Collocation-variational approaches of numerical solution of the Volterra integral equations of the first kind [1]

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Abstract. Linear Volterra equations of the first kind are considered. A class of such problems, which have a single solution, is singled out, and collocation-variational methods are proposed for their numerical solution. The essence of these algorithms is that the approximate solution is found in the nodes of a uniform grid (collocation condition), which give an underdetermined system of linear algebraic equations. The system thus obtained is supplemented by the condition of minimum of the target function, which approximates the square of the norm of the approximate solution. As a result, we obtain a quadratic programming problem: the target function (the square of the norm of the approximate solution) is quadratic, the constraints (collocation conditions) are equal. This problem is solved by the method of Lagrange multipliers. Simple enough methods of the third order are considered in detail. The results of calculations of test problems are given. Further development of this approach for numerical solution of other classes of integral equations is discussed. Bibl. 12. Table 4.

Keywords: *Volterra integral equations, quadrature formulas, collocation, Lagrange multiplier method.*

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1. INTRODUCTION

The paper is devoted to numerical solution of linear Volterra integral equations of the form

$$\int_0^t K(t,\tau)x(\tau)\,d\tau = f(t), \quad 0 \le \tau \le t \le 1, \tag{1}$$

where $f(t)$ and $K(t,\tau)$ are given functions with sufficiently smooth elements, $x(t)$ is the desired function. At

$$K(t,t) \ne 0 \quad \forall t \in [0,1], \;\; f(0) = 0 \tag{2}$$

and continuous functions $K(t,t), K'_t(\tau,t)|_{\tau=t}, f'(t)$ there exists a single continuous solution of this problem (see, e.g., [1], [2]).

Approaches to the numerical solution of equation (1) with condition (2) can be found in monographs [4]-[6] (collocation and multistep methods), [7] (block methods), thesis [8]. In [9], results on this topic and difficulties that arise in developing methods for solving equation (1) are presented.

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In this paper we propose one-step methods for solving the above problems, which have proved themselves perfectly when solving differential-algebraic equations (see [10] and the bibliography given there) and are a generalization of the paper [11].

2. QUADRATURE FORMULAS AND ALGORITHMS

When constructing methods for solving the original problem, we will need some results from the theory of approximate integration. Let us dwell in detail on the four-point quadrature formulas of the third order, which will be required for further presentation.

Let's set a uniform grid on the segment $[0,1] t_i = ih,\ i = 0,1,\dots,N,\ h = 1/N$, and suppose that for a sufficiently smooth function $g(t)$ it is known $g(t_i)$. Then

$$\int_{t_{i-3}}^{t_i} g(t)\,dt \approx h[b_1 g_{i-3} + b_2 g_{i-2} + b_3 g_{i-1} + b_4 g_i], \tag{3}$$

$$\int_{t_{i-3}}^{t_{i-1}} g(t)\,dt \approx h[a_1 g_{i-3} + a_2 g_{i-2} + a_3 g_{i-1} + a_4 g_i], \tag{4}$$

where the coefficients $a_j,\ b_j, j = \overline{1,4}$ satisfy the third-order conditions, i.e., the quadrature formulas (3), (4) are exact for any polynomials of degree not higher than three.

Omitting elementary calculations, we obtain that these coefficients are the solution of the SLAU

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 9 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \\ a_4 & b_4 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 2 & 4.5 \\ 8/3 & 9 \end{pmatrix}. \tag{5}$$

Assuming in (5) $a_1 = a,\ b_1 = b$ - free parameters, we obtain that the solution of SLAU (5) is

$$(a_1, a_2, a_3, a_4) = (a, 7/3 - 3a, -2/3 + 3a, 1/3 - a), \tag{6}$$

$$(b_1, b_2, b_3, b_4) = (b, 2.25 - 3b, 3b, 0.75 - b). \tag{7}$$

Let us proceed to the description of methods for approximate solution of the ELI (1) assuming that $x_0 = x(0)$ is given or calculated in advance. These algorithms are based on the quadrature formulas (3) and (4) with coefficients satisfying relations (6) and (7), respectively. For simplicity, let us assume N to be a multiple of three and denote by

