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PARTIAL DIFFERENTIAL EQUATIONS

MODEL PROBLEM IN A STRIP FOR THE HYPERBOLIC DIFFERENTIAL-DIFFERENCE EQUATION

© 2025 N. V. Zaitseva*

Lomonosov Moscow State University, Russia
**e-mail: zaitseva@cs.msu.ru*

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Abstract. The paper investigates the question of the existence of a classical solution to the initial value problem with incomplete initial data on the boundary of the strip for a hyperbolic differential-difference equation. The equation contains a superposition of a differential operator and a translation operator with respect to a spatial variable that varies along the entire real axis. Using the Gelfand–Shilov operational scheme, a solution to the problem was obtained in explicit form.

Keywords: *hyperbolic equation, differential-difference equation, translation operator, initial problem, operational scheme, Fourier transform*

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1. INTRODUCTION. PROBLEM STATEMENT

The interest in the study of functional-differential and, in particular, differential-difference equations and problems for them is due to two reasons. First, for such generalizations of differential equations some methods "working well" for classical equations are inapplicable, and also there appear qualitatively new effects in the solutions, that have no place in classical cases. Secondly, such equations are encountered in a variety of applications (mechanics of a deformable solid body, processes of vortex formation and formation of complex coherent spots, modeling of crystal lattice vibrations, nonlinear optics, neural networks, etc.), including those that cannot be described by classical models of mathematical physics. Significant results in the study of problems for functional- differential equations of various classes were obtained by A. L. Skubachevskii [1, 2], V. V. Vlasov [3, 4], A. B. Muravnik [5], A. V. Razgulin [6], L. E. Rossovskii [7], V. Zh. Sakbaev [8] and other authors.

We will call according to [1] a *differential-difference* equation containing both differential operators and shift operators.

To date, problems for elliptic (both in bounded and unbounded domains) and parabolic differential-difference equations have been studied in detail. Hyperbolic differential-difference equations have been studied to a much lesser extent. In [9, 10], two-dimensional hyperbolic equations with a shift operator in the senior derivative acting on a spatial variable are considered for the first time. The purpose of this paper is to construct explicitly, using the known operational scheme [11], the solution of the model initial problem in the strip for such an equation.

Let us denote by $D = \{(x,t) : x \in \mathbb{R}, 0 < t < T\}$ the area of the coordinate plane Oxt, where $T > 0$ is a given real number, let $\overline{D} = \{(x,t) : x \in \mathbb{R}, 0 \le t \le T\}$.

We need to find the function $u(x,t) \in C^1(\overline{D}) \cap C^2(D)$, satisfying the equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = a^2 \frac{\partial^2 u(x-h,t)}{\partial x^2}, \quad (x,t) \in D, \tag{1}$$

where $a > 0, h \neq 0$ are given real numbers, and the initial condition

$$u(x,0) = 0, \quad x \in \mathbb{R}. \tag{2}$$

Definition. We will call the *classical solution of the problem* (1), (2) a function $u(x,t)$, continuous and continuously differentiable on the variables x and t in the set $\overline{D}$; twice continuously differentiable on x and t in D; satisfying at each point of the region D the relation (1); such that for each point $x_0 \in \mathbb{R}$ the limit of the function $u(x_0,t)$ at $t \to +0$ exists and is equal to zero.

2. CONSTRUCTING A SOLUTION TO THE PROBLEM

To find the solution of the problem (1), (2) according to the operational scheme [11] we apply, to equation (1) and initial condition (2) (formally), the Fourier transform on the variable x, acting according to the rule

$$\hat{u}(\xi,t) := F_x[u(x,t)] = \int_{-\infty}^{+\infty} u(x,t)e^{i\xi x}dx.$$

As a result, we obtain the problem in Fourier images

$$\frac{d^2\hat{u}(\xi,t)}{dt^2} + a^2\xi^2 e^{ih\xi}\hat{u}(\xi,t) = 0, \tag{3}$$

$$\hat{u}(\xi,0) = 0, \quad \xi \in \mathbb{R}. \tag{4}$$

The characteristic roots of the equation corresponding to equation (3) are determined by the formula

$$k_{1,2} = \pm ia\xi e^{(ih\xi/2)},$$

then the general solution of equation (3) has the form

$$\hat{u}(\xi,t) = C_1(\xi)\cos(a\xi e^{(ih\xi/2)}t) + C_2(\xi)\sin(a\xi e^{(ih\xi/2)}t),$$

where $C_1(\xi)$ and $C_2(\xi)$ are arbitrary constants depending on the parameter $\xi \in \mathbb{R}$. Substituting this function into the initial condition (4), we obtain $C_1(\xi) = 0$. Since problem (3), (4) is a problem with incomplete initial data, let us assume that

$$C_2(\xi) = (a\xi e^{(ih\xi/2)})^{-1}$$

and write down the final form of its solution:

$$\hat{u}(\xi,t) = \frac{\sin(a\xi e^{(ih\xi/2)}t)}{a\xi e^{(ih\xi/2)}}, \quad \xi \in \mathbb{R}.$$

