0$ and $D_2(1) = -1$, i.e., the numbers a, b and c satisfy the conditions of point 1 of Theorem 2 and Theorem 1. Therefore, according to Theorem 1, the function $T_2(t)$ has the form

$$T_2(t) = 2(E_2(t-n) - D_2(t-n)), \quad t \in [n, n+1),$$

and the function $T_1(t)$ can be defined in many ways.

Here are the graphs of $u(x,t)$ for Example 1. In the case when $T_1(t) = F_2(t)$ and $F_2(t)$ are defined by the equality (8), the graph of the function $u(x,t)$ is shown in Fig. 4, *a*; if $F_2(t)$ is defined by expression (9), then in Fig. 4, *b*. When $T_1(t) = F_3(t)$, where $F_3(t)$ is defined by equality (10), the graph of the function $u(x,t)$ is shown in Fig. 5, *a*; and if $F_3(t)$ is defined by equality (11), then in Fig. 5, *b*.

Remark 2. In Example 3, the parameters of the equation do not satisfy the conditions of Corollary 1 in [13], i.e., $a^2 + b + c = 0$. The solution of u is periodic on t. This means that the null solution of the problem (1)–(3) is not asymptotically stable. Therefore, the conditions of Corollary 1 are sufficient for the null solution to be asymptotically stable.

4. EVIDENCE FOR KEY FINDINGS

Proof of theorem 1. Let us denote by $T_{nj}(t)$ the solution of equation (5) on the interval $[n, n+1)$, i.e.

$$T_j(t) = T_{nj}(t), \quad t \in [n, n+1), \quad n = 0, 1, 2, \ldots$$

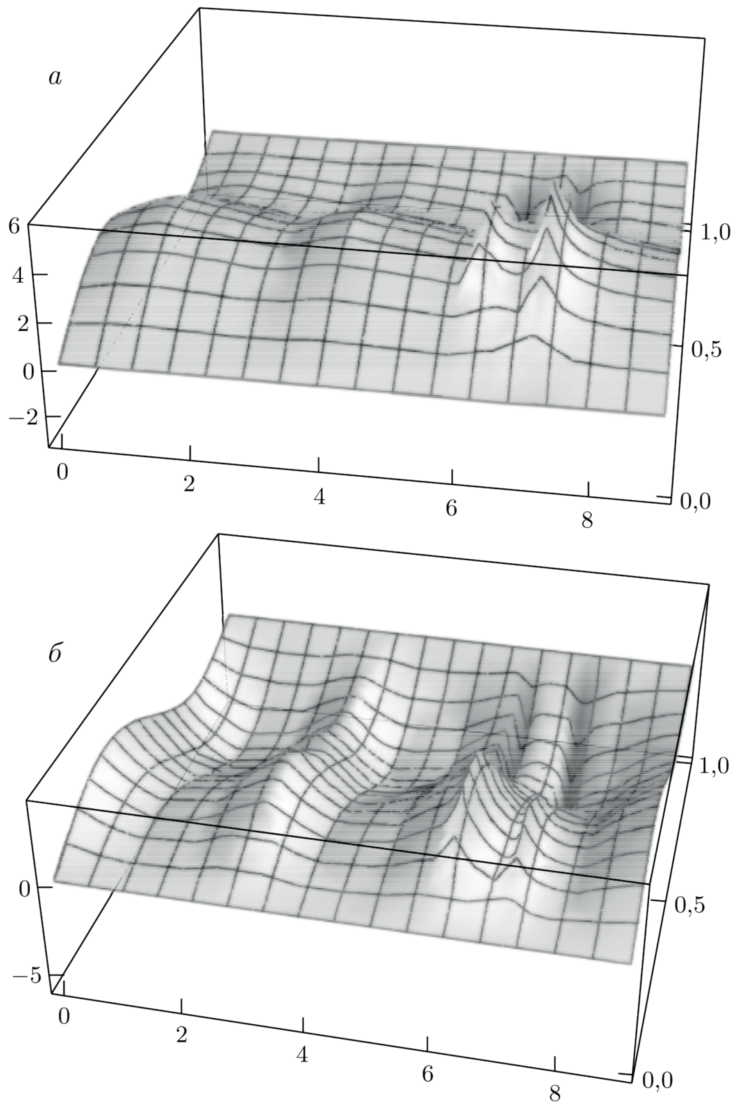

Fig. 4. Graph of the function $u(x,t)$

Then

$$T'_{nj}(t)+a^2\pi^2j^2T_{nj}(t)=-b\pi^2j^2T_{nj}(n)-c\pi^2j^2T_{nj}(n+1),\quad t\in[n,n+1). \tag{13}$$

