IONIC VELOCITY AND ENERGY DISTRIBUTION FUNCTIONS PERTURBED BY ION-ACOUSTIC SOLITONS: ANALYTICAL CALCULATION FOR ARBITRARY AMPLITUDES

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Abstract. Using the Sagdeev pseudopotential method, the distribution functions of background ions disturbed by ion-acoustic solitons for the case of cold ions were calculated. The distribution functions by velocities and kinetic energies were analyzed. Explicit formulas valid for solitons of arbitrary amplitude were obtained. It was shown that solitons form a strongly nonequilibrium plasma in their vicinity. The results were compared with previously obtained analytical calculations and modeling results.

Keywords: *ion-acoustic soliton, distribution functions, soliton currents*

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1. INTRODUCTION

The analysis of the influence of plasma waves on the distribution functions of its charged particles is important from both fundamental and applied points of view [1–4]. For plasma solitons of the acoustic type, this problem was discussed in [5–10]. As is known, ion-acoustic solitons are stable solitary waves of compression or rarefaction of ion density, propagating in space without changes in shape [11–15]. In [5], the problem was solved within the framework of the Vlasov equations (which is the most general approach), the results were obtained in the small-amplitude approximation, the velocity distribution functions for charged particles were used as intermediate calculations and their properties were not analyzed specifically. In [6, 8–10], the velocity and energy distribution functions perturbed by solitons were specifically studied for plasma with cold ions. The initial (unperturbed) velocity of all ions in this case was zero. Solitons perturb the ion velocity in their vicinity. Knowing the soliton profile, all the parameters of the motion of any ion can be calculated at any moment in time, which makes the problem under study deterministic (in contrast to the stochastic problem of warm ions). For the case of cold ions, the Vlasov equations can be replaced by the hydrodynamic equations and the single-particle approximation. In the calculations [6–10], both averaging over an ensemble of particles (by numerical modeling) and time averaging for a single particle using the ergodic hypothesis were used. In the second case, explicit formulas were obtained to describe the perturbed distribution functions. Nevertheless, the obtained expressions either required the use of numerical methods or were valid for small-amplitude solitons. The results obtained showed that the ion distribution function over the components (projections) of velocity (initially equilibrium), perturbed by compression solitons, has an asymmetric nonequilibrium shape in the vicinity of the wave. As it turned out, the shape of the perturbed distribution function is similar to the distribution function of plasma with an ion beam. The integral of the distribution function over the velocity components turned out to be different from zero, which indicates a one-way transfer of ions by ion-acoustic solitons. The latter consequence is completely consistent with the results of [16–20]. In the indicated works, it was shown in various ways that conservative plasma compression solitons carry out a one-way transfer of charged particles over a finite distance in the direction of their motion.

In the proposed work, using time averaging (based on the ergodic hypothesis), analytical formulas have been obtained to describe velocity and energy distribution functions perturbed by solitons, which are valid for arbitrary amplitudes. Graphs of the calculated functions for solitons of different amplitudes have been constructed. The obtained analytical expressions are compared with approximate expressions valid for small amplitudes, as well as with the results of numerical modeling. Only ion distribution functions are considered, while electrons are assumed to be in equilibrium (Boltzmann).

2. THEORETICAL MODEL

To verify our results, we will use comparison with already known results [6, 8, 9]. Let us consider a classical one-dimensional hydrodynamic model of a collisionless plasma containing cold ions T_i=0 and hot equilibrium electrons with temperature $T_e \gg T_i$. We will assume that the magnetic field is absent or parallel to the direction of wave propagation. The system of hydrodynamic equations can be written as

$$\frac{\partial \upsilon_i}{\partial t} + \upsilon_i \frac{\partial \upsilon_i}{\partial X} = -\frac{\partial \Phi}{\partial x}, \tag{1}$$

$$\frac{\partial N_i}{\partial t} + \frac{\partial N_i \upsilon_i}{\partial X} = 0, \tag{2}$$

$$N_e = \exp(\Phi), \tag{3}$$

$$\frac{\partial^2\Phi}{\partial X^2} = N_e - N_i. \tag{4}$$

Here N_i, N_e — denote ion and electron concentrations normalized to the unperturbed ion (electron) concentration $n_0 = n_{0i} = n_{0e}$; υ_i — hydrodynamic ion velocity normalized to the ion-sound velocity $C_s = \sqrt{T_e / m_i}$, m_i — ion mass; $\Phi = e\varphi / T_e$ — normalized electrostatic potential of the wave, e — absolute electron charge, φ — dimensional potential, which can be described by the known expression $E = -\partial\varphi / \partial x$ for the electric field. The time and spatial coordinates *t, X* are normalized to ω_i^{-1} (where $\omega_i = \sqrt{4\pi n_0 e^2 / m_i}$ — ion plasma frequency) and to λ_D , where $\lambda_D = \sqrt{T_e / 4\pi e^2 n_0}$ — Debye radius.

