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INSTABILITY AND STABILIZATION OF SOLUTIONS OF A STOCHASTIC MODEL OF VISCOELASTIC FLUID DYNAMICS

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Abstract. The instability and stability of solutions of the stochastic system describing the flow of a viscoelastic liquid are investigated. It is shown that for certain values of the parameters included in the equations of the system, the existence of unstable and stable invariant spaces. For unstable case, the stabilization problem is solved based on the feedback principle.

Keywords: *Sobolev type stochastic equation, invariant space, stabilization*

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1. INTRODUCTION. PROBLEM STATEMENT

Let $D \subset \mathbb{R}^n$ be a bounded region with boundary ∂D of class C^∞. Let's consider the following model of viscoelastic incompressible fliud flow in $D \times \mathbb{R}$:

$$(\lambda - \nabla^2)u_t = \nu\nabla^2 u - \nabla p, \quad \nabla u = 0; \tag{1}$$

$$u(x,t) = 0, \quad (x,t) \in \partial D \times \mathbb{R}; \quad u(x,0) = u_0, \quad x \in D,$$

where $u(x,t) = (u_1(x,t), u_2(x,t), \ldots, u_n(x,t))$ and p are the velocity and pressure vectors, respectively. System (1) is a linearization of the system

$$(\lambda - \nabla^2)u_t = \nu\nabla^2 u - (u\nabla)u - \nabla p, \quad \nabla u = 0,$$

obtained by A.P. Oskolkov [1] to describe the flow of viscous liquids possessing elasticity property. Redefining ∇p by p, we write the system (1) in the following form

$$(\lambda - \nabla^2)u_t = \nu\nabla^2 u - p, \quad \nabla(\nabla)u = 0. \tag{2}$$

Here, the parameter λ characterizes elastic properties, and ν characterizes viscous properties. In [2], it was shown that the parameter λ can take negative values. In [3], a physical model of fluid flow with negative viscosity was constructed, so we will assume further that $\nu \in \mathbb{R}$.

It has been experimentally shown that the flow of polymer solutions and melts has the property of instability (see the review [4] and the bibliography therein). This instability can have a significant impact on the material processing technologies and the quality of final products. One of the causes of this instability is inlet pulsations ("inlet instability"). Note that polymer solution and melts are viscoelastic fluids. We will investigate the instability and stability of the flow of an incompressible viscoelastic fluid described by system (2) with stochastic initial data. As an initial condition, we choose a random variable

$$\eta(0) = \eta_0, \tag{3}$$

and we will consider the system (2) as a stochastic equation of the Sobolev type

$$L\overset{\circ}{\eta} = M\eta. \tag{4}$$

The solution of the stochastic equation is a stochastic process that is not differentiable at any point. Therefore, as the derivative of the stochastic process η we will consider the Nelson–Glicklich derivative $\overset{\circ}{\eta}$ [5]. At present, a large number of works devoted to the study of stochastic equations of Sobolev type are known. Let us note some of them. The solvability of the Cauchy problem for equation (4) is studied in [6] (in the case of a relatively bounded operator), [7] (in the case of a relatively sectorial operator) and [8] (in the case of a relatively radial operator). In [9], stochastic linear equations of Sobolev type of high order are considered; in [10, 11], the "initial-finite" problem for equation (4) is investigated; in [12], the stability of equation (4) is studied. In [13–15], numerical experiments on finding stable and unstable solutions of stochastic nonclassical equations that can be represented in the form (4) were carried out.

The deterministic system (2) has been studied in various aspects. The study of its solvability was started in [1] under the condition that the parameters $\lambda, \nu \in \mathbb{R}_+$. In [16], the question of existence of solutions was solved using the phase space method at $\lambda \in \mathbb{R}\backslash\{0\}$ and $\nu \in \mathbb{R}_+$; the existence of an exponential dichotomy of solutions was shown. In [17], the initial-final problem for a linear system of Oskolkov equations was studied.

The purpose of this paper is to study the instability and stability of solutions of the stochastic system (2) in the case when the parameters $\lambda, \nu \in \mathbb{R}\backslash\{0\}$, and to solve the problem of stabilization of unstable solutions. In Section 2, we give abstract results on the existence of solutions of equation (4) and their stability. In Section 3, the system (2) in spaces of random K-values is considered, and the solvability of the stochastic system (2) is shown. In Section 4, the existence of stable and unstable invariant spaces is proved, the problem of stabilization of unstable solutions by the feedback principle is solved.