$$f_i = f(t_i),\ K_{ij} = K(t_i, t_j),\ x_i \approx x(t_i).$$

In this case for equation (1) we will have

$$\int_0^{t_{i-1}} K(t_{i-1},\tau)x(\tau)\,d\tau = \int_0^{3h} K(t_{i-1},\tau)x(\tau)\,d\tau + \int_{3h}^{5h} K(t_{i-1},\tau)x(\tau)\,d\tau + \cdots + \tag{8}$$

$$+ \int_{t_{i-3}}^{t_{i-1}} K(t_{i-1},\tau)x(\tau)\,d\tau = h[b_1 K_{i-1,0}x_0 + b_2 K_{i-1,1}x_1 + b_3 K_{i-1,2}x_2 + b_4 K_{i-1,3}x_3) +$$

$$+(b_1 K_{i-1,3}x_3 + b_2 K_{i-1,4}x_4 + b_3 K_{i-1,5}x_5 + b_4 K_{i-1,6}x_6) + \cdots +$$

$$+(a_1 K_{i-1,i-3}x_{i-3} + a_2 K_{i-1,i-2}x_{i-2} + a_3 K_{i-1,i-1}x_{i-1} + a_4 K_{i-1,i}x_i)] =$$

$$= h\sum_{j=0}^{i-3} p_{ij} K_{i-1,j}x_j + h[a_1 K_{i-1,i-3}x_{i-3} + a_2 K_{i-1,i-2}x_{i-2} + a_3 K_{i-1,i-1}x_{i-1} +$$

$a_4K_{i-1,i}x_i] = f_{i-1}$
и

$$\int_0^{t_i} K(t_i,\tau)x(\tau)\,d\tau = \int_0^{3h} K(t_i,\tau)x(\tau)\,d\tau + \int_{3h}^{6h} K(t_i,\tau)x(\tau)\,d\tau + \cdots + \tag{9}$$

$$+\int_{t_{i-3}}^{t_i} K(t_i,\tau)x(\tau)\,d\tau = h[(b_1K_{i,0}x_0 + b_2K_{i,1}x_1 + b_3K_{i,2}x_2 + b_4K_{i,3}x_3) +$$

$$+(b_1K_{i,3}x_3 + b_2K_{i,4}x_4 + b_3K_{i,5}x_5 + b_4K_{i,6}x_6) + \cdots +$$

$$+(b_1K_{i,i-3}x_{i-3} + b_2K_{i,i-2}x_{i-2} + b_3K_{i,i-1}x_{i-1} + b_4K_{i,i}x_i)] =$$

$$= h\sum_{j=0}^{i-3} p_{ij}K_{i,j}x_j + h[b_1K_{i,i-3}x_{i-3} + b_2K_{i,i-2}x_{i-2} + b_3K_{i,i-1}x_{i-1} + b_4K_{i,i}x_i] = f_i,$$
$$i = 3,6,9,\dots,N.$$

The points t_{i-1} and t_i will be called collocation points or collocation nodes.

Assuming $x(0) = x_0$ to be given and using the above quadrature formulas, we obtain that x_{i-2}, x_{i-1} and x_i are solutions of the SLAU

$$\begin{pmatrix} ha_2K_{i-1,i-2} & ha_3K_{i-1,i-1} & ha_4K_{i-1,i} \\ hb_2K_{i,i-2} & hb_3K_{i,i-1} & hb_4K_{i,i} \end{pmatrix}\begin{pmatrix} x_{i-2} \\ x_{i-1} \\ x_i \end{pmatrix} =$$

$$= h\begin{pmatrix} \sum_{j=0}^{i-3} p_{ij}K_{i-1,j}x_j + a_1K_{i-1,i-3}x_{i-3} \\ \sum_{j=0}^{i-3} p_{ij}K_{i,j}x_j + b_1K_{i,i-3}x_{i-3} \end{pmatrix} + \begin{pmatrix} f_{i-1} \\ f_i \end{pmatrix}$$