The system of equations (1)-(4) contains soliton solutions that can be found by various methods. For small amplitudes, it can be described by the KdV equation [21, 22], wherein the profile of the ion-acoustic soliton is expressed as follows:

$$\Phi(x,t) = \Phi_0 \mathrm{sech}^2\left(\frac{X - Mt}{\Delta}\right), \tag{5}$$

where $\Phi_0 = 3(M-1)$ is the soliton amplitude, and $\Delta = \sqrt{6/\Phi_0}$ is its width, $M=V/C_s$ is the Mach number, V is the soliton velocity in the stationary coordinate system. In [9], using equation (5), an explicit formula was obtained for describing the ion velocity distribution function in the vicinity of a compression ion-acoustic soliton, which has the following form:

$$f(\upsilon_i) = \frac{4\sqrt{3}}{T\upsilon_i(2M - \upsilon_i)\sqrt{6(M-1) - \upsilon_i(2M - \upsilon_i)}}. \tag{6}$$

Here T is the averaging time duration. Formula (6) is valid for solitons with small amplitude $\Phi_0 \le 0.5$ [9].

We are interested in solutions of arbitrary amplitude. For our purposes, we will use Sagdeev's pseudopotential method, which is suitable for describing the stationary problem. We will assume that the soliton has passed all stages of evolution and moves at a constant velocity. By introducing a new variable $\xi = X - Mt$, which corresponds to the transition to a coordinate system moving with the wave, the system (1)-(4) can be reduced to a single Poisson equation [23]

$$\frac{\partial^2\Phi}{\partial\xi^2} = e^{\Phi} - N_i(\Phi), \tag{7}$$

where $N_i(\Phi) = M / \sqrt{M^2 - 2\Phi}$ is the normalized ion concentration for the stationary case [16, 23]. A single integration of (7) with respect to Φ, taking into account the boundary conditions $d\Phi/d\xi = 0$ at Φ=0, gives a formula for describing Sagdeev's pseudopotential, $U(\Phi)$ [23]:

$$-U(\Phi) = \frac{1}{2}\left(\frac{\partial\Phi}{\partial\xi}\right)^2, \tag{8}$$

or

$$U(\Phi) = \left(1 - e^{\Phi}\right) - M\left(\sqrt{M^2 - 2\Phi} - M\right). \tag{9}$$

Profiles of ion-acoustic solitons of arbitrary amplitude can be found by numerical integration of equations (7) or (8) using, for example, the 4th-order Runge-Kutta method. Figure 1 shows the soliton potential profiles found using the KdV equation and numerically by the Runge-Kutta (RK) method for different Mach numbers.

Figure 1 demonstrates the classical properties of solitons. Specifically, with increasing soliton velocity (Mach number), its amplitude Φ_0 increases and its width Δ decreases. In the considered two-component model, ion-acoustic solitons can exist in the Mach number range from 1 to 1.6 [6, 8, 23] with amplitudes up to $\Phi_0 \approx 1.6$. The KdV equation adequately describes solitons with amplitude $\Phi_0 \leq 0.5$. Having soliton profiles $\Phi(X)$, we can proceed to calculate the perturbed distribution functions.

3. PERTURBED VELOCITY AND KINETIC ENERGY DISTRIBUTION FUNCTIONS FOR BACKGROUND IONS

To the left and right of a conservative (classical) soliton, the states of the medium (plasma) are identical. As we move away from the soliton center, the plasma quickly (exponentially) returns to an unperturbed state. This means that everywhere except for some vicinity of the solitary wave (tens to hundreds of λ_D), the plasma can be considered in equilibrium. It should be noted that for the case of cold ions, their Maxwellian distribution over velocity components and energy transforms into the Dirac delta function. Following the reasoning in [8-10], we will analyze the perturbation of distribution functions in the plasma region with the soliton located at its center. As noted in [9], in practice it is sufficient for the soliton to be completely within the studied plasma region; however, theoretical analysis is simpler to perform for the case of central symmetry.