The solution of the equation (13) is determined by the formula

$$T_{nj}(t)=-\frac{bT_{nj}(n)}{a^2}(1-e^{-a^2\pi^2j^2(t-n)})+T_{nj}(n)e^{-a^2\pi^2j^2(t-n)}-\frac{cT_{nj}(n+1)}{a^2}(1-e^{-a^2\pi^2j^2(t-n)})$$

or

$$T_{nj}(t)=E_j(t-n)T_{nj}(n)-D_j(t-n)T_{nj}(n+1),\quad t\in[n,n+1). \tag{14}$$

Putting $t=n+1$ in (14) for all $n=0,1,2,\ldots$, we get

$$T_{nj}(n+1)=E_j(1)T_{nj}(n)-D_j(1)T_{nj}(n+1).$$

Hence, taking into account $D_j(1)\neq-1$ we have

$$T_{nj}(n+1)=\frac{E_j(1)T_{nj}(n)}{1+D_j(1)}. \tag{15}$$

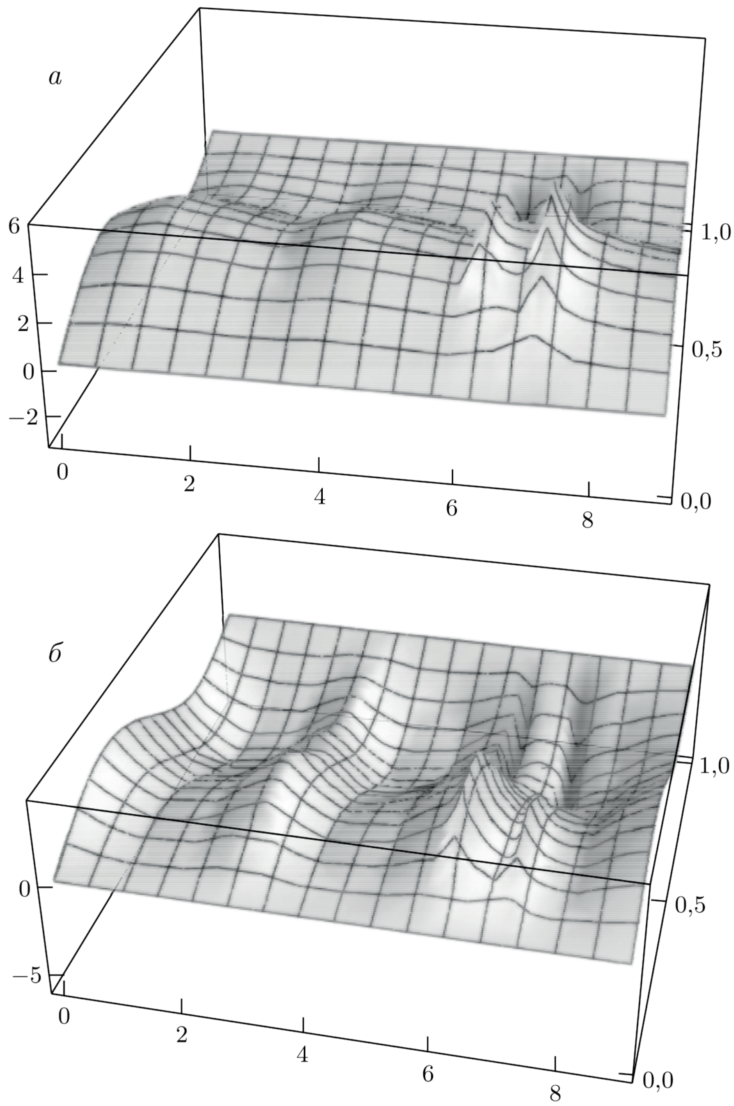

Fig. 5. Graph of the function $u(x,t)$

Then we write (14) as

$$T_{nj}(t) = E_j(t-n)T_{nj}(n) - \frac{D_j(t-n)}{1+D_j(1)}E_j(1)T_{nj}(n). \tag{16}$$

