SOLVABILITY THEORY OF SPECIAL INTEGRODIFFERENTIAL EQUATIONS IN THE CLASS OF GENERALIZED FUNCTIONS

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Abstract. A linear integrodifferential equation with a special differential operator in the principal part is studied. For its approximate solution in the space of generalized functions a special generalized version of the collocation method is proposed and justified. Bibl.16.

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1. INTRODUCTION

This paper is devoted to the approximate solution of the linear integrodifferential equation (IDE)

$$\left(Ax\right)(t) \equiv x^{(q)}(t)\prod_{j=1}^{l}(t-t_j)^{m_j} + \sum_{j=0}^{p}\int_{-1}^{1} K_j(t,s)x^{(j)}(s)ds = y(t), \qquad (1.1)$$

in which $t \in I \equiv [-1,1]$, numbers $t_j \in \left(-1,1\right), m_j \in N$, $j=\overline{1,l}$, and $q, p \in Z^+$ are fixed; $K_j, j=\overline{0,p}$ and y are known "smooth" functions, and x is the desired function. The study of such equations is of undoubted interest both from the point of view of theory (in particular, IMU (1.1) is a generalization of a number of classes of linear integral equations of Fredholm type) and applications. Obviously, the problem of finding the solution of IMU (1.1) in the class of ordinary smooth functions is incorrectly posed. Consequently, the question of constructing the basic spaces ensuring the correctness of this problem is important. When solving this question, it is quite natural to take into account the fact that in the case of $q = p = 0$ the IMU (1.1) is transformed into a linear integral equation of the third kind (UTR) (i.e., in this sense these equations are "related"). It is well known that UTRs are widely used in various fields; in particular, they are found in a number of problems in the theories of neutron transport, elasticity, and particle scattering (see, for example, [1; 2, pp.121-129] and the bibliography given therein), in the theory of partial equations of mixed type [3], and also in the theory of singular integral equations with degenerate symbol [4]. In this case, as a rule, natural classes of solutions of UTRs are special spaces of generalized functions of the type D or V . By D (respectively V) we mean the space of generalized functions constructed by means of the "Dirac delta function" functional (respectively the "finite part of the Adamar integral" functional). A detailed review of the results obtained and an extensive bibliography on UTR can be found in the monograph [5, pp.3-11, 168-173] and in the thesis [6, pp.3-6, 106-114]. Based on the above-mentioned connection between IMU (1.1) and UTR, the corresponding ideas and results for UTR can be successfully used for correct formulation of the problem of solving equation (1.1), development and theoretical justification of approximate methods for its solution in spaces of generalized functions.

The IMU (1.1) at $l = 1,\ t_1 = 0,\ p = 0$ is studied in [7, pp.25-43], in which, using known results on UTR, the Nöter theory for such an equation in the classes of smooth and generalized functions of type D is constructed. In [8] a complete theory of solvability of a general IMU of the form (1.1) is developed at $p = 0$ in some space of generalized functions of the type D . It should be noted that the investigated IMUs are solved exactly only in very rare special cases. Therefore, the development of effective methods of their approximate solution in spaces of generalized functions with the corresponding theoretical justification is especially urgent. Certain results in this direction have been obtained for the IMU (1.1) at $p = 0$. Namely, in [8-11] direct projective methods of its approximate solution based on the application of standard and some special polynomials, as well as splines of the first and second orders are proposed and justified.

In the present paper, for the first time, a complete theory of solvability of IMU (1.1) in some space of the type D of generalized functions (Fredholm equation, solvability conditions, algorithm for finding the exact solution, sufficient conditions for continuous reversibility of the operator A) is constructed. Moreover, a polynomial direct projection method specially adapted to the approximate solution of IMU (1.1) in the class of generalized functions is developed and its justification in the sense of [12; Ch.1, §1-5] is given. Exactly, we prove the existence and uniqueness theorem of the solution of the corresponding approximate equation, establish estimates of error of this solution, and prove convergence of the sequence of approximate solutions to the exact solution in the space of generalized functions. The questions of stability and conditionality of approximating equations are also investigated.

2. SPACES OF BASIC AND GENERALIZED FUNCTIONS

Let $C \equiv C(I)$ be the Banach space of all functions continuous on I with the usual max-norm and $m \in N$. We denote by $C\{m;0\} \equiv C_0^{\{m\}}(I)$ the set of all functions $f \in C$, having at a point $t=0$ a Taylor derivative $f^{\{m\}}(0)$ of order m (see, e.g., [13]). Let us call it a class

point-smooth functions (naturally we consider that $C\{0;0\} \equiv C$). Let us supply the vector space $C\{m;0\}$ with the norm

$$\| f\|_{\{m\}} \equiv \| Tf \|_C + \sum_{i=0}^{m-1} \left| f^{\{i\}}(0) \right|, \tag{2.1}$$

where $T : \mathrm{C}\{m;0\} \to C$ is a "characteristic" operator of class $C\{m;0\}$, defined as follows:

$$Tf \equiv (T^m f)(t) \equiv \left[f(t) - \sum_{i=0}^{m-1} f^{\{i\}}(0) t^i / i! \right] t^{-m} \equiv F(t) \in C, \tag{2.2}$$
$$F(0) \equiv \lim_{t \to 0} F(t).$$

Fair (see, for example, [5, p. 12,14])

Lemma 2.1. i. *The inclusion of $f \in C\{m;0\}$ is equivalent to the expression*

$$f(t) = t^m F(t) + \sum_{i=0}^{m-1} \alpha_i t^i, \tag{2.3}$$

$Tf = F \in C$ *andwith the precision of an avoidable discontinuity at $t=0$, and* $f^{\{i\}}(0) = \alpha_i i!, \quad i = \overline{0, m-1}.$

ii. *The space* $C\{m;0\}$ *is by norm* (2.1) *completely and normally embedded in the space* .C