2. INVARIANT SPACES OF THE STOCHASTIC EQUATION OF SOBOLEV TYPE

By $\mathbf{L}_2$ we denote the space of random variables ξ with zero mathematical expectation and finite variance, and by $\mathbf{CL}_2$ we denote the space of continuous stochastic processes η. We fix $\eta \in \mathbf{CL}_2$ and $t \in \mathfrak{I}$, where $\mathfrak{I}$ is some interval, and through $\mathcal{N}_t^\eta$ we denote the σ-algebra generated by η and $\mathbf{E}_t^\eta = \mathbf{E}(\cdot|\mathcal{N}_t^\eta)$. Let us define the *Nelson–Glicklich derivative* of the stochastic process η at the point $t \in \mathfrak{I}$ as the limit

$$\overset{\circ}{\eta}(\,\cdot\,,\omega) = \frac{1}{2}\left[\lim_{\Delta t\to+0}\mathbf{E}_t^\eta\left(\frac{\eta(t+\Delta t,\cdot)-\eta(t,\cdot)}{\Delta t}\right) + \lim_{\Delta t\to+0}\mathbf{E}_t^\eta\left(\frac{\eta(t,\cdot)-\eta(t-\Delta t,\cdot)}{\Delta t}\right)\right],$$

if it converges in the uniform metric on $\mathbb{R}$. By $\mathbf{C}^l\mathbf{L}_2$ we denote the space of stochastic processes whose Nelson–Glicklich derivatives are a.s. (almost surely) continuous on $\mathfrak{I}$ up to order l inclusive.

Let $\mathfrak{U}$ and $\mathfrak{F}$ be real separable Hilbert spaces, and let $\{\varphi_k\}$ and $\{\psi_k\}$ denote bases in $\mathfrak{U}$ and $\mathfrak{F}$, respectively. Choose a sequence of random variables $\{\xi_k\} \subset \mathbf{L}_2$ ($\{\zeta_k\} \subset \mathbf{L}_2$), such that $\|\xi_k\|_{\mathbf{L}_2} \le \text{const}$ ($\|\zeta_k\|_{\mathbf{L}_2} \le \text{const}$). The elements of the space $\mathbf{U_K L}_2$ ($\mathbf{F_K L}_2$) of ($\mathfrak{U}$-valued ($\mathfrak{F}$-valued)) random $\mathbf{K}$-variables are vectors $\xi = \sum_{k=1}^\infty \lambda_k\xi_k\varphi_k$ ($\zeta = \sum_{k=1}^\infty \lambda_k\zeta_k\psi_k$), where the sequence $\mathbf{K} = \{\lambda_k\} \subset \mathbb{R}_+$ satisfies $\sum_{k=1}^\infty \lambda_k^2 < +\infty$. The following holds:

Lemma 1 [18]. *The operator $A \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$ (linear and continuous) if and only if the operator $A \in \mathcal{L}(\mathbf{U_K K}_2;\mathbf{F_K L}_2)$.*

Let the operators $L \in \mathcal{L}(\mathbf{U_K L}_2;\mathbf{F_K L}_2)$, $M \in Cl(\mathbf{U_K L}_2;\mathbf{F_K L}_2)$. Denote by

$$\rho^L(M) = \{\mu \in \mathbb{C}\colon (\mu L - M)^{-1} \in \mathcal{L}(\mathfrak{F};\mathfrak{U})\}$$

the *L-resolvent set*, and by $\sigma^L(M) = \mathbb{C}\setminus\rho^L(M)$ the *L-spectrum of the operator M*. If the operator M is (L,σ)-bounded, i.e., its L-spectrum is bounded, then there exist projectors

$$P = \frac{1}{2\pi i}\int_\gamma (\mu L - M)^{-1}L\,d\mu \in \mathcal{L}(\mathbf{U_K L}_2), \quad Q = \frac{1}{2\pi i}\int_\gamma L(\mu L - M)^{-1}\,d\mu \in \mathcal{L}(\mathbf{F_K L}_2). \tag{1}$$

Here, the contour $\gamma \subset \mathbb{C}$ bounds a region containing $\sigma^L(M)$.