From the continuity of the function $T_j(t)$ over $t>0$ the following equations follow

$$T_{(n+1)j}(n+1) = T_j(n+1) = \lim_{t\to n+1-0} T_j(t) = T_{nj}(n+1).$$

Consequently, formula (15) can be rewritten in the form

$$T_{(n+1)j}(n+1) = \frac{E_j(1)T_{nj}(n)}{1+D_j(1)},$$

from where

$$T_{nj}(n) = \frac{E_j(1)}{1+D_j(1)}T_{(n-1)j}(n-1) = \frac{E_j^2(1)}{(1+D_j(1))^2}T_{(n-2)j}(n-2) = \cdots = \frac{E_j^n(1)}{(1+D_j(1))^n}T_{0j}(0),$$

or

$$T_{nj}(n) = \frac{E_j^n(1)}{(1 + D_j(1))^n} T_{0j}(0).$$

Thus, the solution $T_{nj}(t)$, defined by the formula (16), is represented only via $T_{0j}(0)$:

$$T_{nj}(t) = \left(E_j(t-n) - D_j(t-n) \frac{E_j(1)}{1 + D_j(1)} \right) \frac{E_j^n(1)}{(1 + D_j(1))^n} T_{0j}(0).$$

The equality $T_{0j}(0) = v_j$ completes the proof of the theorem.

Proof of theorem 2. 1. Let $D_j(1) = -1$, $E_j(1) = 0$. Construct the function $T_j(t) = T_{nj}(t)$, $t \in [n, n+1)$, $n = 0, 1, 2, \ldots$, as follows. Function

$$T_{0j}(t) = E_j(t) T_{0j}(0) - D_j(t) C_{0j}, \quad t \in [0, 1),$$

satisfies the equation (5), where $T_{0j}(0) = v_j$ and C_{0j} are arbitrary numbers. Since $D_j(1) = -1$ and $E_j(1) = 0$, there is an equality $T_{0j}(1) = \lim_{t \to 1} T_{0j}(t) = C_{0j}$. It is easy to check that the function

$$T_{1j}(t) = E_j(t-1) T_{1j}(1) - D_j(t-1) C_{1j}, \quad t \in [1, 2),$$

satisfies equation (5), where C_{1j} is an arbitrary number.

By virtue of continuity of the function $T_j(t)$ we have

$$T_j(1) = T_{1j}(1) = \lim_{t \to 1-0} T_{0j}(t) = T_{0j}(1).$$

The equalities $D_j(1) = -1$ and $E_j(1) = 0$ give $T_{1j}(2) = \lim_{t \to 2} T_{1j}(t) = C_{1j}$.

Function

$$T_{nj}(t) = E_j(t-n) T_{nj}(n) - D_j(t-n) C_{nj}, \quad \text{at } (n, n+1), n \in \mathbb{N},$$

satisfies the equation (5), where C_{nj} is an arbitrary number. Clearly,

$$T_j(n) = T_{nj}(n) = \lim_{t \to n-0} T_{(n-1)j}(t) = T_{(n-1)j}(n).$$

Similarly, from the equalities $D_j(1) = -1$ and $E_j(1) = 0$, we obtain $T_{nj}(n) = \lim_{t \to n+1} T_{nj}(t) = C_{nj}$. After the construction of the function

$$T_j(t) = T_{nj}(t), \quad t \in [n, n+1), n = 0, 1, 2, \ldots,$$

appears the solution of the problem (5), (6). Since the constants $C_{0j}, C_{1j}, \ldots, C_{nj}, \ldots$ are arbitrary, the problem has an infinite number of solutions.

Let $T_j(t)$ be a one-periodic solution of the problem (5), (6), then it can be represented as

$$T_j(t) = T_0 j(t) = E_j(t) T_0 j(0) - D_j(t) C_0 j, \quad t \in [0, 1].$$

Since the function $T_j(t)$ is one-periodic and $T_0 j(1) = C_0 j$, then $T_0 j(0) = T_0 j(1)$, $C_0 j(1) = T_0 j(0) = v_j$. This shows the uniqueness of the one-periodic solution (5), (6).

Let $T_j(t)$ be a two-periodic solution of the problem (5), (6). Then the function $T_j(t)$ on $[0, 2]$ has the form

$$T_j(t) = \begin{cases} E_j(t) T_0 j(0) - D_j(t) T_1 j(1), & t \in [0, 1), \\ E_j(t-1) T_1 j(1) - D_j(t-1) C_1 j, & t \in [1, 2), \end{cases}$$

where $T_0 j(0) = v_j$, $T_1 j(1)$ is an arbitrary number. From the periodicity of $T_j(t)$ it follows that $T_j(0) = T_0 j(0) = T_j(2) = C_1 j$. This shows that the problem (5), (6) has infinitely many two-periodic solutions.