Further, we introduce the following class of "point-smooth" functions:

$$C\{m,q;0\} \equiv \left\{ f \in C\{m;0\} : f^{\{i\}}(0) = 0, \quad i = \overline{0,q-1}, q \in Z^+, q < m \right\}.$$

Hence, given (2.1) - (2.3) (in which we have $i = \overline{q, m-1}$) by norm (2.1), the space $C\{m,q;0\}$ is completely and normally embedded in .C

Let us denote by $C^{(q)} \equiv C^{(q)}(I)$ the vector space of q times continuously differentiable on I functions. By virtue of the Taylor formula with integral residue, it is clear that the function f belongs to the class $C^{(q)}$ if and only if it has the form

$$f(t)=\left(JF\right)\left(t\right)+\sum_{j=0}^{q-1}b_j\left(t+1\right)^j, \tag{2.4}$$

where

$$JF\equiv\left(J_{q-1}F\right)(t)\equiv\left((q-1)!\right)^{-1}\int_{-1}^{t}\left(t-s\right)^{q-1}F(s)ds, \tag{2.5}$$

and $D^q f\equiv f^{(q)}(t)=F(t)\in C,\ f^{(j)}(-1)=b_j j!,\ j=\overline{0,q-1};$ at that $J:C\to C^{(q)},\quad \left(JF\right)^{(j)}=J_{q-1-j}F,\quad j=\overline{0,q-1},\quad D^qJF=F.$

In the vector space $C^{(q)}$ we define a special norm

$$\left\|f\right\|_{(q)}\equiv\left\|D^q f\ \right\|_C+\sum_{j=0}^{q-1}\left|f^{(j)}(-1)\right|,\quad f\in C^{(q)}. \tag{2.6}$$

From relations (2.4), (2.6) and evaluation of the integral (2.5) by max-norm it easily follows

Lemma 2.2. *The space* $C^{(q)}$ *with norm* (2.6) *is complete and embedded in the space* $.C$

Corollary 1. The normal norm $\left\|\cdot\right\|_{C^{(q)}}$ in $C^{(q)}$ and (2.6) are equivalent, i.e., there exists a constant $d_0\geq 1$ such

$$\|\ f\|_{(q)}\leq\left\|f\right\|_{C^{(q)}}\leq d_0\left\|f\right\|_{(q)},\qquad \left\|f\right\|_{C^{(q)}}\equiv\sum_{i=0}^{q}\left\|f^{(i)}\right\|_C,\quad f\in C^{(q)}.$$

Let $C_{-1}^{(q)}\equiv C_{-1}^{(q)}(I)\equiv\left\{f\in C^{(q)}:f^{(i)}(-1)=0,\quad i=\overline{0,q-1}\ \right\}$ be the Banach space of smooth functions with norm $.\|\ f\|_{(q)}\equiv\left\|D^q f\right\|_C$

In further studies we will need one more class of smooth functions:

$$C_{-1}^{(\lambda),(q)}\equiv C_{-1}^{(\lambda),(q)}(I)\equiv C^{(\lambda)}\bigcap C_{-1}^{(q)},\quad \lambda\equiv q+p.$$

By virtue of (2.4) it is obvious that the inclusion of $f\in C_{-1}^{(\lambda),(q)}$ is equivalent to the representation of

$$f(t)=\left(J_{\lambda-1}f^{(\lambda)}\right)\left(t\right)+\sum_{k=q}^{\lambda-1}f^{(k)}(-1)\left(t+1\right)^k\big/k!. \tag{2.7}$$

Hence, based on Lemma 2.2, it is obvious that by the norm

$$\left\|f\right\|_{(\lambda)}\equiv\left\|D^\lambda f\ \right\|_C+\sum_{k=q}^{\lambda-1}\left|f^{(k)}(-1)\right| \tag{2.8}$$

the space $C_{-1}^{(\lambda),(q)}$ is full and nested in C . Therefore, the usual norm in $C^{(\lambda)}$ and (2.8) are equivalent:

$$\left\|f\right\|_{(\lambda)}\leq\left\|f\right\|_{C^{(\lambda)}}\leq d_1\left\|f\right\|_{(\lambda)},\ f\in C_{-1}^{(\lambda),(q)},\ d_1\geq 1. \tag{2.9}$$

Lemma 2.3. *For any function* $f\in C_{-1}^{(\lambda),(q)}$ *the following is true*

$$\left\|\ f^{(q)}\right\|_{(p)}=\left\|f\right\|_{(\lambda)}. \tag{2.10}$$

Proof. By virtue of (2.7) we have

$$f^{(q)}(t)=\left(J_{\lambda-1-q}f^{(\lambda)}\right)(t)+\left[\sum_{k=q}^{\lambda-1}f^{(k)}(-1)\left(t+1\right)^{k}/k!\right]^{(q)}=$$

$$=\left(J_{p-1}f^{(\lambda)}\right)(t)+\sum_{j=0}^{p-1}f^{(q+j)}(-1)\left(t+1\right)^{j}/j!,$$

whence by virtue of (2.4) - (2.6) and (2.8) we find

$$\left\| f^{(q)}\right\|_{(p)}=\left\|D^{p}f^{(q)}\right\|_{C}+\sum_{j=0}^{p-1}\left|f^{(q+j)}(-1)\right|\equiv\left\|f\right\|_{(\lambda)},$$

as requested.