The projectors (5) split the spaces $\mathbf{U_K}\,\mathbf{L}_2 = \mathbf{U}_\mathbf{K}^0\,\mathbf{L}_2 \oplus \mathbf{U}_\mathbf{K}^1\,\mathbf{L}_2$ and $\mathbf{F_K L}_2 = \mathbf{F}_\mathbf{K}^0\mathbf{L}_2 \oplus \mathbf{F}_\mathbf{K}^1\mathbf{L}_2$, where $\mathbf{U}_\mathbf{K}^0\mathbf{L}_2$ ($\mathbf{U}_\mathbf{K}^1\mathbf{L}_2$) $= \ker P$ ($\operatorname{im} P$), $\mathbf{F}_\mathbf{K}^0\mathbf{L}_2$ ($\mathbf{F}_\mathbf{K}^1\mathbf{L}_2$) $= \ker Q$ ($\operatorname{im} Q$). Let L_k (M_k) denote the restriction of the operator L (M) to $\mathbf{U}_\mathbf{K}^k\mathbf{L}_2$, $k = 0,1$. The operators $L_k(M_k) \in \mathcal{L}(\mathbf{U}_\mathbf{K}^k\mathbf{L}_2, \mathbf{F}_\mathbf{K}^k\mathbf{L}_2)$, $k = 0,1$; there exist operators $M_0^{-1} \in \mathcal{L}(\mathbf{F}_\mathbf{K}^0\mathbf{L}_2, \mathbf{U}_\mathbf{K}^0\mathbf{L}_2)$, $L_1^{-1} \in \mathcal{L}(\mathbf{F}_\mathbf{K}^1\mathbf{L}_2, \mathbf{U}_\mathbf{K}^1\mathbf{L}_2)$. Consider the operators $H = L_0^{-1}M_0$ and $S = L_1^{-1}M_1$. If the operator M is (L,p)-bounded and $H \equiv \mathbb{O}$, $p = 0$ or $H^p \ne \mathbb{O}$, $H^{p+1} \equiv \mathbb{O}$, then it is called an *$(L,p)$-bounded operator*.

We call a stochastic $\mathbf{K}$-process $\eta \in \mathbf{C}^1(\mathcal{J};\mathbf{U_K L}_2)$ is called a *solution of equation* (4) if a.s. all its trajectories satisfy equation (4) at all $t \in \mathcal{J}$. A solution $\eta = \eta(t)$ of equation (4) a *solution of the Cauchy problem* (3), (4) if equality (3) holds for some random **L**-variable $\eta_0 \in \mathbf{U_L L}_2$. The set $\mathbf{P} \subset \mathbf{U_L L}_2$ is called the *stochastic phase space* of equation (4) if a.s. any trajectory of the solution $\eta = \eta(t)$ lies in **P** pointwise, i.e., $\eta(t) \in \mathbf{P}$ for all $t \in \mathcal{J}$, and for a.e. $\eta_0 \in \mathbf{P}$ there exists a solution to the problem (3), (4).

Theorem 1 [7]. Let the operator M be (L, p)-bounded, $p \in \{0\} \cup \mathbb{N}$. Then the group

$$U^t = \frac{1}{2\pi i} \int\limits_{\gamma} R^L_\mu(M) e^{\mu t} \, d\mu$$

is the holomorphic resolving group of equation (4)*; the subspace* $\mathbf{U^1_K L}_2$ *is the phase space of equation*(4).

Definition. An invariant subspace $1\mathbf{I}^{s(u)} \subset \mathbf{P}$ is called the *stable (unstable)* invariant space of equation (4) if the condition

$$\|\eta^{s(u)}(t)\|_{\mathbf{U_K L}_2} \le N e^{-\nu(s-t)} \|\eta^{s(u)}(s)\|_{\mathbf{U_K L}_2}$$

holds for $s \ge t$ $(t \ge s)$, $\eta^{s(u)} = \eta^{s(u)}(t) \in \mathbf{I}^1$, and some $N, \alpha \in \mathbb{R}_+$. If the phase space splits into a direct sum $\mathbf{P} = \mathbf{I}^1 \oplus \mathbf{I}^2$, then the solutions $\eta = \eta(t)$ of equation (4) have an *exponential dichotomy*.