Let $T_j(t)$ be the N-periodic solution of the problem (5), (6). The function $T_j(t)$ on the interval $[0, N]$ has the form

$$T_j(t) = \begin{cases} E_j(t) v_j - D_j(t) T_1 j(1), & t \in [0, 1), \\ E_j(t-1) T_1 j(1) - D_j(t-1) T_2 j(2), & t \in [1, 2), \\ \vdots & \\ E_j(t-N+2) T_{(N-1,j)}(N-2) - D_j(t-N+2) T_{(N-1,j)}(N-1), & t \in [N-2, N-1), \\ E_j(t-N+1) T_{(N-1,j)}(N-1) - D_j(t-N+1) v_j, & t \in [N-1, N), \end{cases}$$

where $T_1 j(1), T_2 j(2), \dots, T_{(N-1,j)}(N-1)$ are arbitrary numbers.

2. Suppose that the function $T_j(t)$ is a solution of the problem (5), (6). Then, according to (14), the following equality holds

$$T_n j(t) = E_j(t-n) T_n j(n) - D_j(t-n) T_n j(n+1), \quad t \in [n, n+1).$$

Hence at $t = n+1$ taking into account $D_j(1) = -1$, we have $E_j(1) T_n j(n) = 0$ for all $n = 0, 1, 2, \dots$ Therefore, $T_n j(n) = 0$ for all $n = 0, 1, 2, \dots$, since $E_j(1) \neq 0$, i.e., the equation has only a trivial solution. Hence, if $T_j(0) = v_j = T_0 j(0) \neq 0$, then the problem (5), (6) has no solution.

Proof of Theorem 3. First, prove uniform convergence in any closed set $\overline{\Lambda} \subset [0,1] \times \mathbb{R}_+$ of the following series:

$$\sum_{j=1}^{+\infty} T_j(t) \sin(j\pi x), \tag{17}$$

$$\sum_{j=1}^{+\infty} T_j'(t) \sin(j\pi x), \tag{18}$$

$$\sum_{j=1}^{+\infty} \pi^2 j^2 T_j(t) \sin(j\pi x), \tag{19}$$

where $T_j(t)$ is the solution of the problem (5), (6), and at $[n, n+1)$, $n = 0, 1, 2, \dots$, the functions $T_j(t)$, $T_j'(t)$ are represented, respectively, as (7) and

$$T_j'(t) = -\left(a^2 + b + c\frac{E_j(1)}{1 + D_j(1)}\right) \pi^2 j^2 e^{-a^2\pi^2 j^2 (t-n)} \frac{E_j^n(1)}{(1 + D_j(1))^n} v_j.$$

According to the assumption there is equality

$$v_j = -\frac{2v_j'''}{\pi^3 j^3}, \quad v_j''' = \int_0^1 v'''(x) \cos(j\pi x) dx, \quad j = 1, 2, \dots$$

The continuity of the function $v'''(x)$ implies the convergence of the series $\sum_{j=1}^{+\infty} (v_j''')^2$. Hence, taking into account the Cauchy-Bunyakovsky inequality, we have

$$\left| \sum_{j=1}^{+\infty} j^2 v_j \right| = \frac{2}{\pi^3} \left| \sum_{j=1}^{+\infty} \frac{v_j'''}{j} \right| < +\infty. \tag{20}$$

Since $0 \leq 1 - e^{-a^2\pi^2 j^2 t} \leq 1$, the inequalities are true for all $t \in [0, \infty)$ and $j \in \mathbb{N}$:

$$|E_j(t)| \leq 1 + \frac{|b|}{a^2}, \quad |D_j(t)| < \frac{|c|}{a^2}. \tag{21}$$