Let us introduce the notation $f_\upsilon(\upsilon_i)$ — the ion distribution function with respect to velocity components, $f_W(\upsilon_i)$ — the ion distribution function with respect to kinetic energies. Let us first consider $f_\upsilon(\upsilon_i)$. The required function can be found using the known formulas

$$f_\upsilon(\upsilon_i) = \frac{\Delta N}{N\Delta\upsilon_i}, \tag{10}$$

or

$$f_\upsilon(\upsilon_i) = \frac{\Delta t}{T\Delta\upsilon_i}. \tag{11}$$

Formula (10) is valid when averaging over an ensemble of ions, formula (11) — when averaging over time for a single ion (under the condition of plasma ergodicity). Here υ_i — the velocity of ions along the *x* axis, ΔN — the number of particles with velocities in the range from υ_i to $\upsilon_i + \Delta\upsilon_i$, N — the number of ions in the considered region (in the ensemble), Δt — the time during which the selected ion has a velocity in the range from υ_i to $\upsilon_i + \Delta\upsilon_i$, T — the time over which the averaging is performed. In practice, *T* corresponds to the temporal resolution of measuring instruments. Analysis $f_\upsilon(\upsilon_i)$ using formulas (10) and (11) is in complete agreement, as shown in works [6–10].

In our work, we will focus on finding the exact formula for $f_\upsilon(\upsilon_i)$ using time averaging for the motion of a single test ion (using formula (11)), interacting with an ion-acoustic soliton. The posed problem is schematically shown in Fig. 2.

From Fig. 2, it can be seen that as the soliton moves from left to right, it interacts with an arbitrarily selected background ion for some time and disturbs its dynamic parameters. The ion is displaced forward by several Debye radii after the soliton passes, while its initial and final velocities remain equal to zero [18]. The motion parameters of the selected ion will be used for time averaging. To apply formula (11), we will need the dependence $\upsilon_i(t)$, which we will find from Newton's second law written for a test ion in the electric field of the soliton $m_i\mathbf{a}_i = e\mathbf{E}$. In normalized form, we have

$$\ddot{\xi} = -\frac{\partial\Phi(\xi)}{\partial\xi} \tag{12}$$

or taking into account $\xi = X - Mt$; $\partial / \partial X = \partial / \partial \xi$ for a stationary coordinate system

$$\ddot{X} = -\frac{\partial \Phi(X,t)}{\partial X}. \tag{13}$$

For the analysis of solitons with arbitrary amplitude, it is necessary to use the numerical solution for the potential Φ(*X,t*) in (13). The corresponding solutions are presented in the insets to Fig. 2 as dependencies *X(t)* and $\upsilon_i(t)$ at M=1.05. The problem was solved with the following initial conditions: $X(0)$=40, $\upsilon_i(0) = 0$. In the case under consideration, the ion reaches maximum velocity at t=39. The numerical integration parameters correspond to those presented in Fig. 4 in [8]. Although the numerical solutions are exact, they do not allow obtaining the required analytical expressions. Below we describe a methodology for obtaining the necessary formulas without using numerical methods.

In the limit $\Delta t \to 0$ formula (11) can be written in differential form

$$f_\upsilon(\upsilon_i) = \frac{2dt}{Td\upsilon_i}. \tag{14}$$

The factor "2" corresponds to the case of central symmetry. A detailed derivation of formula (14) can be found in [8]. Formula (14) comprehensively describes the perturbed distribution function for solitons of arbitrary amplitude; however, in the general case, it requires the use of numerical methods, since the dependence $\upsilon_i(t)$ is determined numerically (Fig. 2). For small-amplitude solitons, the desired function $f_\upsilon(\upsilon_i)$ was found in [8, 9] using the Sagdeev pseudopotential expansion method and the KdV equation. Let us proceed to describe the general methodology for obtaining the necessary formulas without using numerical methods.

Let us follow the logic of reasoning [8, 9]. To solve equation (14), we need the dependence of the derivative $dt / d\upsilon_i$ on the parameter υ_i . According to the law of velocity addition, we have $\upsilon_i' = \upsilon_i - M$, where υ_i' is the ion velocity in the moving coordinate system. The conservative nature of the field leads to the conservation of mechanical energy

$$\frac{\upsilon_i'^2}{2} = \frac{M^2}{2} - \Phi \tag{15}$$

or

$$\upsilon_i' = -\sqrt{M^2 - 2\Phi}. \tag{16}$$

In paper [8], formula (16) was obtained by integrating (13). Differentiation of (15) with respect to ξ gives