or in vector-matrix form

$$A_iX_i = B_i, \tag{10}$$

where

$$A_i = \begin{pmatrix} ha_2K_{i-1,i-2} & ha_3K_{i-1,i-1} & ha_4K_{i-1,i} \\ hb_2K_{i,i-2} & hb_3K_{i,i-1} & hb_4K_{i,i} \end{pmatrix}, X_i = (x_{i-2}, x_{i-1}, x_i)^T,$$

$$B_i = -h\begin{pmatrix} \sum_{j=0}^{i-3} p_{ij}K_{i-1,j}x_j + a_1K_{i-1,i-3}x_{i-3} \\ \sum_{j=0}^{i-3} p_{ij}K_{i,j}x_j + b_1K_{i,i-3}x_{i-3} \end{pmatrix} + \begin{pmatrix} f_{i-1} \\ f_i \end{pmatrix}.$$

These systems have dimensionality (2×3) , i.e., they are underdetermined.

We will look at SLAU (10) as constraints of the equality type to find the minimum of the square of the norm of the approximate solution $y_i(t), t \in [t_{i-3}, t_i], y_{i+1}(t_i) = y_i(t_i), t \in [t_{i-3}, t_i]$ $i = 3,4,\dots,N,$ In this case we will have a problem for a conditional minimum

$$\| y \|^2 \to \min \tag{11}$$

under constraints of the equality type (10).

If the norm of the function $y(x_{i-3}, x_{i-2}, x_{i-1}, x_i, t)$ is chosen unsuccessfully, for example, in the space of continuous or continuously differentiable functions, then the problem (11) with constraints (10) will be rather complicated, so we will assume that

1) $y(t) = L_3(x_{i-3}, x_{i-2}, x_{i-1}, x_i, t)$ is an interpolation polynomial of the third degree passing

through the points (x_{i-m}, t_{i-m}), $m = 0,1,2,3$;

2)

$$\| y(t) \|^2 = \| L_3(\cdot) \|^2 = \sum_{m=0}^{r} \int_{t_{i-3}}^{t_i} Ł_3^{(m)}(t) L_3^{(m)}(t)\, dt, \;\; 0 \le r \le 3. \quad (12)$$

Here we will restrict ourselves to a special case of (12), namely, $r = 3$ and to calculate the definite integral in formula (12) we will use some known quadrature formula (see, for example, [12]). Then we have

$$\| L_3(\cdot) \|^2 = \sum_{m=0}^{3} \int_{t_{i-3}}^{t_i} L_3^{(m)}(x_{i-3}, x_{i-2}, x_{i-1}, x_i, t) L_3^{(m)}(x_{i-3}, x_{i-2}, x_{i-1}, x_i, t)\, dt \approx \quad (13)$$

$$\approx h\left[\left|\left|\sum_{m=0}^{3} \alpha_m^0 x_{i-3+m}\right|\right|^2 + \left|\left|\sum_{m=0}^{3} (\alpha_m^1 x_{i-3+m})/h\right|\right|^2 + \left|\left|\sum_{m=0}^{3} (\alpha_m^2 x_{i-3+m})/h^2\right|\right|^2 + \right.$$

$$\left. + \left|\left|\sum_{m=0}^{3} (\alpha_m^3 x_{i-3+m})/h^3\right|\right|^2\right] = \varphi(x_{i-2}, x_{i-1}, x_i),$$

where $\sum_{m=0}^{3} (\alpha_m^q x_{i-3+m})/h^q \approx x^{(q)}(\xi_i^p)$, $\xi_i^p \in [t_{i-3}, t_i]$, and the norm of the finite-dimensional vector is here understood as Euclidean.

The coefficients α_m^q depend on the choice of the quadrature formula and the approximation formula .$L_3^{(m)}(x_{i-3}, x_{i-2}, x_{i-1}, x_i, t)$

The coefficients α_m^3 are determined uniquely from the apparent equality $\Delta^3 x_i = (x_i - 3x_{i-1} + 3x_{i-2} - x_{i-3})$, i.e. $\alpha^3 = (1, -3, 3, -1)$.

For example, at $\bar{t} = t_{i-3}$ the coefficients $\alpha^0 = (0,0,0,1), \alpha^1 = 1/6(2, -9, 18, -11), \alpha^2 = (-1, 4, -5, 2)$.