In the further study of the regular integrodifferential operator we will need one important property of "point-smooth" functions. In this connection, let us introduce the following class of "smooth" functions:

$$C_0^{\{n\},(r)}\equiv C_0^{\{n\},(r)}(I)\equiv\left\{\varphi\in C\left\{n;0\right\}:T^{n}\varphi\in C^{(r)},\quad r=0,1,2,...\right\},$$

where T^n is a "characteristic" operator of the class $C\left\{n;0\right\}$, defined according to the rule (2.2). We will use the family

$$Y_j\equiv C_0^{\{m-q-1+j\},(j)},\quad j=\overline{0,p},\ q<m,$$

where m, q and p are fixed parameters appearing in IMU (1.1) at $.l=1$

Lemma 2.4. *For any function* $\varphi\in Y_j$, $j=\overline{0,p}$ *there is the equality*

$$\left(\varphi^{(j)}\right)^{\{k\}}(0)=\varphi^{\{k+j\}}(0),\ j=\overline{0,p},\ k=\overline{0,m-q-1}. \tag{2.11}$$

Proof. At $j=0$ the property is obvious. By virtue of the structure (2.3) of the "point-smooth" function we have

$$\varphi(t)=t^{m-q-1+j}\cdot\Phi_j(t)+\sum_{k=0}^{m-q-2+j}a_k t^k, \tag{2.12}$$

where

$$\Phi_j\equiv T^{m-q-1+j}\varphi\in C^{(j)},\ \varphi^{\{k\}}(0)=a_k k!,\ k=\overline{0,m-q-2+j},\ j=\overline{1,p}.$$

Differentiating (2.12) successively j times by applying the usual Leibniz formula, we easily obtain the following representation:

$$\varphi^{(j)}(t)=t^{m-q-1}\left(T^{m-q-1}\varphi^{(j)}\right)(t)+\sum_{k=0}^{m-q-2}\tau_{k,j}\,a_{k+j}t^k=$$

$$=t^{m-q-1}\left[\tau_{m-q-1,j}\Phi_j(t)+g_j(t)\right]+\sum_{k=0}^{m-q-2}\tau_{k,j}\,a_{k+j}t^k, \tag{2.13}$$

in which g_j is expressed in a certain way through Φ_j , with $g_j(t)=o(1)$ at $t\to 0$, a $\tau_{k,j}\equiv\prod_{l=1}^{j}\left(k+l\right),\ j=\overline{1,p},\ \tau_{k,0}\equiv 1.$

According to (2.13), (2.3), (2.12) and the definition of the Taylor derivative (see, for example, [5, p.12]) we find the derivatives of the corresponding orders:

$$\left(\varphi^{(j)}\right)^{\{k\}}(0)=\tau_{k,j}\,a_{k+j}k!=a_{k+j}\left(k+j\right)!\,,\,k=\overline{0,m-q-2}; \tag{2.14}$$

$$\left(\varphi^{(j)}\right)^{\{m-q-1\}}(0)\equiv\left(m-q-1\right)!\lim_{t\to 0}\left(T^{m-q-1}\varphi^{(j)}\right)\left(t\right)=$$
$$=\left(m-q-1\right)!\tau_{m-q-1,j}\lim_{t\to 0}\Phi_j\left(t\right)=\left(m-q-1+j\right)!\lim_{t\to 0}\Phi_j\left(t\right)\equiv$$
$$\equiv\varphi^{\{m-q-1+j\}}(0),\quad j=\overline{1,p}. \tag{2.15}$$

On the other hand, by virtue of (2.12) and (2.3) we have

$$\varphi^{\{k+j\}}(0)=\mathrm{a}_{k+j}\left(k+j\right)!,\;\; k=\overline{0,m-q-2},\, j=\overline{1,p}. \tag{2.16}$$

From (2.14) to (2.16) follows (2.11), which was required.

Let us now construct the basic space in our research:

$$Y\equiv C^{(p)}\left\{m,q;0\right\}\equiv\left\{y\in C\left\{m,q;0\right\}:Ty\equiv T^m y\in C^{(p)}\right\}.$$

Let's set the norm in it

$$\|\, y\|_Y\equiv\|\, Ty\,\|_{(p)}+\sum_{i=q}^{m-1}\left|y^{\{i\}}(0)\right|,\quad y\in Y. \tag{2.17}$$

Lemma 2.5(see [14]). i. *Inclusion of* $\varphi\in Y$ *is equivalent to the representation*

$$\varphi(t)=\left(UJ_{p-1}\Phi\right)(t)+t^m\sum_{j=0}^{p-1}\alpha_j\left(t+1\right)^j+\sum_{i=q}^{m-1}\beta_i t^i, \tag{2.18}$$

with $D^pT\varphi=\Phi\in C,\;\; \left(T\varphi\right)^{(j)}(-1)=\alpha_j j!,\, j=\overline{0,p-1},\; \varphi^{\{i\}}\left(0\right)=\beta_i i!,\, i=\overline{q,m-1};$ $Uf\equiv t^m f(t),$ *operator* J_{p-1} *defined according to* (2.5).

ii. The space Y *with respect to the norm* (2.17*) is complete and embedded in the space* $C\left\{m,q;0\right\}$.

The criterion of compactness of sets in the space Y establishes

Lemma 2.6 (see[14]). *The set* $M\subset Y$ *is relatively compact in* Y *if and only if:* (i) M *is bounded;* (ii) *the family* $D^pT(M)$ *of functions continuous on* I *is equally continuous.*

Further over the space Y *of basic functions we construct the family* $X\equiv D_{-1}^{(\lambda),(q)}\left\{m;0\right\}$ *of generalized functions* $x(t)$ *of the following form*

$$x(t)\equiv z(t)+\sum_{i=0}^{m-q-1}\gamma_i\delta^{\{i\}}(t), \tag{2.19}$$

where $t\in I,\, z\in C_{-1}^{(\lambda),(q)},\, \lambda\equiv q+p,\, \gamma_i\in R$ *are arbitrary constants, and* δ *and* $\delta^{\{i\}}$ *are respectively the Dirac delta function and its "Taylor" derivatives, acting on the space* Y *of principal functions according to the following rule:*

$$\left(\delta^{\{i\}},y\right)\equiv\int_{-1}^{1}\delta^{\{i\}}\left(t\right)y(t)dt\equiv(-1)^i y^{\{i\}}(0),\qquad i=\overline{0,m-q-1}. \tag{2.20}$$

Obviously, the vector space X *is Banach with respect to the norm*

$$\| x\|_X \equiv \| z\|_{(\lambda)} + \sum_{i=0}^{m-q-1} |\gamma_i|. \qquad (2.21)$$

We conclude this section by giving a property about the "mixed" derivatives of the delta function, which is needed later on.