$$\upsilon_i' \frac{d\upsilon_i'}{d\xi} = -\frac{d\Phi}{d\xi}. \tag{17}$$

Now, taking into account (8), as well as $\frac{dt}{d\upsilon_i'} = \frac{dt}{d\xi}\frac{d\xi}{d\upsilon_i'} = \frac{1}{\upsilon_i'}\frac{d\xi}{d\upsilon_i'}$ we can obtain

$$\frac{dt}{d\upsilon_i'} = \frac{1}{\sqrt{-2U(\Phi)}}.$$

Considering (9), we have

$$\frac{dt}{d\upsilon_i'} = \frac{1}{\sqrt{-2\left[\left(1-e^{\Phi}\right) - M\left(\sqrt{M^2-2\Phi} - M\right)\right]}}. \tag{18}$$

Further, taking into account the law of velocity addition $\upsilon_i' = \upsilon_i - M$, and expressing Φ from (15), and $\sqrt{M^2-2\Phi}$ from (16), we get

$$\frac{dt}{d\upsilon_i} = \frac{1}{\sqrt{-2\left(1-e^{M\upsilon_i-\upsilon_i^2/2} + M\upsilon_i\right)}} \tag{19}$$

or, considering (14), finally

$$f_\upsilon(\upsilon_i) = \frac{2}{T\sqrt{-2\left(1-e^{M\upsilon_i-\upsilon_i^2/2} + M\upsilon_i\right)}}. \tag{20}$$

Formula (20) can be rewritten as $f_\upsilon(\upsilon_i) = 2/(T\sqrt{-2U(\Phi)})$, where $\Phi(\upsilon_i)$ is determined by equation (15). Figure 3 shows graphs of the function $f_\upsilon(\upsilon_i)$, obtained using three methods: using the approximate formula (6) (first obtained in [9]); using the exact formula (20); and by particle ensemble modeling using the methodology from [6, 8]. The graph of the function $f_\upsilon(\upsilon_i)$, obtained from expression (20), valid for arbitrary amplitudes, is filled in since it is considered the reference. The approximate dependencies are represented by dashed curves.

As can be seen from Fig. 3, the results obtained using formula (20) completely coincide with the simulation results [8]. The approximate formula (6) remains valid for small amplitudes. The parameters of the function $f_\upsilon(\upsilon_i)$ are the Mach number M and the averaging time T. The domain of definition $f_\upsilon(\upsilon_i)$ is in the range $0<\upsilon_i<M$. Indeed, in papers [6, 16, 18], it is shown that in the electric field of a classical ion-acoustic compression soliton, ions can only move with a positive velocity $\upsilon_i > 0$. On the other hand, the upper boundary $\upsilon_i < M$ is determined by the subcriticality of the considered solitons. At $\upsilon_i \geq M$, wave breaking occurs and multi-stream motion forms.

Using (20), it is possible to find the average value of the ionic current density induced by solitons, J_i . In normalized form, we have $J_i = \int_0^M \upsilon_i f_\upsilon(\upsilon_i)\,d\upsilon_i$. For M=1.05, T=71 we get $J_i = 0.26$, which is in complete agreement with the results of [6], where the value J_i was calculated for a group of identical solitons with a period of T=71 at M=1.05. There is also agreement with the results of [16], where the value J_i was obtained using hydrodynamic equations. In turn, knowing J_i it is easy to calculate the total electric ionic charge transferred by a soliton through a unit area, $Q_i = J_i T$. It is straightforward to verify that the obtained dependence $Q_i(M) \sim Q_i(\Phi_0)$ is consistent with the dependence $\Delta X(\Phi_0) = Q_i(\Phi_0)$, calculated in [18] by different methods (here ΔX is the distance of ion transfer by a soliton).

Let's find an expression for the distribution function by kinetic energies. Knowing the function $f_\upsilon(\upsilon_i)$ we can find the function $f_W(W_i)$, using the known relationship [7, 10]

$$f_t(\upsilon_i)\,d\upsilon_i = f_t(W_i)\,dW_i. \tag{21}$$

Considering that $W_i = \upsilon_i^2/2$ we have

$$f_W(W_i) = \frac{\sqrt{3}}{T\sqrt{W_i}\left(e^{\left(\sqrt{2W_i}M - W\right)} - \sqrt{2W_i}M - 1\right)^{1/2}}. \tag{22}$$

The domain of the function $f_{W_i}(W_i)$ is determined by the inequality $0 < W_i < M^2/2$.

