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STUDY OF DYNAMICS REGULARITIES FOR MORPHOLOGICAL PATTERN OF ABRASION SHORES OF CRYOLITHOZONE BASED ON COMPLEXING MATHEMATICAL MODELING AND SPACE IMAGERY

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Abstract. The article is devoted to the study of dynamics regularities of abrasion shores of the cryolithozone based on complex mathematical modeling and space imagery including their significance for obtaining information on dynamic parameters of ongoing processes based on remote sensing data. The studied landscape of abrasion shores is a combination of thermal cirques of different ages and preservation, it develops under the action of processes of both the appearance of new thermal cirques and partial or complete erasure of existing ones due to the formation of new ones. The characteristic feature of thermal cirques is a clear arc-shaped boundary with the adjacent watershed surface, which is well detected on remote sensing data. The technique includes creating and analyzing a mathematical model of the morphological pattern changes of abrasion shores within the cryolithozone. The model uses the approach of the random process theory and empirical measurement of thermal cirques in different physiographic conditions on space imagery. The combination of mathematical modeling with space imagery interpretation allowed us to show that in different physiographic and geocryological conditions, a stable stationary distribution of thermal cirque sizes of abrasion shores of the Arctic cryolithozone is formed with a significant development time in homogeneous areas. The physiographic and geocryological variety of different sites does not prevent the existence of the limiting stationary distribution. Thus, the morphological pattern of the abrasion shore, being in constant change, nevertheless has a stationary distribution of thermal cirque sizes, their average size, and average location density, i.e., it is in a state of dynamic balance. The research gave a mathematical dependence between the limiting thermal cirque size distribution for abrasion shores and the size distribution for forming young thermal cirques. The sites' physical-geographical, geological-geomorphological, and geocryological conditions influence the character of the stationary limit distribution through the size distribution of forming young thermal cirques. The results obtained allow us to predict quantitative characteristics of the thermal cirques (and consequently landslides) formation process, namely, the size distribution of emerging new thermal cirques and landslides, based on measurements of the observed thermal cirque sizes using high-resolution single-shot remote sensing data. This is essential in predicting the development, in particular, of shore retreat.

Keywords: *mathematical morphology of landscapes, abrasion shores, cryolithozone, mathematical models of landscape morphological patterns, remote sensing data, mathematical modeling*

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INTRODUCTION

Many studies are devoted to morphological peculiarities of abrasion banks of cryolithozone. An extensive group of works is devoted to the study of ongoing processes in connection with bank retreat (Belova et al., 2001; Novikova, 2002; Pizhankova, Dobrynina, 2010; Aleksyutina et al., 2018; Belova et al., 2020), landscape factors influencing the development of processes (Sovershaev, 1998; Khomutov, Leibman, 2008; Kizyakov, 2005; Vasiliev et al. 2001), their connection with climatic characteristics (Leibman et al., 2021). However, most researchers study the processes of development of abrasion banks of cryolithozone in connection with retreat, and, accordingly, the extent of morphological elements (thermal cirques) and its changes in the direction perpendicular to the shoreline are analyzed. At the same time, little attention has been paid to the study of the extent of thermal cirques along the coastal slope and its quantitative parameters.

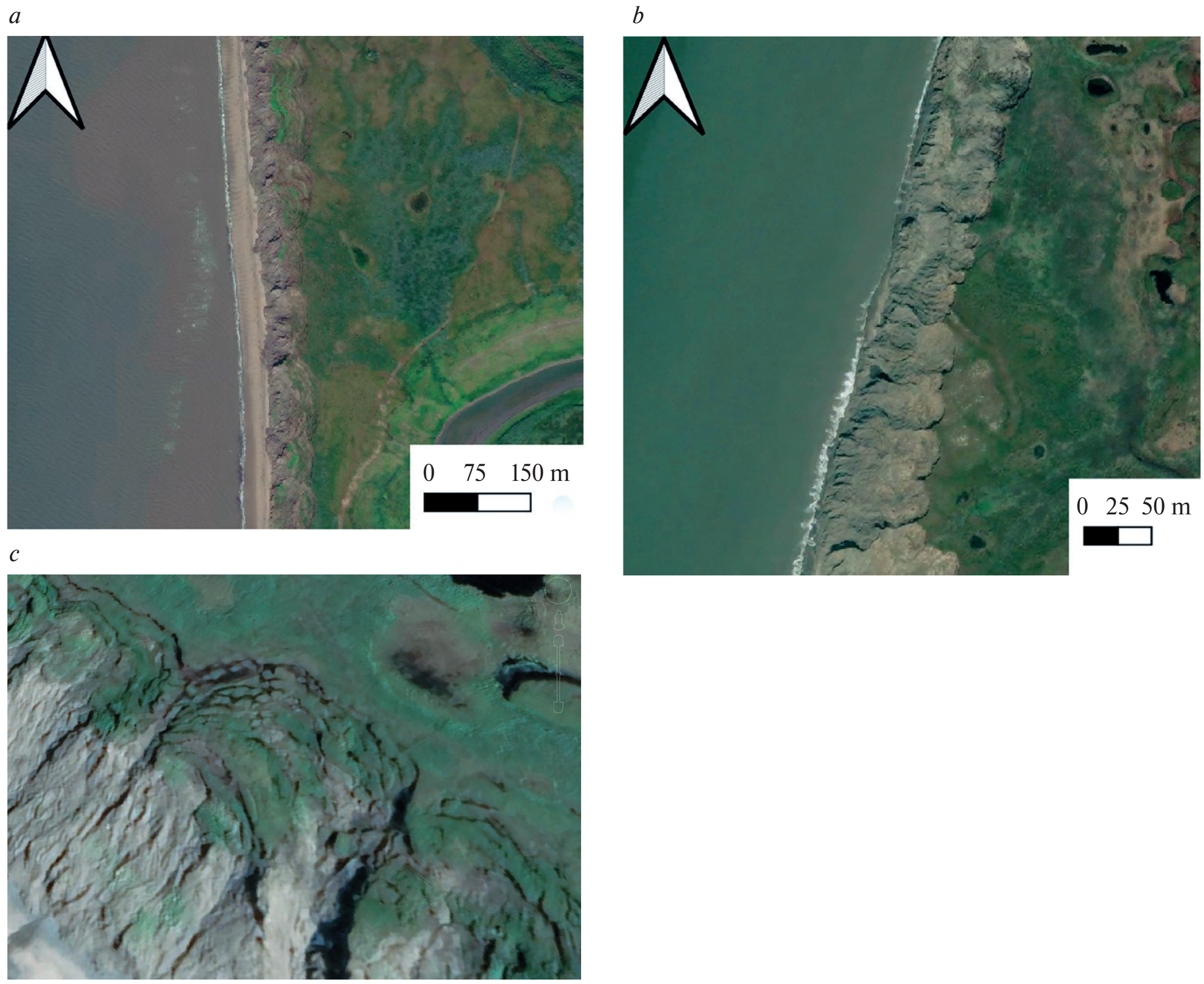

Fig. 1. Typical image of abrasion banks with the development of thermal cirques in the cryolithozone on the materials of high-resolution visible space imagery: *a, b* — general view (Victorov et al., 2023), *c* — example of a thermal cirque.