Note that $\lim_{j\to\infty} D_j(1) = c/a^2$, so given $D_j(1) \neq -1$ and $c \neq -a^2$ there exists a number $\rho > 0$ such that

$$|1 + D_j(1)| \geq \rho, \quad j \in \mathbb{N}. \tag{22}$$

Using inequalities (21) and (22), we obtain uniform estimates for $T_j(t)$ and $T_j'(t)$:

$$|T_j(t)| \leq C_1 \left(\frac{1 + \frac{|b|}{a^2}}{\rho} \right)^n |v_j|, \quad t \in [n, n+1), \tag{23}$$

$$|T_j'(t)| \leq C_2 \left(\frac{1 + \frac{|b|}{a^2}}{\rho} \right)^n \pi^2 j^2 |v_j|, \quad t \in [n, n+1), \tag{24}$$

where

$$C_1 = 1 + \frac{|b|}{a^2} + \frac{|c|}{a^2} \frac{1 + \frac{|b|}{a^2}}{\rho},$$

$$C_2 = a^2 + |b| + |c| \frac{1 + \frac{|b|}{a^2}}{\rho}.$$

Let $m = 1 + \sup_{(x,t)\in\overline{\Lambda}} t$. Then from (23) and (24) for all $(x,t) \in \overline{\Lambda}$, the series (17)–(19) are evaluated as follows:

$$\left|\sum_{j=1}^{+\infty} T_j(t)\sin(j\pi x)\right| \le C_1 \left(\frac{1+\frac{|b|}{a^2}}{\rho}\right)^m \sum_{j=1}^{+\infty} |v_j|,$$

$$\left|\sum_{j=1}^{+\infty} T_j'(t)\sin(j\pi x)\right| \le C_2 \left(\frac{1+\frac{|b|}{a^2}}{\rho}\right)^m \pi^2 \sum_{j=1}^{+\infty} j^2|v_j|,$$

$$\left|\sum_{j=1}^{+\infty} \pi^2 j^2 T_j(t)\sin(j\pi x)\right| \le C_1 \left(\frac{1+\frac{|b|}{a^2}}{\rho}\right)^m \pi^2 \sum_{j=1}^{+\infty} j^2|v_j|.$$

Hence and from (20), we obtain uniform convergence of series (17)–(19) in any closed set $\overline{\Lambda} \subset [0,1]\times\mathbb{R}_+$.

Thus, the function $u(x,t) = \sum_{j=1}^{+\infty} T_j(t)\sin(j\pi x)$ is continuous on the set $\Omega = [0,1]\times\mathbb{R}_+$; and the partial derivatives $u_t = \sum_{j=1}^{+\infty} T_j'(t)\sin(j\pi x)$, $u_{xx} = \sum_{j=1}^{+\infty} \pi^2 j^2 T_j(t)\sin(j\pi x)$ exist and are continuous on Ω with a possible exception at points $(x,[t]) \in \Omega$, where one-sided derivatives exist on the second argument.

Since $D_j(1) \ne -1$ for each $j \in \mathbb{N}$, then by Theorem 1 the problem (5), (6) has a single solution $T_j(t)$ for each $j \in \mathbb{N}$. Hence, the function $u(x,t)$, defined by the formula (4), satisfies the equalities (5), (6) in Ω with possible exceptions at the points $(x,[t]) \in \Omega$ and is the only solution of the problem (1)–(3).

Proof of Theorem 4. 1. Let $D_{j_0}(1) = -1$ and $E_{j_0}(1) = 0$ for some $j = j_0$. Then $D_j(1) > -1$ at $j < j_0$ and $D_j(1) < -1$ at $j > j_0$. Hence we have

$$|1 + D_j(1)| \ge \rho_1$$

for some number $\rho_1 > 0$ and for all $j \in \mathbb{N}\backslash\{j_0\}$.

By Theorem 1, the problem (5), (6) is solvable for $j \ne j_0$ and the solution of $T_j(t)$ at $j \ne j_0$ is of the form (7). Since $D_{j_0}(1) = -1$; and $E_{j_0}(1) = 0$, then by point 1 of Theorem 2, the problem (5), (6) has infinitely many solutions. Let us denote by $T_{j_0}(\cdot)$ the solution of the problem (5), (6) for $j = j_0$. Then from (4) the solution of the boundary value, problem (1)–(3) has the form (12). The uniform convergence of this series to a continuous function $u(x,t)$ in any closed set $\overline{\Lambda} \subset [0,1]\times\mathbb{R}_+$ and the existence of continuous partial derivatives of u_t and u_{xx} on Ω with a possible exception at the points $(x,[t]) \in \Omega$, where one-sided derivatives exist on the second argument, are proved similarly as in the proof of Theorem 3.

2. If $D_{j_0}(1) = -1$, $E_{j_0}(1) \ne 0$ and $v_{j_0} \ne 0$, then by Theorem 2 the problem (5), (6) has no solution at $j = j_0$. Hence, according to (4), the boundary value problem (1)–(3) has no solution.

CONFLICT OF INTERESTS

The authors of this paper declare that they have no conflict of interests.

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