Figure 4 shows the graphs of $f_W(W_i)$ for different Mach numbers. For comparison, the graphs present the results from [10], obtained through ensemble modeling and using an approximate formula derived from the KdV equation.

From Fig. 4, it can be seen that the results obtained using formula (22) are in complete agreement with the simulation results from [10]. The approximate formula (14) from [10] remains valid for small amplitudes.

As can be seen from Fig. 3 and 4, the distribution functions of background ions become highly non-equilibrium in the vicinity of solitons. They correspond to the transport of ions by the soliton (excitation of soliton currents) and have a "beam-like" form. The presence of a charged particle flow in the vicinity of a soliton can cause the development of streaming instabilities [24]. In particular, electron drift with velocity $\upsilon_e > C_s$ can cause drift ion-acoustic instability. With increasing drift velocity, Buneman instability can develop. However, in our model, electrons were assumed to be in equilibrium, and their flows were set to zero. The soliton currents considered by us can cause streaming dust-acoustic instabilities [25]. This situation is possible in dusty plasma in the presence of ion-acoustic solitons [13]. Ion flows can also affect the charge of dust particles in dusty plasma, which is one of the causes of instabilities [26]. Detailed analysis of such problems is a topic for future work.

CONCLUSION

Based on the Sagdeev pseudopotential method, an analytical expression is obtained describing the distribution functions of background ions perturbed by an ion-acoustic soliton in terms of velocity components $f_\upsilon(\upsilon_i)$ and kinetic energies $f_W(W_i)$. It was previously shown [5-10] that ion-acoustic solitons strongly perturb the initially equilibrium distribution function of ions. In the region occupied by solitons, such a function has a "beam-like" form [6-10]. The obtained results are valid only for cold plasma fractions. For the case of warm ions, they can only be used as estimates. It is expected that accounting for the thermal motion of ions will lead to broadening of the maxima of the distribution functions. The analytical formulas (20), (22) are simple to apply, they can be used to interpret experimental data, as well as to develop new methods of plasma diagnostics. It is worth noting that the approach we used is quite universal and can be used to describe the properties of electron- and dust-acoustic solitons.

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FIGURE CAPTIONS

Рис. 1. Soliton potential profiles at different Mach numbers: "RK" - numerical simulation using the Runge-Kutta method; "KdV" - analysis using the Korteweg-de Vries equation according to formula (5).

Рис. 2. Scheme of interaction between a test ion and an ion-acoustic soliton; the insets show the dependencies of $X(t)$ and $\upsilon_i(t)$ for the test ion. Circles represent the initial and final positions of the test ion.

Рис. 3. Perturbed distribution functions $f_\upsilon(\upsilon_i)$ at T =38 and at different Mach numbers, calculated using various methods: using approximate formula (6) - dashed curve; using exact formula (20) - filled solid curve; using particle ensemble simulation according to the methodology [6, 8] - triangles.

Рис. 4. Perturbed distribution functions $f_W(W_i)$ at T =38 and at different Mach numbers, calculated using various methods: using approximate formula (14) from [10] - dashed curve; using exact formula (22) - filled solid curve; using particle ensemble simulation according to the methodology [10] - triangles.

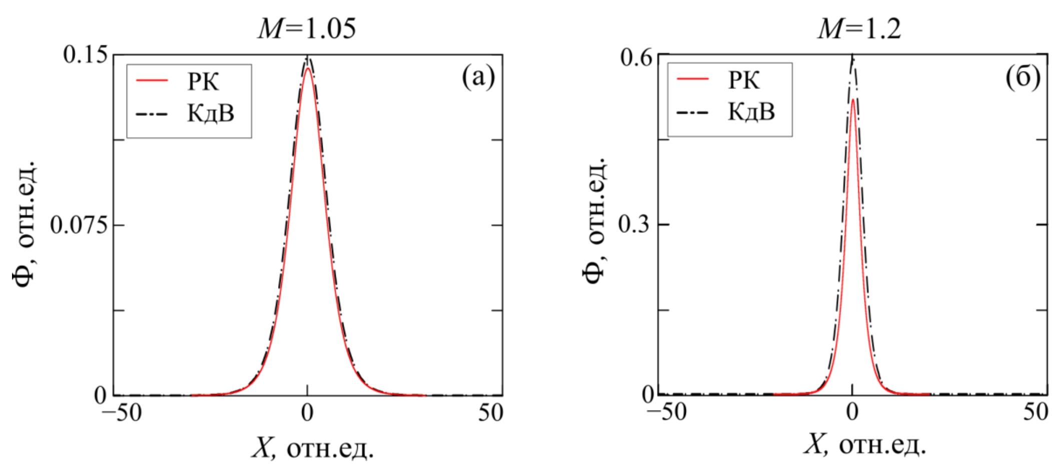

Fig. 1.