The aim of the research was to study the regularities of dynamics of abrasion banks of cryolithozone based on the complex of mathematical modeling and space imagery and their significance for obtaining information on ongoing processes, including their quantitative parameters, using remote sensing data.

The landscape of abrasion banks with the development of thermal cirques is a combination of thermal cirques, including landslide bodies of different ages, surfaces with the development of intensive thermo-abrasion, erosion, rockfall and thermodenudation processes. A characteristic feature of thermal cirques is a clear arc-shaped boundary with the adjacent watershed surface, well interpreted on the materials of space surveys, the appearance of such a boundary is associated with the fact that the formation of a thermal cirque begins, as a rule, with the development of landslide process. It is not uncommon to also observe arc-shaped residual areas of the watershed surface on the slope, corresponding to different stages of landsliding. A typical view of the bank is presented in Fig. 1.

The study is oriented to investigate the change in the size of thermal cirques along the shoreline; the length of the chord closing the arc-shaped boundary of the thermal cirque with the adjacent watershed surface was taken as the size of the thermal cirque.

The development of shores occurs under the action of a complex of processes involved in the formation of values of the thermal cirque size, including an increase in the number of thermal cirques due to the formation of a new thermal cirque inside the boundaries of an existing thermal cirque with the older thermal cirque splitting into two parts, a decrease in the number of thermal cirques due to the complete erasure of thermal cirques (and, accordingly, their chords) when overlapping younger ones, partial erasure of already existing thermal cirques with preservation of their number due to the overlapping of younger ones ("lateral erasure"), the appearance

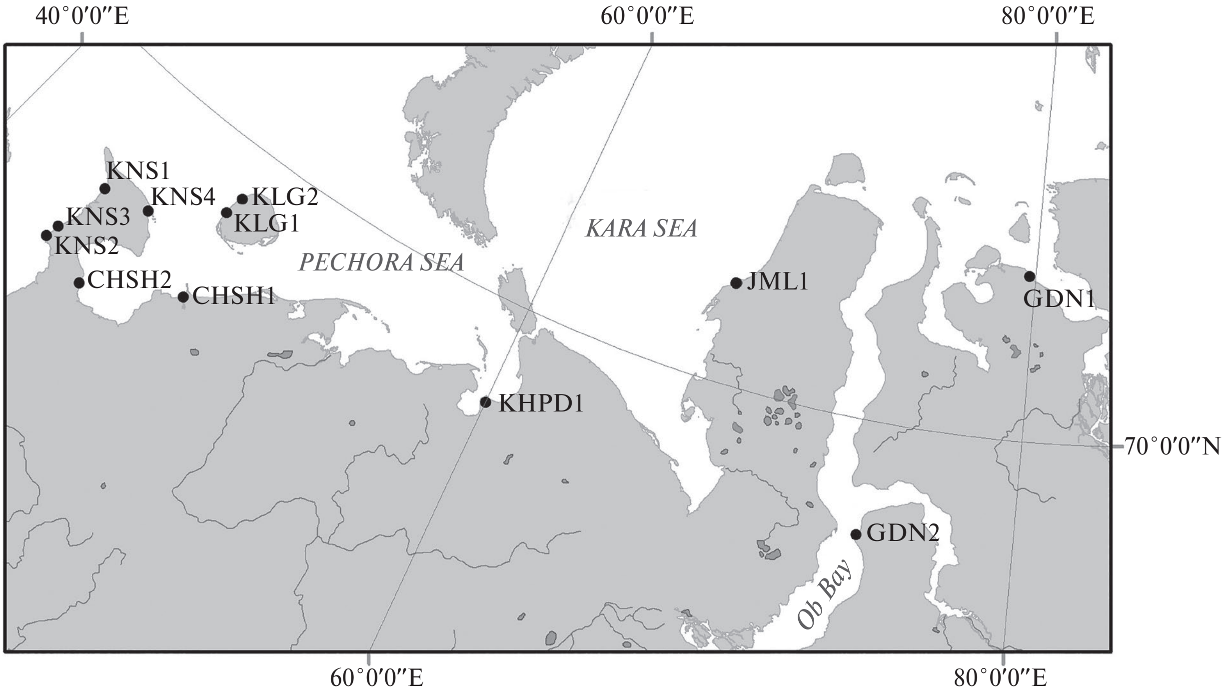

Fig. 2. Schematic of key site locations.

of a new thermal cirque on one or another thermal cirque ("lateral erasure"), and the appearance of a new thermal cirque on one or another thermal cirque.

In this case, the boundary of the coastal slope and the adjacent watershed surface is a system of arcs of thermal cirques, some of which are new, unaffected by subsequent erasures, and the other part is residual, preserved after one-, two-, three-, etc. multiple partial erasures of existing thermal cirques by new ones. Thus, in general, on the image the researcher observes at a random moment of time (survey) a system of different-aged formations in constant change. In this situation the problem arises — how to correlate quantitative characteristics of the image observed on the space image with quantitative dynamic parameters of the ongoing process of thermal cirques formation, and accordingly what information about the process the image allows to extract.

METHODOLOGY

The methodology included the following steps:

• creation and analysis of a mathematical model of changes in the morphological pattern of abrasion banks in the cryolithozone,

• study of thermal cirque sizes in different physiographic conditions using satellite data.

The mathematical model of changes in the morphological pattern of abrasion banks in the cryolithozone was based on the consideration of ongoing changes as a random process

• study of thermocircuit sizes in different physiographic settings using satellite data included:

• selection of key sites,

• measurement of thermal cirque sizes using space imagery materials,

• statistical processing of the obtained characteristics of the morphological pattern of the coastal slope and analysis of the obtained results.