Applying now the inverse Fourier transform to the found function (formally), we obtain by analogy with [12] the following relations:

$$\begin{aligned}
F_\xi^{-1}[\hat{u}(\xi,t)] &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{u}(\xi,t)e^{-ix\xi}d\xi = \\
&= \frac{1}{2\pi a}\int_{-\infty}^{+\infty}\frac{\sin(a\xi t e^{(ih\xi/2)})}{\xi e^{(ih\xi/2)}}e^{-ix\xi}d\xi = \\
&= \frac{1}{2\pi a}\left[\int_0^{+\infty}\frac{\sin(a\xi e^{(-ih\xi/2)}t)}{\xi}e^{i(x+h/2)\xi}d\xi \right. + \\
&\quad \left. + \int_0^{+\infty}\frac{\sin(a\xi e^{(ih\xi/2)}t)}{\xi}e^{-i(x+h/2)\xi}d\xi\right] = \\
&= \frac{1}{2\pi a}\int_0^{+\infty}\left[\frac{\sin((at\cos(h\xi/2)+x+h/2)\xi)}{\xi e^{(-at\xi\sin(h\xi/2))}} \right. + \\
&\quad \left. + \frac{\sin((at\cos(h\xi/2)-x-h/2)\xi)}{\xi e^{(at\xi\sin(h\xi/2))}}\right]d\xi.
\end{aligned} \tag{5}$$

Remark 1. If we put $h = 0$ in (5), then we obtain $\theta(at-|x|)/(2a)$ — the *fundamental solution* of the wave operator $\partial^2/\partial t^2 - a^2\partial^2/\partial x^2$, where θ is the Heaviside function.

Since the obtained improper integral in (5) diverges, we introduce, according to [11], the *regularizer* $f(\xi)$ for expression (5) — a function satisfying the conditions:

1) $f(\xi)$ is positively defined and continuous on the set $[0,+\infty)$;

2) for any number $\epsilon>0$ there are the following equations

$$\lim_{\xi\to+\infty} f(\xi)e^{at\xi\sin(h\xi/2)}\xi^{\epsilon}=0,\quad \lim_{\xi\to+\infty} f(\xi)e^{-at\xi\sin(h\xi/2)}\xi^{\epsilon}=0; \tag{6}$$

3) the integrals converge at any value of $t\in[0,T]$

$$\int_0^{+\infty} f(\xi)e^{at\xi\sin(h\xi/2)}d\xi,\quad \int_0^{+\infty} f(\xi)e^{-at\xi\sin(h\xi/2)}d\xi; \tag{7}$$

4) the integrals converge at any value of $t\in(0,T]$

$$\int_0^{+\infty} f(\xi)\xi e^{at\xi\sin(h\xi/2)}d\xi,\quad \int_0^{+\infty} f(\xi)\xi e^{-at\xi\sinh\xi/2}d\xi. \tag{8}$$

An example of such a function satisfying conditions 1)–4) is, for example, the function $f(\xi)=\xi^{\beta}e^{(-CT\xi)}$, where $\beta\geq 0$ and $C>a>0$ are any real constants.

Remark 2. The fulfillment of the equations (6) entails [13, p. 102] the convergence of the integral integrals

$$\int_0^{+\infty} \frac{f(\xi)}{\xi}e^{at\xi\sin(h\xi/2)}d\xi,\quad \int_0^{+\infty} \frac{f(\xi)}{\xi}e^{-at\xi\sin(h\xi/2)}d\xi. \tag{9}$$

3. KEY FINDINGS

Lemma. *If conditions* 1)–4) *are satisfied, the function*

$$G(x,t):=\int_0^{+\infty}\left[\frac{f(\xi)\sin((at\cos(h\xi/2)+x+h/2)\xi)}{\xi e^{-at\xi\sin(h\xi/2)}}+\frac{f(\xi)\sin((at\cos(h\xi/2)-x-h/2)\xi)}{\xi e^{at\xi\sin(h\xi/2)}}\right]d\xi \tag{10}$$

satisfies equation (1) *in the classical sense*.

Proof. The integrand in (10) is continuous on the set $[0,+\infty)$ as a composition of continuous functions (there is no singularity at the point $\xi=0$ due to the limit relation $\sin\alpha/\alpha\to 0$ at $\alpha\to 0$).