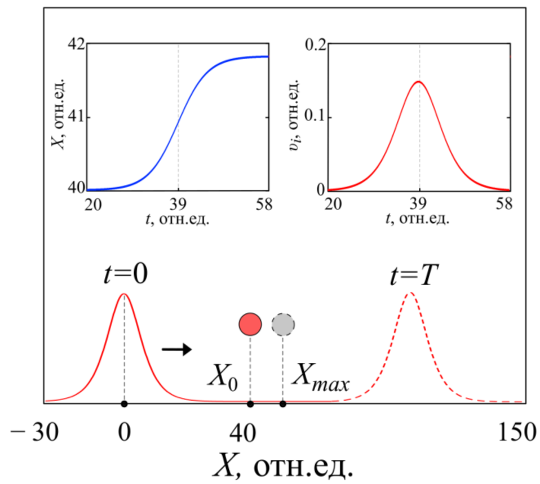

Fig. 2.

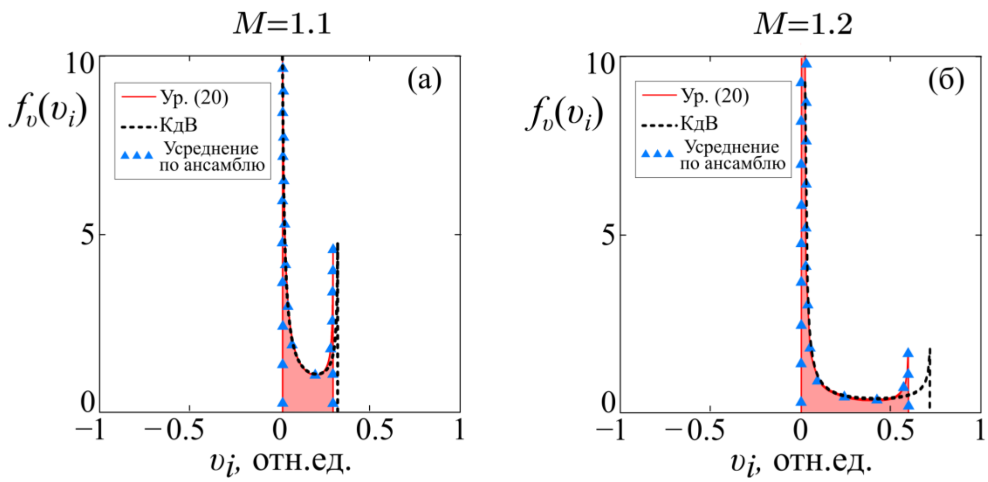

Fig. 3.

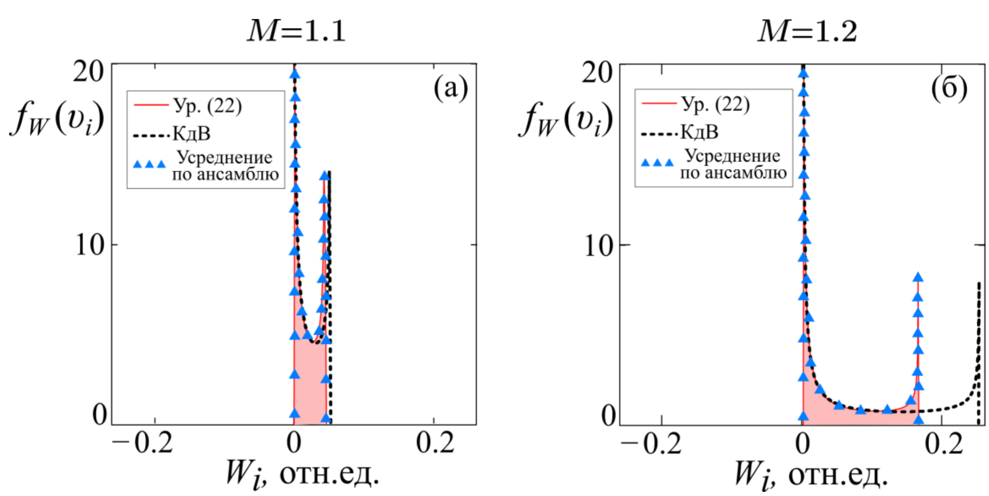

Fig. 4.