The key sites were selected based on the requirements of relative morphological homogeneity of the area and homogeneity of physiographic, primarily geological-geomorphological and geocryological conditions. As a result, 12 sites with a length of 2.5–10.1 km were selected, which generally have a rectilinear strike (Fig. 2) and are located within the coastline of the Kanin Nose, Gydansky, Tazovsky and Yamal peninsulas, Kolguyev Island, and Khaipudyrskaya Bay.

High resolution satellite images from WorldView 3 (0.3 m/pix resolution), GeoEye 1 (0.5 m/pix resolution), WorldView 2 (0.5 m/pix resolution) and several other satellites were used to analyze morphological features.

RESULTS

The first part of the solution to the problem at hand is to determine the change in the probability distribution of thermal cirque sizes over time to assess its behavior with significant time of abrasion bank development.

For this purpose, the model proposed earlier (Victorov, 2022) for the formation of the morphological pattern of a rectilinear long abrasion bank (L) with homogeneous physiographic and geocryological conditions was used; relative constancy in time of climatic conditions is also assumed. The model is based on the following assumptions:

— the probability of appearance of new thermal cirques (in the number k = 1,2...) for the time Δu on the coastline segment[1] Δl is determined only by the values of the time interval and the segment[2]

$$p_1(\Delta\ell, \Delta u) = \lambda\Delta\ell\Delta u + o(\Delta\ell\Delta u),$$
$$p_k(\Delta\ell, \Delta u) = o(\Delta\ell\Delta u),\ k > 1,$$

where λ is the parameter corresponding to the average number of thermal cirques formed per unit time on the unit length of the coastline;

— sizes (arc chord length) of emerging thermal cirques do not depend on the place of their appearance on the site and have a constant probability distribution $F_0(x)$ independent of time[3].

The analysis allowed us to show (Victorov, 2022) that in this case the appearance of new thermal cirques corresponds to a Poisson random process, i.e., the probability of appearance of k thermal cirques at length l for time u is given by the expression

$$P_\mu(k) = \frac{(\lambda ul)^k}{k!} e^{-\lambda ul} \quad (1)$$

Hence, if the coastal segment of interest has the size Δv the probability of a single hit of the right border of the forming thermal cirque inside this segment during the time interval Δu based on the model assumptions and the Poisson character of the process of appearance of new thermal cirques (1), is as follows

$$q = \lambda\Delta v\Delta u + o(\Delta u) \quad (2)$$

and the probability of the right border of the forming thermal cirque not falling inside this segment is equal to

$$p_0^1 = e^{-\lambda\Delta vu} \quad (3)$$

In addition, it is shown that the probability that the considered thermal cirque will be neither hit nor erased during the time u by a forming thermal cirque with an initial point outside the considered thermal cirque is given by the expression

$$p_0^2 = e^{-\lambda au} \quad (4)$$

where a is the mathematical expectation (average size) of forming (young) thermal cirques.

In the first step, we obtain an equation for the variation of the probability distribution of the thermal cirque size (chord length of the thermal cirque arc) in time. Let the thermal cirque dimensions at time u have a probability distribution $F(x,u)$[4]. Consider the behavior of thermal cirque for the time interval $(u, u + \Delta u)$ the following cases are possible (Fig. 2):

a) the thermocircus remains unchanged,

b) splitting an older thermal cirque into two elements with erasure of some part by forming a new thermal cirque inside the boundaries of the existing one ("internal erasure"), thus increasing the total number of thermal cirques,

c) erasure of part of the thermal cirque (and consequently part of the chord) due to the superposition of younger ones ("lateral erasure"),

d) complete erasure of the thermal cirque (and consequently of the chordae) at the superposition of the formed younger one.

The last three cases are accompanied by the appearance of a newly formed thermal cirque (e) at one or another site.

Let the total number of thermal cirques at time u be n_0 and the number of thermal cirques with chord less than x is respectively

$$n_1(x,u) = n_0 F(x,u)$$

Consider an existing thermal cirque with dimension y (chord length), introduce a coordinate system with zero at the right end of the chord and directed to the left. Determine the mathematical expectation of the number of thermal cirques with chord less than x at the moment $u + \Delta u$.

In case (a), the fact that the thermal cirque of size y will be neither erased nor damaged during the time interval Δu corresponds to the fulfillment of the condition that the initial point of the forming thermal cirque is outside the existing thermal cirque, but the thermal cirque is not damaged (Fig. 3a). Taking into account the Poisson character of the process, formulas (3) and (4), as well as the independence of the events under consideration, the required probability is given by the expression

$$p_0 = e^{-\lambda(a+y)\Delta u} = 1 - \lambda(a + y)\Delta u + o(\Delta u).$$

[1] The position of the thermal cirque is conditionally accepted as the position of the point at the right boundary of its chord (starting point).

[2] $o(\Delta x)$ here and below in accordance with the usual notations — an infinitely small quantity of a higher order than Δx.

[3] It is assumed that the distribution $F_0(x)$ and other distributions in the work have finite mean and variance.

[4] It is assumed that the distribution has a finite mathematical expectation and variance, as well as partial derivatives and continuous mixed derivatives.