Let the operator M be (L, p)-bounded, $p \in \{0\} \cup \mathbb{N}$ and the relative spectrum has the form

$$\sigma^L(M) = \sigma^L_s(M) \oplus \sigma^L_u(M), \tag{6}$$

where

$$\sigma^L_s(M) = \{\mu \in \sigma^L(M) : \mathrm{Re}\,\mu < 0\} \ne \varnothing, \quad \sigma^L_u(M) = \{\mu \in \sigma^L(M) : \mathrm{Re}\,\mu > 0\} \ne \varnothing.$$

Then there are projectors

$$P_{l(r)} = \frac{1}{2\pi i} \int\limits_{\gamma_{l(r)}} R^L_\mu(M) \, d\mu \in \mathcal{L}(\mathbf{U_K L}_2),$$

where the contour $\gamma_{l(r)}$ lies in the left (right) half-plane of the complex plane and bounds a part of the L-spectrum of the operator M $\sigma^L_{s(u)}(M)$. Let us denote by $\mathbf{I}^{(s(u))} = \mathrm{im}\, P_{l(r)}$.

Let the operator M be (L, p)-bounded and condition (6) be satisfied, then $\mathbf{U^1_K L}_2 = \mathbf{I}^s \oplus \mathbf{I}^u$. Equation (4) will be considered as a system

$$H\overset{\circ}{\eta}{}^0 = \eta^0, \tag{7}$$

$$L_s\overset{\circ}{\eta}{}^s = M_s \eta^s, \tag{8}$$

$$L_u\overset{\circ}{\eta}{}^u = M_u \eta^u. \tag{9}$$

Remark 1. The operator M is (L, p)-bounded, so the operator H is nilpotent of degree p. Then the solution of equation (7) $\eta^0 = 0$ and the stochastic process $\eta = \eta^s + \eta^u$ is a solution of equation (4), where η^s and η^u are solutions of equations (8) and (9), respectively. Thus, the question of stability and instability of solutions of equation (4) is reduced to the study of stability and instability of solutions of η^s and η^u.

Theorem 2. *Let the operator* M *be* (L, p)*-bounded,* $p \in \{0\} \cup \mathbb{N}$ *and condition* (6) *be satisfied, then the solutions* $\eta = \eta(t)$ *of equation* (4) *have an exponential dichotomy.*

Proof. The solving groups of equations (8) and (9) have the form

$$U^t_l = \frac{1}{2\pi i} \int\limits_{\gamma_l} (\mu L_s - M_s)^{-1} L_s e^{\mu t} \, d\mu, \quad U^t_r = \frac{1}{2\pi i} \int\limits_{\gamma_r} (\mu L_u - M_u)^{-1} L_u e^{\mu t} \, d\mu.$$

Let's denote $\alpha = -\max_{\mu \in \sigma^L_l(M)} \mathrm{Re}\,\mu$ and $\beta = \min_{\mu \in \sigma^L_r(M)} \mathrm{Re}\,\mu$. Then

$$\|U^t_l\|_{\mathcal{L}(\mathbf{U_K L}_2)} \le e^{-\alpha t} \int\limits_{\gamma_l} \|(\mu L_s - M_s)^{-1} L_s\|_{\mathcal{L}(\mathbf{U_K L}_2)} \, |d\mu| \le N_l e^{-\alpha t}, \tag{10}$$

$$\|U^t_r\|_{\mathcal{L}(\mathbf{U_K L}_2)} \le e^{\beta t} \int\limits_{\gamma_r} \|(\mu L_r - M_r)^{-1} L_r\|_{\mathcal{L}(\mathbf{U_K L}_2)} \, |d\mu| \le N_r e^{\beta t}. \tag{11}$$

Let $s \geq t$. Then the solution η^s of equation (8) can be written as $\eta^s(t) = U_l^{s-t}\eta^s(s)$. By virtue of (10) we have the relations

$$\|\eta^s(t)\|_{U_K L_2} = \|U_l^{s-t}\eta^s(s)\|_{U_K L_2} \leq N_l e^{-\alpha(s-t)}\|\eta^s(s)\|_{U_K L_2}.$$

Further, let $t \geq s$. Then the solution η^u of equation (9) is: $\eta^u(t) = U_r^{t-s}\eta^u(s)$. By virtue of (11) we have

$$\|\eta^u(t)\|_{U_K L_2} = \|U_r^{t-s}\eta^u(s)\|_{U_K L_2} \leq N_r e^{\beta(t-s)}\|\eta^u(s)\|_{U_K L_2} = N_r e^{-\beta(s-t)}\|\eta^s(s)\|_{U_K L_2}.$$

The theorem is proved.