Let's investigate the convergence of the integral

$$\int_0^{+\infty} F(x,t;\xi)d\xi:=\int_0^{+\infty}\frac{f(\xi)\sin((at\cos(h\xi/2)+x+h/2)\xi)}{\xi e^{-at\xi\sin(h\xi/2)}}d\xi. \tag{11}$$

In view of condition 1)

$$\left|\int_0^{+\infty} F(x,t;\xi)d\xi\right|\leqslant\int_0^{+\infty}\frac{f(\xi)}{\xi}e^{at\xi\sin(h\xi/2)}d\xi,$$

then by virtue of the fulfillment of condition 2) and, as a consequence, of Remark 2, the integral (11) converges.

Let us now check that function (11) satisfies equation (1). For this purpose, we differentiate (11) formally under the sign of the integral over the variables t and x up to the second order:

$$\int_0^{+\infty} F_x(x,t;\xi)d\xi=\int_0^{+\infty} f(\xi)\cos((at\cos(h\xi/2)+x+h/2)\xi)e^{at\xi\sin(h\xi/2)}d\xi; \tag{12}$$

$$\int_0^{+\infty} F_{xx}(x,t;\xi)d\xi=-\int_0^{+\infty} f(\xi)\xi\sin((at\cos(h\xi/2)+x+h/2)\xi)e^{at\xi\sin(h\xi/2)}d\xi,$$

then

$$\int_0^{+\infty} F_{xx}(x-h,t;\xi)d\xi=-\int_0^{+\infty} f(\xi)\xi\sin((at\cos(h\xi/2)+x-h/2)\xi)e^{at\xi\sin(h\xi/2)}d\xi. \tag{13}$$

Next,

$$\int_0^{+\infty} F_t(x,t;\xi)d\xi = a\int_0^{+\infty} f(\xi)\big[\cos(h\xi/2)\cos((at\cos(h\xi/2)+x+h/2)\xi)+ \\ +\sin(h\xi/2)\sin((at\cos(h\xi/2)+x+h/2)\xi)\big]e^{at\xi\sin(h\xi/2)}d\xi = \\ = a\int_0^{+\infty} f(\xi)\cos((at\cos(h\xi/2)+x)\xi)e^{at\xi\sin(h\xi/2)}d\xi; \tag{14}$$

$$\int_0^{+\infty} F_{tt}(x,t;\xi)d\xi = -a^2\int_0^{+\infty} f(\xi)\xi\big[\cos(h\xi/2)\sin((at\cos(h\xi/2)+x)\xi)- \\ -\sin(h\xi/2)\cos((at\cos(h\xi/2)+x)\xi)\big]e^{at\xi\sin(h\xi/2)}d\xi = \\ = -a^2\int_0^{+\infty} f(\xi)\xi\sin((at\cos(h\xi/2)+x-h/2)\xi)e^{at\xi\sin(h\xi/2)}d\xi. \tag{15}$$

Substituting the found derivatives (13) and (15) into the relation (1), we are convinced of its validity.

Let us examine the integral (12) for uniform convergence. We have

$$\int_0^{+\infty} |F_x(x,t;\xi)|d\xi \le \int_0^{+\infty} f(\xi)e^{(at\xi\sin(h\xi/2))}d\xi.$$

Since the integral in the right-hand side of the inequality converges due to condition 3), and the integrand in it does not depend on the variable x, then by virtue of the Weierstrass sign the integral (12) converges uniformly on the variable x at any finite interval $[x_1, x_2] \subset \mathbb{R}$.

Similarly, from the estimation

$$\int_0^{+\infty} |F_{xx}(x-h,t;\xi)|d\xi \le \int_0^{+\infty} f(\xi)\xi e^{at\xi\sin(h\xi/2)}d\xi,$$

condition 4) and the independence of the integrand from x in the right-hand side of the last inequality results in the uniform convergence of the integral (13) on the variable x on any interval $[x_1, x_2] \subset \mathbb{R}$. This means that the differentiation under the sign of the integral in (11) on the variable x up to and including the second order was legitimate.

Let us now evaluate the integral (14):

$$\int_0^{+\infty} |F_t(x,t;\xi)|\,d\xi \le a\int_0^{+\infty} f(\xi)e^{at\xi\sin(h\xi/2)}\,d\xi \le \begin{cases} a\int\limits_0^{+\infty} f(\xi)e^{at_2\xi\sin(h\xi/2)}\,d\xi, & \sin(h\xi/2)\ge 0, \\ a\int\limits_0^{+\infty} f(\xi)e^{at_1\xi\sin(h\xi/2)}\,d\xi, & \sin(h\xi/2)< 0. \end{cases}$$

The integrals in the right-hand side of the relations converge according to condition 3), and the integrand expressions in them do not depend on t, hence, the integral (14) converges uniformly on any interval $[t_1, t_2] \subset [0, T]$.