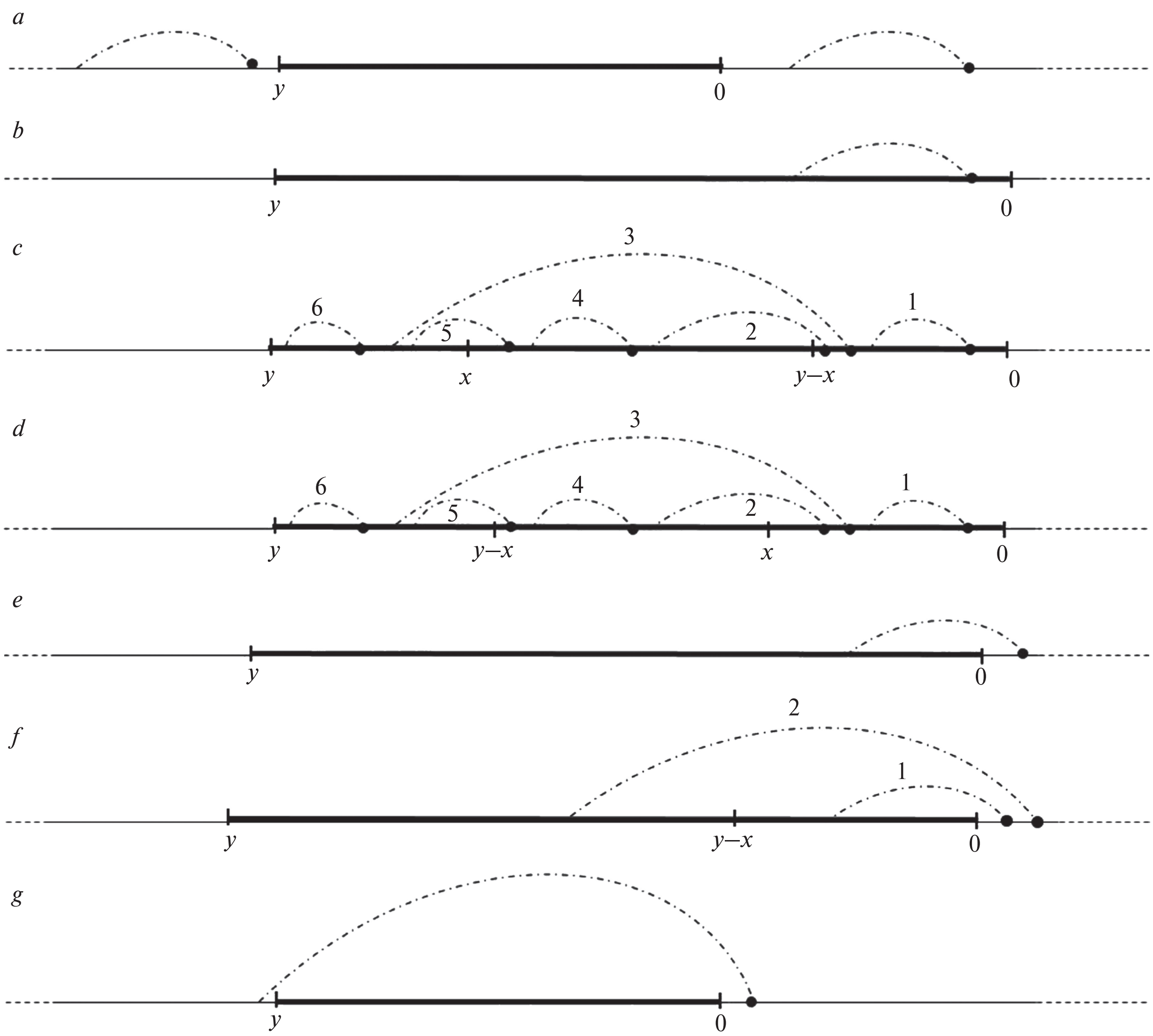

Fig. 3. Different types of interaction between the existing thermal cirque and the new thermal cirque being formed (explanations in the text); cases of interaction: *a* — no erasure, *b, c, d* — internal erasure, *e, f* — lateral erasure, *g* — complete erasure. Symbols: thin line — shoreline, thick line — chord of the existing thermal cirque under consideration, black point — initial point of the arc of the forming thermal cirque, dashed arc — arc of the forming thermal cirque, numbers — different variants of the mutual arrangement of thermal cirques.

Accordingly, the mathematical expectation of the number of thermal cirques with chord less than x at time $u + \Delta u$ given the number of thermal cirques n_0 at time u in case (a) can be obtained by integrating the probability obtained above to take into account all possible sizes of thermal cirques and then multiplying by the number of thermal cirques n_0 at the moment u,

$$N_0(x, u + \Delta u | n_0) =$$
$$= n_0 \int_0^x f(y,u)\left[1 - \lambda(a + y)\Delta u\right] dy + o(\Delta u),$$

where $f(y, u)$ is the density distribution of the thermal cirque size at time u.

The internal obliteration probability ((b) see above) of a thermal cirque of size y is determined by the fact that both first and second boundary points of the new thermal cirque are inside the existing thermal cirque, with three possibilities b1–b3:

(b1) $0 < y < x$ (Fig. 3b), in any case splitting into two residual thermal cirques with chord less than x, and taking into account the equal probability of location of the initial point on any segment Δv of the chord (assumption 1) with the probability given in expression (2), and taking into account the condition that the chord of the new thermal cirque does not overlap the end point of the existing one (so that there is no lateral erasure), by integration we obtain the probability of internal erasure for the time interval Δu

$$p_1 = \lambda\Delta u\int_o^y F_0(y-v)dv = \lambda\Delta u\int_0^y F_0(v)dv;$$

respectively, the mathematical expectation of the number of thermal cirques given the number of thermal cirques n_0 at the moment u can be obtained by integrating the probability obtained above to take into account all possible sizes of the existing thermal cirque (y) and taking into account the formation of two new thermal cirques and then multiplying by the number of thermal cirques n_0 at the moment u,

$$N_1^1(x,u+\Delta u|n_0) = 2n_0\lambda\Delta u\int_0^x f(y,u)\int_0^y F_0(v)dvdy$$

(b2) $x < y < 2x$ (with $y - x < x$ (Fig. 3c), in this case, if the start and end points[5] of the forming thermal cirque are on the segment $[0, y-x]$ (Fig. 3c arcs 1 and 6), or the start and end points are on the segment $[x, y]$ only one residual thermal cirque with chord less than x is formed; if the initial point is on the segment $[0, x]$ and the end point is on the segment $[y-x, y]$ (Fig. 3c arcs 2–5), two thermal cirques with chord less than x are formed. Accordingly, determining the above probabilities by integration, further the mathematical expectation of the number of thermal cirques with chord less than x at time $u + \Delta u$ given the number of thermal cirques n_0 at time u can be obtained by integrating the obtained probabilities to take into account all possible sizes of the thermal cirque existing at moment u and then multiplying by the number of thermal cirques n_0 at moment u,

$$N_1^2(x,u+\Delta u|n_0) = n_0\lambda\Delta u\int_x^{2x} f(y,u)\left\{1\cdot\int_0^{y-x} F_0(y-x-\right.$$
$$\left.-v)dv + 2\int_0^{y-x}[F_0(y-v)-F_0(y-x-v)]dv\right\}dy +$$
$$+n_0\lambda\Delta u\int_x^{2x} f(y,u)\left\{2\int_{y-x}^{x} F_0(y-v)dv +\right.$$
$$\left.+1\int_x^y F_0(y-v)dv\right\}dy + o(\Delta u)$$