Corollary 1. *By the conditions of Theorem* 2, *any trajectory of the solution* $\eta^{s(u)} = \eta^{s(u)}(t)$ *of equation* (8) (*equation* (9)) *a.c. lies in the stable (unstable) invariant space* $I^{s(u)}$ *pointwise, i.e.,* $\eta^{s(u)}(t) \in \mathbf{I}^{s(u)}$ *at all* $t \in \mathbb{R}$.

Remark 2. If $\sigma^L_{s(u)}(M) = \varnothing$, то $\mathbf{I}^{s(u)} = \{0\}$.

3. STOCHASTIC SYSTEMTYPE

We will consider the system (2) in the spaces of random K-values. For this purpose we denote by $\mathbb{H}^2 = (W_2^2(D))^n$, $\mathring{\mathbb{H}}^1 = (\mathring{W}_2^2(D))^n$, $\mathbb{L}^2 = (L_2(D))^n$. The closure $\{u \in C^\infty : \nabla u = 0\}$ of the lineal $\mathbb{L}^2$ is denoted by $\mathbb{H}_\sigma$, and there exists a splitting $\mathbb{L}^2 = \mathbb{H}_\sigma \oplus \mathbb{H}_\pi$, where $\mathbb{H}_\pi$ is an orthogonal complement to $\mathbb{H}_\sigma$, and $\Pi\colon \mathbb{L}^2 \to \mathbb{H}_\pi$ is an otroprojector corresponding to this complement. The contraction of the projector Π onto $\mathbb{H}^2 \cap \mathring{\mathbb{H}}^1 \subset \mathbb{L}^2$ is a continuous operator $\Pi\colon \mathbb{H}^2 \cap \mathring{\mathbb{H}}^1 \to \mathbb{H}^2 \cap \mathring{\mathbb{H}}^1$. Let us represent the space $\mathbb{H}^2 \cap \mathring{\mathbb{H}}^1 = \mathbb{H}^2_\sigma \oplus \mathbb{H}^2_\pi$, where $\ker \Pi = \mathbb{H}^2_\sigma$, $\operatorname{im} \Pi = \mathbb{H}^2_\pi$. Let us denote $\Sigma = \mathbb{I} - \Pi$. Let us put $\mathfrak{U} = \mathbb{H}^2_\sigma \times \mathbb{H}^2_\pi \times \mathbb{H}_\pi$ and $\mathfrak{F} = \mathbb{H}_\sigma \times \mathbb{H}_\pi \times \mathbb{H}_\pi$. The element $u \in \mathfrak{U}$ has the form $u = (u_\sigma, u_\pi, p)$.

Lemma 2 [2]. *The formula* $A = (-\nabla^2)^n : \mathbb{H}^2 \cap \mathring{\mathbb{H}}^1 \to \mathbb{L}^2$ *defines a linear continuous operator with positive discrete spectrum* $\sigma(A)$, *condensing to the point* $+\infty$, *and the mapping* $A\colon \mathbb{H}^2_{\sigma(\pi)} \to \mathbb{H}^2_{\sigma(\pi)}$ *is bijective.*

The formula $B\colon u \to -\nabla(\nabla u)$ *defines a linear continuous surjective operator* $B\colon \mathbb{H}^2 \cap \mathring{\mathbb{H}}^1 \to \mathbb{H}^2_\pi$, *with* $\ker B = \mathbb{H}^2_\sigma$.