Lemma 2.7. *On the space* Y_j *of basic functions the equality is true*

$$\left(\delta^{\{i\}}(t)\right)^{(j)} = \delta^{\{i+j\}}(t), \quad j=\overline{0,p}, \quad i=\overline{0,m-q-1}. \qquad (2.22)$$

Proof. Note that (see, e.g., [15, p.419]) for any function $\varphi \in Y_j$ there is a relation

$$\left(\left(\delta^{\{i\}}\right)^{(j)}, \varphi\right) \equiv (-1)^j \left(\delta^{\{i\}}, \varphi^{(j)}\right) \equiv (-1)^{j+i} \left(\varphi^{(j)}\right)^{\{i\}}(0), \qquad (2.23)$$

$$j=\overline{0,p}, \quad i=\overline{0,m-q-1}.$$

On the other hand, by virtue of (2.20) we have

$$\left(\delta^{\{i+j\}}, \varphi\right) \equiv (-1)^{i+j} \varphi^{\{i+j\}}(0), \quad j=\overline{0,p}, \quad i=\overline{0,m-q-1}. \qquad (2.24)$$

Hence, the required equality (2.22) follows from (2.23), (2.24) and (2.11).

3. THE FREDHOLMICITY OF THE STUDIED IDU

Let IMU (1.1) be given. For the sake of reducing cumbersome calculations and simplifying the formulations, without limiting the generality of ideas, methods and results, we will hereafter assume $l=1,\ t_1=0$, i.e., we consider the IMU of the form

$$\left(Ax\right)(t) \equiv \left(Vx\right)(t) + \left(Kx\right)(t) = y(t), \quad t \in I, \qquad (3.1)$$

$$V \equiv UD^q,\ D^q f \equiv f^{(q)}(t),\ Ug \equiv t^m\, g(t),\ Kx \equiv \sum_{j=0}^{p} \int_{-1}^{1} K_j(t,s) x^{(j)}(s)ds,$$

where $q, p \in Z^+,\ m \in N,\ q < m;\ y \in Y \equiv C^{(p)}\{m,q;0\}$, K_j are known kernels with the following properties:

$$K_j(t,\cdot) \in Y,\ K_j(\cdot,s) \in Y_j,\ \varphi_{jk}(s) \equiv \left(K_j\right)_t^{\{k\}}(0,s) \in C,$$

$$\psi_{ji}(t) \equiv \left(K_j\right)_s^{\{i+j\}}(t,0) \in Y, \quad j=\overline{0,p},\ k=\overline{q,m-1},\ i=\overline{0,m-q-1}; \qquad (3.2)$$

and $x \in X$ is the element you're looking for.

Theorem 1. *Under the conditions* (3.2), *the operator* $A: X \to Y$ *fredholms*.

Proof. Let us study the

$$Vx \equiv t^m x^{(q)}(t) = y(t), \quad y \in Y. \qquad (3.3)$$

Let us show that the operator $V: X \to Y$ is bounded. By virtue of (2.19) and (3.3) we have

$$\left(D^q x\right)(t) = \left(D^q z\right)(t) + \sum_{i=0}^{m-q-1} \gamma_i \delta^{\{i+q\}}(t) = \left(D^q z\right)(t) + \sum_{k=q}^{m-1} \gamma_{k-q} \delta^{\{k\}}(t). \qquad (3.4)$$

Then, given the property

$$\left(t^m \cdot \delta^{\{k\}}(t), \varphi(t)\right) \equiv \left(\delta^{\{k\}}, t^m \varphi(t)\right) \equiv (-1)^k \left(t^m \cdot \varphi\right)^{\{k\}}(0) = 0, \qquad (3.5)$$

$$k = \overline{0, m-1}, \varphi \in C,$$

we obtain $Vx \equiv UD^q x = UD^q z$, whence, on the basis of relations (2.17), (2.18), (2.21) and (2.10) it follows that

$$\|Vx\|_Y = \|UD^q z\|_Y \equiv \|TUD^q z\|_{(p)} = \|D^q z\|_{(p)} = \|z\|_{(\lambda)} \le \|x\|_X,$$

i.e. $\|V\| \equiv \|V\|_{X \to Y} \le 1$.

Now in the space $X \equiv D_{-1}^{(\lambda),(q)}\{m;0\}$ we find the solution of equation (3.3) and the index of the operator V. From Equations (3.4) and (3.5) it follows that in the space X the general solution of the homogeneous equation $Vx = 0$ has the form

$$\tilde{x}(t) \equiv \sum_{i=0}^{m-q-1} \gamma_i \delta^{\{i\}}(t), \quad \gamma_i \in R;$$

hence, $\alpha(V) \equiv \dim\ker V = m - q$. On the other hand, the inhomogeneous equation (3.3) is solvable in X if and only if the additional conditions $\left(\delta^{\{i\}}(t), y\right) = 0, \quad i = \overline{q, m-1}$. If they are satisfied, the general solution of equation (3.3) is represented by the formula

$$x^*(t) = \left(J_{q-1} Ty\right)(t) + \sum_{i=0}^{m-q-1} \gamma_i \delta^{\{i\}}(t), \quad \gamma_i \in R.$$

This means that $\beta(V) \equiv \dim\operatorname{coker} V = m - q$. Thus, $ind\, V \equiv \alpha(V) - \beta(V) = 0$, i.e., the operator $V : X \to Y$ fredholms.

Next, let us discuss the properties of the integrodifferential operator K. By virtue of relations (3.1), (3.2), (2.19), (2.22) and (2.20) we have

$$(Kx)(t) = (Kz)(t) + \sum_{j=0}^{p} \sum_{i=0}^{m-q-1} (-1)^{i+j} \gamma_i \psi_{ji}(t). \qquad (3.6)$$

Hence, taking into account the conditions (3.2), we see that $Kx \in Y, x \in X$.