Transforming and simplifying, we obtain

$$N_1^2(x,u+\Delta u|n_0) =$$
$$= n_0\lambda\Delta u\int_x^{2x} f(y,u)\left[2\int_{y-x}^{y} F_0(v)dv\right]dy + o(\Delta u)$$

(b3) $2x < y$ (with $x < y - x$ (Fig. 3g), in this case, if the starting point of the forming thermal cirque is on the segment $[0, x]$ and the end point is on the segment $[0, y-x]$ (Fig. 3d arcs 1 and 2), or the start point is on the segment $[x, y]$ and the end point is on the segment $[y-x, y]$ (Fig. 3d arcs 5 and 6), then only one residual thermal cirque with chord less than x is formed; in the case if the initial point of the forming thermal cirque is on the segment $[0, x]$ and the end point is on the segment $[y-x, y]$ (Fig. 3d arc 5 and 6), then only one residual thermal cirque with chord less than x is formed (Fig. 3d arc 3), then two residual thermal cirques with chord less than x will be formed; finally, if the initial and the end points of the forming thermal cirque are on the segment $[x, y-x]$ (Fig. 3d arc 4), then no residual thermal cirques with chord less than x will be formed. Accordingly, determining the above probabilities by integration, further the mathematical expectation of the number of thermal cirques with chord less than x at time $u + \Delta u$ given the number of thermal cirques n_0 at time u can be obtained by integrating the obtained probabilities to take into account all possible sizes of the thermal cirque existing at moment u and then multiplying by the number of thermal cirques n_0 at moment u,

$$N_1^3(x,u+\Delta u|n_0) = Nn_0\lambda\Delta u\int_{2x}^{+\infty} f(y,u)\left\{1\cdot\int_0^x F_0(y-\right.$$
$$\left.-x-v)dv + 2\cdot\int_0^x[F_0(y-v)-F_0(y-x-v)]dv\right\}dy +$$
$$+n_0\lambda\Delta u\int_{2x}^{+\infty} f(y,u)\left\{1\cdot\int_x^{y-x}\left[F_0(y-v)-\right.\right.$$
$$\left.\left.-F_0(y-x-v)\right]dv + 1\cdot\int_{y-x}^{y} F_0(y-v)dv\right\}dy + o(\Delta u)$$

Transforming and simplifying, we obtain

$$N_1^3(x,u+\Delta u|n_0) =$$
$$= n_0\lambda\Delta u\int_{2x}^{+\infty} f(y,u)\left[2\int_{y-x}^{y} F_0(v)dv\right]dy + o(\Delta u)$$

Summing up the mathematical expectation of the number of thermal cirques with chord less than x given the number of thermal cirques n_0 at time u for all three variants and simplifying, we obtain for internal erasure

[5] Let us recall that, in accordance with the above assumption, the endpoint of the chord is always located to the left of the starting point.

the value of the mathematical expectation of the number of thermal cirques with chord less than x

$$N_1(x,u+\Delta u|n_0) = n_0 2\lambda\Delta u\int_0^x f(y,u)\int_0^y F_0(v)dvdy + $$
$$+ n_0 2\lambda\Delta u\int_x^{+\infty} f(y,u)\int_{y-x}^{y} F_0(v)dvdy + o(\Delta u)$$

The lateral erasure probability ((c) see above) of a thermal cirque of size y is determined by having one endpoint of the newly forming thermal cirque outside the existing thermal cirque and the other endpoint inside the existing thermal cirque, with the following possibilities (c1–c2):

(c1) $0 < y < x$ (Fig. 3d), where, if the starting point is to the right of the existing thermal cirque, the probability of lateral erasure is determined by the condition that the size of the forming thermal cirque must be sufficient to encroach on the existing thermal cirque, but at the same time not overlap its end point, so that there is no complete erasure. Given the Poisson nature of the process of emergence of new thermal cirques and the long length of the bank, the equal probability of lateral erasure on both the right and left sides, simplifying, then multiplying by the number of thermal cirques n_0 at moment u, we obtain the mathematical expectation of the number of thermal cirques with chord less than x given the number of thermal cirques n_0 at moment u

$$N_2^1(x,u+\Delta u|n_0) = 2n_0\lambda\Delta u\int_0^x f(y,u)\int_0^{+\infty}\left[F_0(y+v) - \right.$$
$$\left. - F_0(v)\right]dvdy = 2n_0\lambda\Delta u\int_0^x f(y,u)\int_0^y\left[1 - F_0(v)\right]dvdy + o(\Delta u)$$

(c2) $x < y$

(Fig. 3e), in this variant, if the named conditions are fulfilled, if the end point is on the segment $[y-x, y]$ then only one thermal cirque with chord less than x is formed, in otherwise no residual thermal cirque with chord less than x is formed. Similarly to the previous one, we obtain the mathematical expectation of the number of thermal cirques with chord less than x under the condition of the number of thermal cirques n_0 at the moment u

$$N_2^2(x,u+\Delta u|n_0) = 2n_0\lambda\Delta u\int_x^{+\infty} f(y,u)\int_0^{+\infty}\left[F_0(y+v) - \right.$$
$$\left. - F_0(y-x+v)\right]dvdy =$$
$$= 2n_0\lambda\Delta u\int_x^{+\infty} f(y,u)\int_{y-x}^{y}\left[1 - F_0(v)\right]dvdy + o(\Delta u)$$

Summing up, we obtain for lateral erasure the mathematical expectation of the number of thermal cirques with chord less than x given the number of thermal cirques n_0 at time u expression

$$N_2(x,u+\Delta u|n_0) = 2n_0\lambda\Delta u\int_0^x f(y,u)\int_0^y\left[1 - F_0(v)\right]dvdy + $$
$$+2n_0\lambda\Delta u\int_x^{+\infty} f(y,u)\int_{y-x}^{y}\left[1 - F_0(v)\right]dvdy + o(\Delta u)$$

In case (d) (Fig. *3j*) of complete erasure, the mathematical expectation of the number of thermal cirques is zero.