The spaces $W_2^2(D)$, $L_2(D)$ are separable Hilbert spaces, so the spaces $\mathfrak{U}$, $\mathfrak{F}$ are separable Hilbert spaces as their finite products. Let us construct the spaces $\mathbf{U_K L}_2$ and $\mathbf{F_K L}_2$. The operators $L, M \in \mathcal{L}(\mathbf{U_K K}_2; \mathbf{F_K L}_2)$ are defined as

$$L = \begin{pmatrix} \Sigma(\lambda\mathbb{I} + A) & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \Pi(\lambda\mathbb{I} + A) & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} \end{pmatrix}, \quad M = \begin{pmatrix} -\nu\Sigma A & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & -\nu\Pi A & -\Pi \\ \mathbb{O} & \Pi B & \mathbb{O} \end{pmatrix}.$$

Then the stochastic system of equations (2) can be viewed as a stochastic linear equation (4). The following is true

Lemma 3. *Operators* $L, M \in \mathcal{L}(\mathbf{U_K K}_2; \mathbf{F_K L}_2)$.

Proof. Clearly, the operators $L, M \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$, with $\operatorname{im} L = \mathbb{H}^2_\sigma \times \mathbb{H}^2_\pi \times \{0\}$, $\ker L = \{0\} \times \{0\} \times \mathbb{H}^2_\pi$, so by virtue of lemma 1, $L, M \in \mathcal{L}(\mathbf{U_K K}_2; \mathbf{F_K L}_2)$.

Lemma 4. *For any* $\lambda \in \mathbb{R} \setminus \sigma(A)$, $\nu \in \mathbb{R}$ *the operator* M is $(L, 1)$-*limited.*

Proof. In [2] it is shown that the operator M is $(L, 1)$-bounded if the operators $L, M\colon \mathfrak{U} \to \mathfrak{F}$, so by virtue of lemma 1, the statement of this lemma follows.

Theorem 3. For any $\lambda \in \mathbb{R} \setminus \sigma(A)$, $\nu \in \mathbb{R}$ *and for any random variable* $\eta_0 \in \mathbf{U}^1_\mathbf{K}\mathbf{L}_2$ *there exists a solution to problem* (3), (4) *which is of the form* $\eta(t) = U^t\eta_0$, $t \in \mathcal{J}$.

Proof. By virtue of lemmas 3 and 4„ the stochastic system of equations (2) satisfies all the requirements of Theorem 1. The phase space has the form

$$\mathbf{U}^1_\mathbf{K}\mathbf{L}_2 = \begin{cases} \mathbf{U_K L}_2, & \text{if } \lambda \neq \nu_k \text{ for } k \in \mathbb{N}; \\ \eta \in \mathbf{U_K L}_2 : \langle\,\cdot\,, \varphi_k\rangle\varphi_k = 0, & \text{if } \lambda = \nu_k, \end{cases}$$

where ν_k is the spectrum of the operator $\tilde{A}\colon \mathbb{H}^2_\pi \to \mathbb{H}^2_\pi$, that is the contraction of the operator A onto $\mathbb{H}^2_\pi$. The resolving group can be represented as

$$U^t = \begin{pmatrix} \displaystyle\sum_{\nu_k \neq \lambda} \exp\left\{\frac{\nu\nu_k}{\nu_k - \lambda}\right\}\langle\cdot, \varphi_k\rangle\varphi_k & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} \end{pmatrix}.$$

4. EXPONENTIAL DICHOTOMIES AND STABILIZATION OF SOLUTIONS OF A STOCHASTIC SYSTEM OF EQUATIONS

The relative spectrum has the form $\sigma^L(M) = \left\{\frac{\nu\nu_k}{(\nu_k-\lambda)}\right\}$. Note that the spectrum $\sigma(\tilde{A}) = \{\nu_k\}$ is positive discrete finite and condensed to the point $+\infty$ (Solonnikov–Vorovich–Yudovich theorem). The following holds

Theorem 4. *For any $\lambda \in \mathbb{R}\backslash\sigma(A)$, $\lambda > \nu_1$ and $\nu \in \mathbb{R}\backslash\{0\}$, solutions $\eta = \eta(t)$ of the stochastic system of equations* (2) *have an exponential dichotomy.*