Before proceeding to the evaluation of the image (3.6) of the operator K we assume the following notations:

$$d_2 \equiv \max_{j=\underline{0,p}} \left\|D_t^p T_t K_j\right\|_C, \quad d_3 \equiv \max_{j=\underline{0,p}} \sum_{l=0}^{p-1} \left\|\left(T_t K_j\right)_t^{(l)}(-1,s)\right\|_C,$$

$$d_4 \equiv \max_{j=\underline{0,p}} \sum_{k=q}^{m-1} \left\|\varphi_{jk}\right\|_C, \quad d_5 \equiv \max_{i=\underline{0,m-q-1}} \sum_{j=0}^{p} \left\|\psi_{ji}\right\|_Y.$$

Then, using definition (2.17), evaluation (2.9) and definition (2.21), we successively find that

$$\|Kx\|_Y \le \|Kz\|_Y + \sum_{j=0}^{p} \sum_{i=0}^{m-q-1} |\gamma_i| \left\|\psi_{ji}\right\|_Y \equiv$$

$$\equiv \left\| \sum_j \int_{-1}^{1} \left(D_t^p T_t K_j \right)(t,s) z^{(j)}(s)ds \right\|_C + \sum_{l=0}^{p-1} \left| \sum_j \int_{-1}^{1} \left(T_t K_j \right)_t^{(l)} (-1,s) z^{(j)}(s)ds \right| +$$

$$+ \sum_{k=q}^{m-1} \left| \sum_j \int_{-1}^{1} \varphi_{jk}(s) z^{(j)}(s)ds \right| + \sum_j \sum_i |\gamma_i| \|\psi_{ji}\|_Y \le$$

$$\le 2d_2 d_1 \|z\|_{(\lambda)} + 2d_3 d_1 \|z\|_{(\lambda)} + 2d_4 d_1 \|z\|_{(\lambda)} + d_5 \sum_i |\gamma_i| \le$$

$$\le d_6 \|x\|_X, \quad d_6 \equiv 2d_1 \left(d_2 + d_3 + d_4 \right) + d_5.$$

Consequently, the operator K acts from X to Y in a restricted manner, whereby $\|K\| \equiv \|K\|_{X \to Y} \le d_6$.

Further, let $L \equiv \{x\} \subset X$ be an arbitrary bounded set. Reasoning analogously to the case of integral equations of the third kind (see [5, p.52, 53]), using Lemma 2.6 it is easy to show that the set $M \equiv K(L)$ is relatively compact in Y . In other words, the operator $K : X \to Y$ is quite continuous. Then the statement of Theorem 1 follows directly from the fact that the perturbation of a nonetherian operator by a completely continuous operator preserves nonetherianism and does not change its index.

4. CONTINUOUS REVERSIBILITY OF THE INTEGRODIFFERENTIAL OPERATOR

Consider the IMU (3.1) in which the kernels K_j are subject to conditions (3.2), $y \in Y$, and $x \in X$ is the desired generalized function of the form (2.19). Taking into account relations (2.19), (3.4) - (3.6), we transform equation (3.1) to the form

$$\left(Az \right)(t) = y(t) - \sum_{i=0}^{m-q-1} c_i f_i(t), \tag{4.1}$$

where $f_i(t) \equiv \sum_{j=0}^{p} \left(-1\right)^j \psi_{ji}(t), c_i \equiv \left(-1\right)^i \gamma_i, \quad i = \overline{0, m-q-1}.$ Our problem is to find the function $z \in C_{-1}^{(\lambda),(q)}$ and arbitrary constants c_i.

Lemma 4.1. *Let the following requirements be satisfied*:

$$K_j(t,\cdot) \in Y, \varphi_{jk}(s) \equiv \left(K_j \right)_t^{\{k\}}(0,s) \in C, y \in Y, \quad j = \overline{0,p}, k = \overline{q, m-1}.$$

Then IMU (3.1)$\left(A : C_{-1}^{(\lambda),(q)} \to Y \right)$ *is equivalent in the space* $C_{-1}^{(\lambda),(q)}$ *to IMU*

$$Bx \equiv \left(D^q x \right)(t) + \sum_{j=0}^{p} \int_{-1}^{1} \left(T_t K_j \right)\left(t,s\right) x^{(j)}(s)ds = (Ty)(t)$$

and ratios

$$\sum_{j=0}^{p} \int_{-1}^{1} \varphi_{jk}(s) x^{(j)}(s)ds = y^{\{k\}}(0), \quad k = \overline{q, m-1}.$$

Proof. By virtue of expression (2.3) it is obvious that for any function $g \in Y$ there is an equivalence:

$$g = 0 \Leftrightarrow Tg = 0,\ g^{\{k\}}(0) = 0, \quad k = \overline{q, m-1}. \tag{4.2}$$

Then, taking in (4.2) $g \equiv Ax - y \in Y,\ x \in C_{-1}^{(\lambda),(q)},\ y \in Y,$ we see that the statement of the lemma is true.

It follows from this lemma that equation (4.1) is equivalent to IMU

$$(Bz)(t) = (Ty)(t) - \sum_{i=0}^{m-q-1} c_i (Tf_i)(t) \tag{4.3}$$

in the space $C_{-1}^{(\lambda),(q)}$ and relations

$$y^{\{k\}}(0) - \sum_{j=0}^{p} \int_{-1}^{1} \varphi_{jk}(s) z^{(j)}(s) ds - \sum_{i=0}^{m-q-1} c_i f_i^{\{k\}}(0) = 0, \quad k = \overline{q, m-1}. \tag{4.4}$$

Let us first study in detail the IMU of the form (4.3) with the operator $: B$

$$(Bz)(t) \equiv z^{(q)}(t) + \sum_{j=0}^{p} \int_{-1}^{1} \mu_j(t,s) z^{(j)}(s) ds = f(t), \tag{4.5}$$

in which $\mu_j \equiv T_t K_j,\ j = \overline{0, p},\ f \in C^{(p)}.$ We will use the substitution $z^{(q)} \equiv u(t) \in C^{(p)}$. By virtue of (2.4), (2.5) and the definition of the class $C_{-1}^{(q)}$ we have

$$z = J_{q-1} u,\ z^{(j)} = J_{q-1-j} u, \quad j = \overline{0, q-1}. \tag{4.6}$$