In case (e) of appearance of new thermal cirques, the mathematical expectation of the number of thermal cirques with chord less than x, given the number of thermal cirques n_0 at the moment u appearing during the considered time interval on the whole length of the coast L, in accordance with the Poisson character of the process is equal to

$$N_4(x,u+\Delta u|n_0) = \lambda L\Delta u F_0(x)$$

Summing over all cases, and over all values of $|n_0$ taking into account their probabilities and simplifying, we obtain the mathematical expectation of the number of thermal cirques with chord less than x at time $u + \Delta u$

$$N(x,u+\Delta u) = N(u)\left[F(x,u) - \lambda a F(x,u)\Delta u + \right.$$
$$\left. + \lambda\Delta u\int_0^x yf(y,u)dy + 2\lambda\Delta u x[1 - F(x,u)]\right] +$$
$$+ \lambda L\Delta u F_0(x) + o(\Delta u)$$

where $N(u)$ is the mathematical expectation of the total number of thermal cirques at time u

Going to the limit at $x \to +\infty$ given the equality for the average size of the thermal cirque

$$h(u) = \frac{L}{N(u)} = \int_0^{+\infty} yf(y,u)dy = \frac{1}{\gamma(u)} \quad (5)$$

where $\gamma(u)$ is the average linear density of thermal cirque locations along the shoreline, as well as the following equality from the finiteness of the second order moment of the thermal cirque size distribution function

$$\lim_{x\to+\infty}\left[x\left[1 - F(x,u)\right]\right] = 0$$

we obtain the mathematical expectation of the total number of thermal cirques at time $u + \Delta u$

$$N(u+\Delta u) = N(u) - N(u)\lambda a\Delta u + $$
$$+ 2N(u)\lambda\Delta u\int_0^{+\infty} yf(y,u)dy + o(\Delta u).$$

Hence, dividing by L and Δu and going to the limit at $\Delta u \to 0$ it is not difficult to obtain, considering (5), the differential equation for the variation of the average linear density of the circus arrangement

$$\frac{d\gamma(u)}{du} = 2\lambda - \lambda a\gamma(u).$$

After its solution by standard methods for the initial condition $\gamma(0) = \frac{1}{a}$ (since at the initial moment there is no superposition and erasure of thermal cirques, and the average size is equal to the average size of forming thermal cirques) the change of mathematical expectation of the thermocircle size with time is given by the following expression

$$h(u) = \frac{a}{2 - e^{-\lambda a u}}. \quad (6)$$

Let's take as probability of thermocircle of size not more than x at moment $u + \Delta u$ at large number of thermocircles the ratio of mathematical expectation of number of thermal cirque with size less than x to mathematical expectation of total number of thermal cirques at the moment $u + \Delta u$

$$F(x,u+\Delta u) = N(u)\left[F(x,u) - \lambda a F(x,u)\Delta u + \right.$$
$$\left. + \lambda\Delta u\int_0^x yf(y,u)dy + 2\lambda\Delta u x[1 - F(x,u)]\right] +$$
$$+ \lambda L\Delta u F_0(x) + o(\Delta u) \Big/ N(u) - N(u)\lambda a\Delta u +$$
$$+ 2N(u)\lambda\Delta u\int_0^{+\infty} yf(y,u)dy + o(\Delta u).$$

Subtracting the probability value $F(x,u)$ at time u simplifying, dividing by Δu $N(u)$ and going to the limit at $\Delta u \to 0$ we obtain the equation for the thermal cirque size distribution

$$\frac{1}{\lambda}\frac{\partial F(x,u)}{\partial u} = \int_0^x yf(y,u)dy + 2x\left[1 - F(x,u)\right] +$$
$$+ F_0(x)h(u) - 2F(x,u)h(u)]$$

with the conditions at the initial moment $F(x,0) = F_0(x)$ and $F(0,u) = 0$ arising from the fact that the size cannot be negative, the thermal cirques do not overlap at the initial moment, and at that moment the distribution corresponds to the size distribution of the new thermal cirques forming.

The next step is to solve the obtained equation. Passing to a new unknown function $\phi(x,u)$

$$\phi(x,u) = \int_0^x F(v,u)dv \quad (7)$$

changing the order of differentiation in the resulting mixed derivatives[6] and integrating *over x,* we reduce the equation to the following one

$$\frac{1}{\lambda}\frac{\partial\phi(x,u)}{\partial u} + [x + 2h(u)]\phi(x,u) = x^2 + h(u)I(x) + C(u),$$

where

$$I(x) = \int_0^x F_0(v)dv$$

$C(u)$ is some function depending only on u. This equation can be regarded as a linear inhomogeneous differential equation on u, and when solved in standard way, taking into account the initial moment conditions, we finally obtain

$$\phi(x,u) = \exp(-\lambda xu - 2\lambda\int_0^u h(v)dv)\left[\lambda\int_0^u [x^2 + \right.$$
$$\left. + h(s)I(s)]\exp(\lambda xu + 2\lambda\int_0^u h(v)dv)ds + \int_0^x F_0(v)dv\right].$$

The desired probability distribution of thermal cirque sizes can be formed according to (7) by differentiation of the obtained solution

$$F(x,u) = \varepsilon(x,u)\left[-\lambda^2 u\int_0^u \frac{x^2 + h(s)I(x)}{\varepsilon(x,s)}ds + \right.$$
$$\left. + \lambda\int_0^u \frac{2x + h(s)F_0(x) + \lambda s x^2 + \lambda s h(s)I(x)}{\varepsilon(x,s)}ds\right] +$$
$$+\varepsilon(x,u)\left[-\lambda u\int_0^x F_0(v)dv + F_0(x)\right] \quad (8)$$

where

$$\varepsilon(x,u) = \exp[-\lambda xu - 2\lambda\int_0^u h(v)dv].$$

Finally, the last step, in accordance with the task at hand, is to evaluate the behavior of the obtained distribution at large development time. Passing to the limit at $u \to +\infty$ in expression (8), twice using the Lopital rule (the conditions, as it is easy to see, are satisfied (Fichtenholtz, vol. 1, 1970. p. 151)) and the variation of

[6] The necessary conditions (Fichtenholtz, vol. 1, 1970, para. 190) are satisfied (see footnote 3).