From the assessment

$$\int_0^{+\infty} |F_{tt}(x,t;\xi)|\,d\xi \le \begin{cases} a^2\int\limits_0^{+\infty} f(\xi)\xi e^{at_2\xi\sin(h\xi/2)}\,d\xi, & \sin(h\xi/2)\ge 0, \\ a^2\int\limits_0^{+\infty} f(\xi)\xi e^{at_1\xi\sin(h\xi/2)}\,d\xi, & \sin(h\xi/2)< 0 \end{cases}$$

and condition 4) it follows that the integral (15) converges uniformly on any segment $[t_1, t_2] \subset (0, T]$. Thus, the differentiation (15) under the sign of the integral over the variable t up to and including the second order is valid.

Similarly it can be shown, in view of conditions 1) and 2), that the non-singular integral converges

$$\int_0^{+\infty} H(x,t;\xi)d\xi := \int_0^{+\infty} \frac{f(\xi)\sin((at\cos(h\xi/2)-x-h/2)\xi)}{\xi e^{at\xi\sin(h\xi/2)}}d\xi \tag{16}$$

and that function (16) satisfies equation (1), differentiating directly (16) under the sign of integral on variables x and t up to the second order inclusive and substituting the found derivatives $H_{tt}(x,t;\xi)$ and $H_{xx}(x-h,t;\xi)$ into (1). In this case, by virtue of conditions 3) and 4), the integrals $H_x(x,t;\xi)$ and $H_{xx}(x,t;\xi)$ converge uniformly on the variable x at any segment $[x_1,x_2]\subset\mathbb{R}$ and the integrals $H_t(x,t;\xi)$ and $H_{tt}(x,t;\xi)$ converge uniformly at any segment $[t_1,t_2]$ of the sets $[0,T]$ and $(0,T]$, respectively.

Thus, it is shown that function (10) exists at every point of the domain D and satisfies equation (1) in the classical sense. The lemma is proved.

On the basis of the lemma the following is true.

Theorem. *If conditions* 1)–4) *are satisfied, the function*

$$u(x,t)=\frac{1}{2\pi a}\int_{-\infty}^{+\infty}G(x-\tau,t)u_0(\tau)d\tau, \tag{17}$$

where $G(x,t)$ *is defined by equality* (10), $u_0(x)$ *is any integrable function on the whole number line, satisfies equation* (1) *in the classical sense and the limit relation*

$$\lim_{t\to+0}u(x_0,t)=0$$

for any value of $x_0\in\mathbb{R}$.

Proof. Function (17) has the form

$$u(x,t)=\frac{1}{2\pi a}\int_{-\infty}^{+\infty}u_0(\tau)\int_0^{+\infty}\left[\frac{\sin((at\cos(h\xi/2)+x-\tau+h/2)\xi)}{\xi e^{-at\xi\sin(h\xi/2)}}+\right.$$
$$\left.+\frac{\sin((at\cos(h\xi/2)-x+\tau-h/2)\xi)}{\xi e^{at\xi\sin(h\xi/2)}}\right]d\xi d\tau.$$

Since $u_0(x)\in L_1(\mathbb{R})$, it is sufficient to show that $|G(x-,t)|\leqslant const$, that is true, due to condition 2) and Remark 2, for the existence of the function (17) in the domain D. In view of the proved lemma, function (17) is a classical solution of equation (1). Note also that, by virtue of the same lemma, the function (17) belongs to the class $C^1(\overline{D})\cap C^2(D)$ (the integrand in (17) is continuous), the integrals $u_x(x,t)$ and $u_{xx}(x,t)$ converge uniformly on the variable x at any finite segment $[x_1,x_2]\subset\mathbb{R}$, the integrals $u_t(x,t)$ and $u_{tt}(x,t)$ converge uniformly on t at any finite segment $[t_1,t_2]$ of the sets $[0,T]$ and $(0,T]$, respectively (the integral $u_t(x,t)$ converges on the boundary $t=0$).

Let $x_0\in\mathbb{R}$. In (17) we substitute the variable by the formula $(x_0-\tau)/t=\eta$ and get

$$u(x_0,t)=\frac{t}{2\pi a}\int_{-\infty}^{+\infty}G(t\eta,t)u_0(x_0-t\eta)d\eta,$$

whence at $t\to+0$ follows the evaluation of $|u(x_0,t)|<\varepsilon$ for any arbitrarily small number $\varepsilon>0$. The theorem is proved.

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CONFLICT OF INTERESTS

The author of this paper declares that he has no conflict of interests.

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