Let us now study the operator

$$Mz \equiv \sum_{j=0}^{p} \int_{-1}^{1} \mu_j(t,s) z^{(j)}(s) ds.$$

Let us first consider the case $p < q$. By changing the order of integration in the double integral, we find that

$$\left(Mz\right)(t) = \sum_j \left((q-1-j)!\right)^{-1} \int_{-1}^{1} \mu_j(t,s) \left(\int_{-1}^{s} \left(s-\rho\right)^{q-1-j} u(\rho) d\rho \right) ds =$$

$$= \sum_j \left((q-1-j)!\right)^{-1} \int_{-1}^{1} u(\rho) \left(\int_{\rho}^{1} \mu_j(t,s)(s-\rho)^{q-1-j} ds \right) d\rho.$$

Hence, in this case IMU (4.5) is equivalent to the following Fredholm equation of the second kind in space $C^{(p)}$:

$$Gu \equiv u(t) + \int_{-1}^{1} G_p(t,\rho) u(\rho) d\rho = f(t), \tag{4.7}$$

where

$$G_p(t,\rho) \equiv \sum_{j=0}^{p} \left((q-1-j)!\right)^{-1} \int_{\rho}^{1} \mu_j(t,s)(s-\rho)^{q-1-j} ds. \tag{4.8}$$

At $p \geq q$ taking into account (4.8) we have

$$(Mz)(t)=\int_{-1}^{1}G_{q-1}(t,\rho)u(\rho)d\rho+\sum_{j=q}^{p}\int_{-1}^{1}\mu_j(t,s)u^{(j-q)}(s)ds=$$

$$=\int_{-1}^{1}G_{q-1}\cdot u(\rho)d\rho+\int_{-1}^{1}\mu_q(t,\rho)u(\rho)d\rho+\sum_{k=1}^{p-q}\int_{-1}^{1}\mu_{q+k}(t,\rho)u^{(k)}(\rho)d\rho. \qquad (4.9)$$

Next, let us introduce the nuclei into consideration:

$$g_k(t,\rho)\equiv\begin{cases}G_{q-1}(t,\rho)+\mu_q(t,\rho) & при \quad k=0;\\ \mu_{q+k}(t,\rho), & если \quad k=\overline{1,p-q}.\end{cases}$$

Then taking into account (4.9) IMU (4.5) takes the form

$$Lu\equiv u(t)+\sum_{k=0}^{p-q}\int_{-1}^{1}g_k(t,\rho)u^{(k)}(\rho)d\rho=f(t), \qquad (4.10)$$

moreover $g_k(t,\cdot)\in C^{(p)}$.

Thus, at $p<q$, substitution $z^{(q)}\equiv u$ equivalently leads IMU (4.3) to the equation of the second kind

$$(Gu)(t)=(Ty)(t)-\sum_{i=0}^{m-q-1}c_i(Tf_i)(t). \qquad (4.11)$$

Let $\nu=-1$ not be an eigenvalue of equation (4.11) (or the kernel G_p) and R be the solving operator of this equation. Then the function

$$u^*(t)\equiv(RTy)(t)-\sum_i(RTf_i)(t)$$

is the only smooth solution of equation (4.11). Hence,

$$z^*(t)\equiv(J_{q-1}u^*)(t)=(J_{q-1}RTy)(t)-\sum_i c_i(J_{q-1}RTf_i)(t)$$

is the only smooth solution of IMU (4.3), which will be the solution of the initial equation (4.1) if by virtue of (4.4) the constants $\{c_i\}$ satisfy the quadratic system of linear algebraic equations (SLAE)

$$\sum_{i=0}^{m-q-1}c_i(Qf_i)^{\{k\}}(0)=(Qy)^{\{k\}}(0), \quad k=\overline{q,m-1}, \qquad (4.12)$$

where the operator $Q\equiv E-KJ_{q-1}RT$ maps Y to Y , and E is a unit operator in $.Y$

In the case $p\geq q$, taking into account (4.9) and (4.10), IMU (4.3) is equivalent to the Fredholm equation of the II kind

$$(Lu)(t)=(Ty)(t)-\sum_{i=0}^{m-q-1}c_i(Tf_i)(t) \qquad (4.13)$$

with the enabling operator $.\tilde{R}:C^{(p)}\to C^{(p)}$

Thus, it's been proven

Theorem 2. *Let the following conditions be satisfied*:

a) *kernels* K_j, $j=\overline{0,p}$ *, satisfy the requirements* (3.2), *and the* $y \in Y$;

b) the *number* $\nu=-1$ *is not an eigenvalue of equation* (4.11) *at* $p<q$ (*respectively, of equation* (4.13) *in the case of); $p \ge q$*

c) *the determinant of SLAU* (4.12) *is different from zero* (*at* $p \ge q$ *the role of the operator* R *is played by* $\tilde{R}$ *).*

Then for any right part $y \in Y$ the IMU (3.1) has a single generalized solution $x^* \in X$, represented by the formula

$$x^*(t)=\left(J_{q-1}STy\right)(t)-\sum_{i=0}^{m-q-1} c_i^*\left(J_{q-1}STf_i\right)(t)+\sum_{i=0}^{m-q-1}(-1)^i c_i^* \delta^{\{i\}}(t),$$

where $S=R$ at $, p<q$ $S=\tilde{R}$ in the case of $p \ge q$, and $\left\{c_i^*\right\}$ is the only solution of SLAU of the form (4.12).

Corollary 2. Under the conditions of Theorem 2, the integrodifferential operator $A: X \to Y$, defined by equality (3.1), is continuously reversible.