Table 1. Results of evaluation of compliance of samples of thermal cirque sizes at key sites with different types of theoretical distributions

Plot	Sample size	Normal distribution			Lognormal distribution			Gamma distribution		
		average м	standard м	*p*	logarithmic mean	logarithm standard	*p*	λ	α	*p*
KNS1	183	50	43	0.000	3.675	0.666	0.288	22.66	2.22	0.015
KNS2	181	60	43	0.000	3.881	0.639	0.241	23.18	2.57	0.010
KNS3	181	23	15	0.000	2.951	0.586	0.782	7.63	2.99	0.131
KNS4	159	46	27	0.000	3.678	0.529	0.339	12.40	3.68	0.153
JML1	108	18	11	0.000	2.751	0.504	0.574	4.71	3.81	0.314
KLG1	113	24	89	0.057	3.108	0.156	0.254	3.51	6.87	0.331
KLG2	108	25	105	0.020	3.16	0.151	0.923	3.74	6.80	0.640
KPD1	111	31	118	0.309	3.36	0.14	0.694	3.93	7.82	0.842
CSH1	290	22	115	0.000	3.004	0.235	0.015	4.91	4.60	0.037
CSH2	278	15	100	0.000	2.532	0.312	0.545	4.622	3.21	0.113
GDN1	190	14	40	0.000	2.564	0.195	0.658	2.637	5.42	0.305
GDN2	319	22	344	0.000	2.832	0.449	0.235	9.56	2.25	0.006

Note. λ — scale parameter, α — shape parameter, *p* — parameter of agreement of distributions (the difference between empirical and theoretical distributions is statistically significant at the level of 0.99 in the case of $p < 0.01$).

the average thermal cirque size with time (6), we finally obtain that there is a limit to the probability distribution of thermal cirque sizes, and it is equal to

$$\lim_{u\to+\infty} F(x,u) = F(x) = 1 - \frac{a}{2}\frac{\left[1 - F_0(x)\right]}{\left[x + a\right]} - \frac{a}{2}\frac{a - \int_0^x \left[1 - F_0(v)\right] dv}{\left[x + a\right]^2} \quad (9)$$

where $a, F_0(x)$ are, respectively, the mathematical expectation (mean size) and size distribution of young thermal cirques forming.

The second part of the task was to carry out studies of thermocircle sizes (chord lengths) at specific sites, including obtaining samples of chord lengths for thermal cirque of each site and comparing them with theoretical distributions to identify characteristic features of the distributions[7]. As mentioned above, the key sites were selected based on the requirements of relative morphological homogeneity of the site and homogeneity of physiographic, primarily geological and geomorphologic, conditions; thus, there are no significant physiographic differences within each site.

The slope edges were delineated on the images and the arc-shaped boundaries of thermal cirques with the adjacent watershed surface were delineated, and the chords of the arcs were drawn. The chords were measured using ArcGIS. The obtained samples were compared with theoretical distributions of various types; the comparison was performed using the chi-square test (Pearson's criterion) in the Statistica program in compliance with the standard requirements of the methodology in terms of the sample volume and the size of partition intervals.

The sites vary considerably in terms of conditions. For example, according to the state geologic maps at a scale of 1:200 000 are composed of marine, glacial, lacustrine-glacial, alluvial-marine and lacustrine-alluvial sediments from the surface; the sediments are represented by both sands and siltstones, gravel-pebble sediments, sandy loams and loams with boulders and pebbles, as well as silty-fine sandy sediments; permafrost rocks have both discontinuous and massive-island and continuous distribution. In climatic terms, the areas belong to either the Arctic or Subarctic belts.

The results of statistical processing for all sites are summarized in Table 1.

The analysis of the similarity of empirical distributions of chord lengths with different types of theoretical distributions (normal, lognormal, and gamma distribution) gives interesting results. All twelve plots show agreement with the same type of distribution, the lognormal distribution, at a significance level of 0.99. This is fulfilled when there is a significant difference in the values of the distribution parameters at different sites. The gamma distribution also agrees with the empirical data, but slightly worse; this can be explained by the general similarity of the lognormal distribution and the gamma distribution. The normal distribution does not agree with the empirical data at any site.

[7] Performed jointly with M.V. Arkhipova, V.V. Bondar, T.V. Gonikov (Victorov et al., 2023).

Figure 4 shows examples of the correspondence between empirical distributions and theoretical lognormal distributions for different sites.

OBSERVATION

Thus, mathematical modeling of the process of change in the morphological pattern of abrasion banks of the cryolithozone has shown that in different physical-geographical and geocryological conditions, a stable stationary distribution of the sizes of thermal cirques is formed under significant development time. This distribution is formed in the conditions of constant appearance of new thermal cirques, as well as complete or partial (internal and lateral) erasure of existing ones; therefore, the distribution of thermal cirque sizes observed at each moment, first of all, on the materials of space imagery, coincides with the distribution of sizes of new thermal cirques being formed. Thus, the obtained result allows us to conclude that the morphological pattern of the abrasion coast, being in constant change, nevertheless, with a significant development time, has a stationary distribution of the sizes of thermal cirques, their constant average size and average density, i.e., it is in a state of dynamic equilibrium. Interestingly, the analysis shows that the limiting distribution does not depend on the distribution of thermal cirque sizes at the initial moment $F(x,0)$.

The physical-geographical, geological-geomorphological, and geocryological conditions do not affect the existence of the limiting distribution, but affect the character of this distribution through the size distribution of young thermal cirques formed; the relationship is described by expression (9). At the same time, the density of thermal cirque generation (λ) does not influence the limiting size distribution, apparently affecting only the rate of convergence to the limiting distribution.

It should be emphasized that we have previously proposed a variant of the model for the development of morphological pattern of abrasion banks (Victorov, 2022), but it used a significant simplification — it neglected internal erosion, which greatly facilitated the analysis, but made it less accurate. Thus, the present model is a new and much better one.

Comparison of empirical distributions of observed thermal cirque sizes with the results of mathematical modeling allows us to conclude that the same type of size distributions of forming young thermal cirques is characteristic of different physiographic, in particular geological-geomorphological and geocryological conditions. This follows from the uniformity of the observed distributions of thermal cirque sizes (lognormal distributions) and the established dependence between distribution of the sizes of forming young thermal cirques and the observed distributions of thermal cirque sizes described by expression (9). The same expression allows us to predict, in a more detailed analysis, the quantitative characteristics of the thermal cirque formation process (and, accordingly, of the landslides that start the formation of thermal cirque), namely, the size distribution of emerging new thermal cirques and landslides, by measurements of the observed sizes from the materials of a single high-resolution space imagery and, accordingly, by the probability distribution of these sizes.