Proof. Let $\lambda \in \mathbb{R}\backslash\sigma(A)$ and $\lambda > \nu_1$, then $\sigma^L(M) = \sigma_1^L(M)\cup\sigma_2^L(M)$, where $\sigma_1^L(M) = \{\mu \in \sigma^L(M) : \nu_k < \lambda\}$, $\sigma_2^L(M) = \{\mu \in \sigma^L(M) : \nu_k > \lambda\}$. This spectral decomposition is accompanied by invariant spaces

$$\mathbf{I}^1 = \{\eta \in \mathbf{U}^1_{\mathbf{K}}\mathbf{L}_2 \colon \langle\,\cdot\,,\varphi_k\rangle\varphi_k = 0,\ \nu_k < \lambda\},\quad \mathbf{I}^2 = \{\eta \in \mathbf{U}^1_{\mathbf{K}}\mathbf{L}_2 \colon \langle\,\cdot\,,\varphi_k\rangle\varphi_k = 0,\ \nu_k > \lambda\}.$$

The space $\mathbf{I}^1$ is finite-dimensional, $\dim \mathbf{I}^1 = \max\{k \colon \nu_k < \lambda\}$, , and the space $\mathbf{I}^2$ is infinite-dimensional, $\operatorname{codim} \mathbf{I}^2 = \dim \mathbf{I}^1 + \dim\ker L$.

If $\nu > 0$ ($\nu < 0$), then $\sigma^L_{1(2)}(M)$ lies in the left half-plane and $\sigma^L_{2(1)}(M)$ lies in the right half-plane of the complex plane. By virtue of Theorem 2, $\mathbf{I}^{1(2)}$ is a stable invariant space, $\mathbf{I}^{2(1)}$ is an unstable invariant space, and the solutions of the stochastic system of equations (2) have exponential dichotomy. The theorem is proved.

Corollary 2. *If $\lambda < \nu_1$ and $\nu < 0$, then the phase space of the stochastic system of equations (2) coincides with the stable invariant space. If $\lambda < \nu_1$ and $\nu > 0$, then the phase space of the stochastic system of equations (2) coincides with the unstable invariant space.*

Let us proceed to the problem of stabilization of unstable solutions. For this purpose, we will consider equation (4) in the form of the system (7)–(9). For definiteness, let us assume $\nu > 0$ and $\lambda > \nu_1$. It follows from Theorem 4 that $\mathbf{I}^s = \mathbf{I}^1$ and $\mathbf{I}^u = \mathbf{I}^2$. The space $\mathbf{I}^s$ is a stable invariant space, so for the solutions $\eta_l = \eta_l(t)$ of equation (8) the following is true

$$\lim_{t\to+\infty} \|\eta_l(t)\|_{\mathbf{U_K L}_2} = 0.$$

By virtue of Remark 1, consider the following stabilization problem. It is required to find such a stochastic process χ, so that for the solutions of Eq.

$$L_r\overset{\circ}{\eta}_r = M_r\eta_r + \chi \tag{12}$$

the following condition was satisfied

$$\lim_{t\to+\infty} \|\eta_r(t)\|_{\mathbf{U_K L}_2} = 0. \tag{13}$$

We will find χ using the inverse of $\chi = B\eta_r$, where B is some linear bounded operator. Equation (12) will take the form

$$L_r\eta_r = M_r\eta_r + B\eta_r = (M_r + B)\eta_r.$$

Let's find $m = \max \mu_k \in \sigma_2^L(M)\{\mu_k\}$ and the number n of the obtained maximum value. Let's put

$$B = -\nu(\varepsilon + \nu_n)\mathbb{I},$$

where ε can be chosen as small as desired. Then the relative spectrum

$$\sigma^{L_u}(M_u + B) = \left\{\frac{\nu\nu_k - \nu(\varepsilon + \nu_n)}{\lambda - \nu_k}\right\}$$

lies in the left half-plane, of the complex plane and by virtue of Theorem 2, equality (13) is satisfied for the solution of $\eta_r = \eta_r(t)$.

CONCLUSION

It is planned to continue studies on stability and instability of solutions for stochastic semilinear equations of Sobolev type with a relatively spectral operator. It is planned to carry out numerical experiments on finding stable and unstable solutions of the stochastic system (2) and stabilization of unstable solutions.

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CONFLICT OF INTERESTS

The author of this paper declares that she has no conflict of interests.

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