5.GENERALIZED COLLOCATION METHOD (GCM)

Let IMU (3.1) be given, in which the kernels K_j, $j=\overline{0,p}$, have properties (3.2), $y \in Y$, and $x \in X$ is the desired element. We will look for its approximate solution in the form

$$x_n \equiv x_n\left(t;\left\{c_j\right\}\right) \equiv z_n(t)+\sum_{i=0}^{m-q-1} c_{i+n+\lambda} \delta^{\{i\}}(t), \tag{5.1}$$

$$z_n(t) \equiv \sum_{i=q}^{n+\lambda-1} c_i t^i, \quad \lambda \equiv q+p,\ n=2,3,\ldots. \tag{5.2}$$

The unknown parameters $c_j=c_j^{(n)}$, $j=\overline{q, n+m+p-1}$, are found, according to OMK, from the quadratic SLAU $(n+m+p-q)$ -th order:

$$\left(D^p T \rho_n\right)\left(\nu_k\right)=0, \quad k=\overline{1,n}, \quad \left(T \rho_n\right)^{(j)}(-1)=0,\ j=\overline{0,p-1},$$
$$\rho_n^{\{i\}}(0)=0,\ i=\overline{q,m-1}, \tag{5.3}$$

where $\rho_n(t) \equiv \rho_n^A(t) \equiv (Ax_n-y)(t)$ is the non-convexity of the approximate solution, and $\left\{\nu_k\right\} \subset I$ is a system of Chebyshev knots of genus I(or II)

For the computational algorithm (3.1), (5.1) - (5.3) is true

Theorem 3: *Suppose that* the *homogeneous IMU* $Ax=0$ *has only zero solution in* X *(e.g., under the conditions of Theorem* 2), *and the functions* $h_j \equiv D_t^p T_t K_j$ *(by* t *),* $g_{ji} \equiv D^p T \psi_{ji}$, $j=\overline{0,p}, i=\overline{0,m-q-1}$ *and* $D^p Ty$ *belong to the Dini-Lipschitz class. Then for all* $n \in N, n \ge n_0$, *the SLAU* (5.3) *has a single solution* $\left\{c_j^*\right\}$ *and the sequence of approximate solutions* $x_n^* \equiv x_n\left(t;\left\{c_j^*\right\}\right)$ *converges* *to the exact solution* $x^* = A^{-1}y$ *of equation* (3.1) *by the norm of the space* X *with rate*

$$\Delta x_n^* = \left\| x_n^* - x^* \right\| = O\left\{ \left[\sum_{j=0}^{p} \left(E_{n-1}^t(h_j) + \sum_{i=0}^{m-q-1} E_{n-1}(g_{ji}) \right) + E_{n-1}\left(D^p Ty\right) \right] \ln n \right\}, \qquad (5.4)$$

where $E_l(f)$ *is the best uniform approximation of the function* $f \in C$ *by algebraic polynomials of degree no higher than* l , *and* $E_l^t(\cdot)$ *denotes the functional* $E_l(\cdot)$ *applied on the variable* t.

Proof. Obviously, the IMU (3.1) is represented as a linear operator equation

$$Ax \equiv Vx + Kx = y, x \in X \equiv D_{-1}^{(\lambda),(q)}\{m;0\}, y \in Y \equiv C^{(p)}\{m,q;0\}, \quad (5.5)$$

in which the operator $A: X \to Y$ is continuously reversible.

Let us also write the system (5.1) - (5.3) in operator form. For this purpose, we construct the corresponding finite-dimensional subspaces. Namely, we denote by $X_n \subset X$ the $(n+m+p-q)$ -dimensional subspace of elements of the form (5.1), and by $Y_n \subset Y$ we take the class $\Pi_q^{n+m+p-1} \equiv span\left\{t^i\right\}_q^{n+m+p-1}$. Then we introduce the linear operator $\Gamma_n \equiv \Gamma_{n+m+p-q}: Y \to Y_n$ according to the rule

$$\Gamma_n y \equiv \Gamma_{n+m+p-q}(y;t) \equiv (UJ_{p-1}L_n D^p Ty)(t) +$$
$$+\sum_{j=0}^{p-1} (Ty)^{(j)}(-1)\frac{t^m(t+1)^j}{j!} + \sum_{i=q}^{m-1} y^{\{i\}}(0)\frac{t^i}{i!}, \qquad (5.6)$$

where $L_n: C \to \Pi_0^{n-1} \equiv \Pi_{n-1} \equiv span\left\{t^i\right\}_0^{n-1}$ is an interpolation operator Lagrangian over the system of nodes $\{v_k\}_1^n$. Then the system (5.1) - (5.3) is equivalent to the following linear equation:

$$A_n x_n \equiv Vx_n + \Gamma_n Kx_n = \Gamma_n y, \quad x_n \in X_n, \Gamma_n y \in Y_n. \qquad (5.7)$$

It is not difficult to verify this by carrying out the corresponding reasoning given in the proof of Theorem 3 [8].

Thus, to prove Theorem 3, it is sufficient to establish the existence, singularity and convergence of solutions of equations (5.7). For this purpose, we need the approximating property of the operator Γ_n.

Lemma 5.1. *For any function* $y \in Y$ *the following estimation is valid*

$$\left\| y - \Gamma_n y \right\|_Y \le d_7 E_{n-1}\left(D^p Ty\right) \ln n, \quad n = 2,3,... \qquad (5.8)$$

(*hereinafter* d_i $\left(i = \overline{7,9}\right)$ *are some constants whose values do not depend on the number* n).