The results obtained are also significant in practical terms for predicting the development, in particular, the retreat, of coasts, due to the previously established correlations between the sizes of thermal cirques and the arrows of the arcs bounding them (Victorov et al., 2023), which are closely related to the rate of retreat.

CONCLUSIONS

Abrasion shores of the cryolithozone develop under the action of a complex of processes including both increase and decrease in the number of thermal cirques due to the formation of new thermal cirques and, to a greater or lesser extent, erasure of existing ones; therefore, the distribution of thermal cirque sizes observed at each moment, first of all on the materials of space imagery, generally does not coincide with the distribution of sizes of new thermal cirques being formed.

The complex of mathematical modeling and space methods allowed us to show that in different physiographic and geocryological conditions, a stable stationary distribution of the sizes of thermal cirques of abrasion shores of the Arctic cryolithozone is formed in different physiographic and geocryological conditions with a significant development time at homogeneous sites. Differences in the conditions of different sites do not affect the fact of existence of the limiting stationary distribution. Thus, the morphological pattern of the abrasion shore, being in constant change, nevertheless has a stationary distribution of the sizes of thermal cirques, their constant average size and average density, i.e. it is in a state of dynamic equilibrium.

The dependence of stable stationary distribution of abrasion bank thermal cirque sizes on the size distribution of forming young thermal cirques is obtained. Physical-geographical, geological-geomorphological and geocryological conditions of the sites influence the character of the stationary limiting distribution through the size distribution of young thermal cirques forming.

Comparison of empirical distributions of thermal cirque sizes observed from space imagery with the results of mathematical modeling allows us to conclude that different physiographic, in particular, geological-geomorphological and geocryological conditions are characterized by the same type of size distribution of forming young thermal cirques; the conditions affect only the values of distribution parameters.

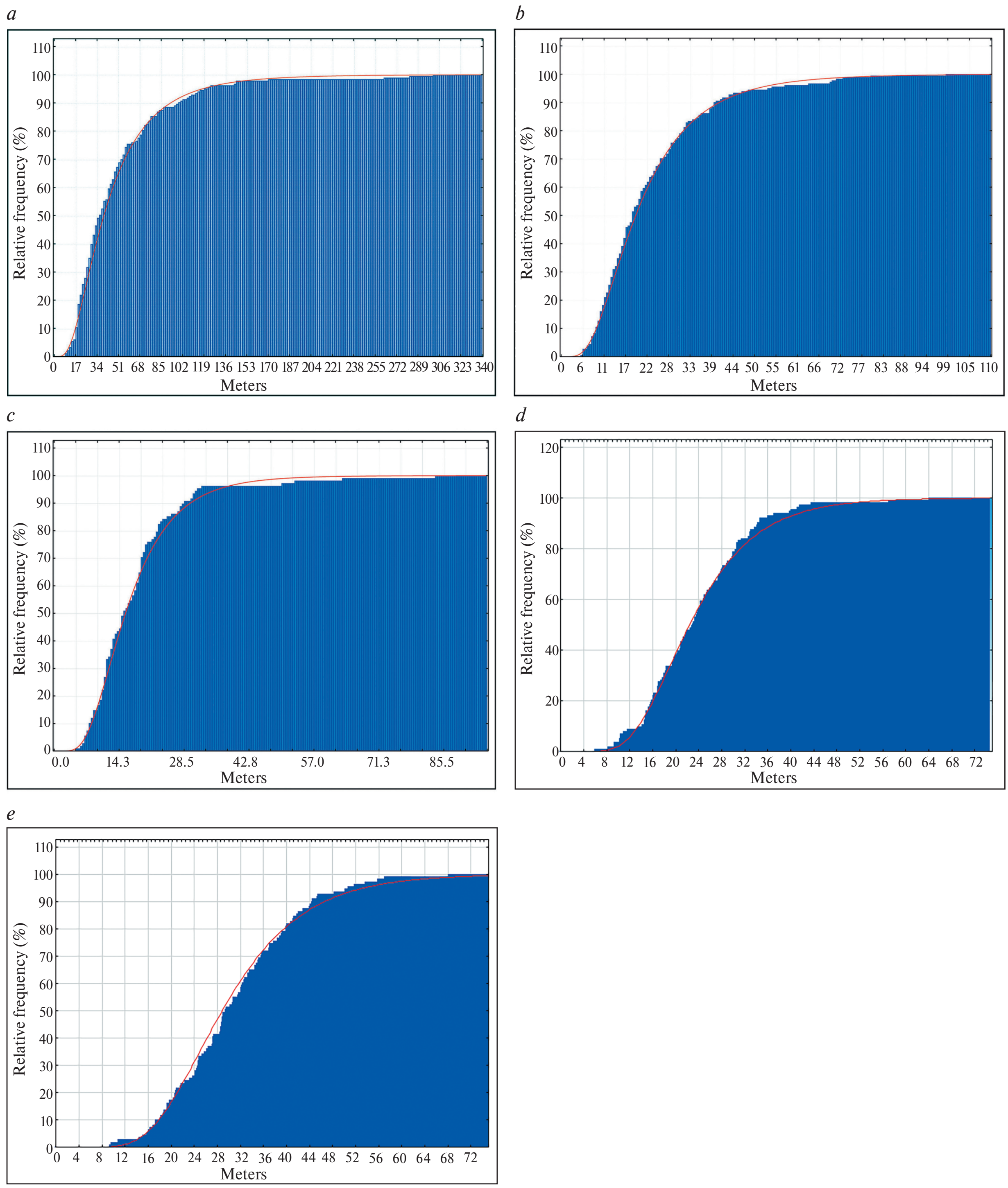

Fig. 4. Examples of correspondence between empirical distributions (blue contour) and theoretical lognormal distributions (red line) for key sites (*a* — KNS1, *b* — KNS3, *c* — JML1, *d* — KLG1, *e* — KHPD1)

The obtained results make it possible to predict the quantitative characteristics of the thermal cirque formation process, namely, the size distribution of emerging new thermal cirques, based on measurements of the observed thermal cirque sizes using the materials of a single high-resolution space imagery; this is essential in forecasting the development, in particular, the coastal retreat due to the existing correlation between the thermal cirque sizes and the arrows of the arcs bounding them.

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