The fairness of this lemma follows easily from the representation (2.18), definitions (5.6), (2.17), and evaluation (see, e.g., [12, p.107])

$$\left\| f - L_n f \right\|_C \le d_7 E_{n-1}(f) \ln n, \quad f \in C.$$

Let us now discuss the "closeness" of the operators A and A_n on the subspace X_n. Using equations (3.1), (5.7) and the estimate (5.8), for an arbitrary element $x_n \in X_n$ we find that

$$\|Ax_n - A_n x_n\|_Y = \|Kx_n - \Gamma_n K x_n\|_Y \le d_7 E_{n-1}\left(D^p TKx_n\right)\ln n. \tag{5.9}$$

By virtue of (3.6) and (5.1) we have

$$(Kx_n)(t) = (Kz_n)(t) + \sum_{j=0}^{p}\sum_{i=0}^{m-q-1}(-1)^{i+j} c_{i+n+\lambda}\psi_{ji}(t).$$

Hence,

$$D^p TKx_n = \sum_{j=0}^{p}\int_{-1}^{1} h_j(t,s) z_n^{(j)}(s)ds + \sum_j\sum_i (-1)^{i+j} c_{i+n+\lambda} g_{ji}(t). \tag{5.10}$$

In order to polynomialize the function $D^p TKx_n \in C$, we construct the following element:

$$\left(P_{n-1}x_n\right)(t) \equiv \sum_j \int_{-1}^{1} h_{n-1}^j(t,s) z_n^{(j)}(s)ds + \sum_j\sum_i (-1)^{i+j} c_{i+n+\lambda} g_{n-1}^{ji}(t), \tag{5.11}$$

where h_{n-1}^j and g_{n-1}^{ji} are polynomials of degree $n-1$ of the best uniform approximation for h_j (by t) and g_{ji}, respectively. According to the structure (5.11) it is clear that $P_{n-1}x_n \in \Pi_{n-1}$.

Based on expressions (5.10) and (5.11), evaluation (2.9), and definition (2.21), we successively derive the intermediate evaluation

$$E_{n-1}\left(D^p TKx_n\right) \le \left\|D^p TKx_n - P_{n-1}x_n\right\|_C \equiv$$

$$\equiv \max_{t\in I}\left|\sum_j \int_{-1}^{1}\left(h_j - h_{n-1}^j\right)(t,s) z_n^{(j)}(s)ds + \sum_j\sum_i (-1)^{i+j} c_{i+n+\lambda}\left(g_{ji} - g_{n-1}^{ji}\right)(t)\right| \le$$

$$\le 2\|z_n\|_{C^{(\lambda)}} \sum_j E_{n-1}^t(h_j) + \sum_j\sum_i |c_{i+n+\lambda}| E_{n-1}\left(g_{ji}\right) \le$$

$$\le 2d_1 \|z_n\|_{(\lambda)} \sum_j E_{n-1}^t(h_j) + \|x_n\|_X \sum_j\sum_i E_{n-1}\left(g_{ji}\right) \le$$

$$\le 2d_1 \|x_n\|_X \sum_j E_{n-1}^t(h_j) + 2d_1\|x_n\|_X \sum_j\sum_i E_{n-1}\left(g_{ji}\right) =$$

$$= d_8\left\{\sum_j\left[E_{n-1}^t(h_j) + \sum_i E_{n-1}(g_{ji})\right]\right\}\|x_n\|, \quad d_8 \equiv 2d_1. \tag{5.12}$$

From inequalities (5.9) and (5.12) follows the desired "closeness" estimate of the operators A and A_n:

$$\varepsilon_n \equiv \|A - A_n\|_{X_n\to Y} \le d_9\left\{\sum_j\left[E_{n-1}^t(h_j) + \sum_i E_{n-1}(g_{ji})\right]\right\}\ln n. \tag{5.13}$$

Then, based on the estimates (5.13) and (5.8) from Theorem 7 (see [12; Ch.1, §4]), we obtain the statement of Theorem 3 with error estimate (5.4).

Corollary 3. If the functions h_j (by t), g_{ji} and D^pTy belong to the class $H_\alpha^r(S)$, then under the conditions of Theorem 3 the is true

$$\Delta x_n^* = O\left(n^{-r-\alpha}\ln n\right), \quad r+1\in N, \alpha\in(0,1],$$

where

$$H_\alpha^r(S)\equiv\left\{f\in C^{(r)}(I):\omega\left(f^{(r)};\Delta\right)\le S\Delta^\alpha, \quad S\equiv const>0\right\},$$

and $\omega(f;\Delta)$ is the modulus of continuity of the function $f\in C$ with step $\Delta, \quad 0<\Delta\le 2.$

6. CONCLUDING REMARKS

Remark 1. According to the definition of norm in the space $X\equiv D_{-1}^{(\lambda),(q)}\{m;0\}$ it is easy to see that the convergence of the sequence (x_n^*) of approximate solutions to the exact solution $x^*=A^{-1}y$ in the metric X implies the usual convergence in the space of generalized functions, i.e. weak convergence.

Remark 2. When numerically solving operator equations $Ax=y$, a natural question arises about the rate of convergence of the nonconvexity $\rho_n^*(t)\equiv(Ax_n^*-y)(t)$ of the method under study. One of the results in this direction follows easily from the main theorem 3, namely: if the initial data h_j, g_{ji} and D^pTy of the equation (3.1) belong to the class $H_\alpha^r \quad (0<\alpha\le 1, \quad r=0,1,2,...)$, then under the conditions of Theorem 3 the estimate is valid $\left\|\rho_n^*\right\|_Y = O(n^{-r-\alpha}\ln n).$

Remark 3. At $q=0$ the studied IMU (3.1) is an IMU of the third kind with the operator $A:D^{(p)}\{m;0\}\to C_0^{\{m\},(p)}$, and the direct projection method (5.1) - (5.3) is a special variant of OMK for IMUs of the third kind. Consequently, Theorem 3 contains the known results [16] on the justification of a special variant of OMK for approximate solution of third-order equations in the class of generalized functions.

Remark 4. Since, under the conditions of Theorem 3, the approximating operators A_n have the property of the $\left\|A_n^{-1}\right\|=O(1), \quad A_n^{-1}:Y_n\to X_n, \quad n\ge n_1,$

it is obvious (see [12; Ch.1, §5]) that the direct method for the IMU (3.1) proposed in this paper is stable with respect to small perturbations of the initial data. This allows us to find a numerical solution of the studied equations on a computer with any predetermined degree of accuracy. Moreover, if the IMU (3.1) is well-conditioned, then the SLAE (5.3) is also well-conditioned.

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