At $\bar{t} = t_{i-2}$ coefficients $\alpha^0 = (0,0,1,0), \alpha^1 = 1/6(-1, 6, -3, -2)$, $\alpha^2 = (0, 1, -2, 0)$.

At $\bar{t} = t_{i-1}$ coefficients $\alpha^0 = (0,1,0,0), \alpha^1 = 1/6(2, 3, -6, 1)$, $\alpha^2 = (1, -2, 1, 0)$.

At $\bar{t} = t_i$ coefficients $\alpha^0 = (1,0,0,0), \alpha^1 = 1/6(11, -18, 9, -2)$, $\alpha^2 = (2, -5, 4, -1)$.

Thus, given that x_0 is set, at each integration segment $[t_{i-3}, t_i]$, $i = 3, 6, \ldots, N$, we have a quadratic programming problem: to find the minimum of the target function $\varphi(x_{i-2}, x_{i-1}, x_i)$ under constraints of the equality type (10).

Since multiplication of the target function (13) by an arbitrary non-zero number does not affect the finding of the argument of the conditional minimum, this problem is equivalent to the problem

$$\min \psi(x_i, x_{i-1}, x_{i-2}) = h^6 \left|\left|\sum_{m=0}^{3} \alpha_m^0 x_{i-3+m}\right|\right|^2 + \quad (14)$$

$$+ h^4 \left|\left|\sum_{m=0}^{3} (\alpha_m^1 x_{i-3+m})/h\right|\right|^2 + h^2 \left|\left|\sum_{m=0}^{3} (\alpha_m^2 x_{i-3+m})/h^2\right|\right|^2 + \| \Delta^3 x_i \|^2$$

with constraints (10).

Since the first, second and third summands in (14) contain small summands of order h^6, h^4, h^2 respectively, they can be discarded (or part of them). For example, restricting (14) to the third and fourth summands only or to the last summand only, we obtain two mathematical programming problems:

1. Find

$$\min \psi_1(x_i, x_{i-1}, x_{i-2}) = h^2 \left|\left|\sum_{m=0}^{3} (\alpha_m^2 x_{i-3+m})/h^2\right|\right|^2 + \| \Delta^3 x_i \|^2. \quad (15)$$

2. Find

$$\min \psi_2(x_i, x_{i-1}, x_{i-2}) = \| \Delta^3 x_i \|^2 \quad (16)$$

under constraints of the equality type (10).

Problems (15), (10) and (16), (10) can be solved by the method of Lagrange multipliers. Since the target functions (15) and (16) are quadratic and the constraints (10) are equalities, the solution of these problems is the solution of the corresponding SLAEs. For example, the solution of problems (16) with constraints (10) is the solution of SLAU of the following form

$$A_i X_i = B_i, \tag{17}$$

where

$$A_i = \begin{pmatrix} 3 & -3 & 1 \\ hb_2 K_{i-1,i-2} & hb_3 K_{i-1,i-1} & hb_4 K_{i-1,i} \\ ha_2 K_{i,i-2} & ha_3 K_{i,i-1} & ha_4 K_{i,i} \end{pmatrix}, \tag{18}$$

.

$$X_i = (x_{i-2}, x_{i-1}, x_i)^T,$$

$$B_i = (x_{i-3}, B_i^T)^T,$$

where the vector B_i is defined by formula (10).

Assertion. Let the conditions for the integral equation (1) be satisfied:

1) the elements of $K(t,\tau)$, $f(t)$ belong to the class ; $C^4_{[0,1]}$

2) $K(t,t) \neq 0 \forall t \in [0,1]$, $.f(0) = 0, x_0 = x(0)$

Then it is fair to estimate , $\| x_i - x(t_i) \| = O(h^3) i = 3,4,\dots,N$. where x_{i-2}, x_{i-1}, x_i are solutions of problems (17)

The proof is based on the discrete analog of the Gronwall-Bellman lemma (see [3], [6]).

Note that if we put $r < 3$, in (12), we obtain a different family of algorithms. For example, at $r = 2$ (by analogy with problem (14)) we will have problems on the conditional minimum of a quadratic function:

1. Find

$$\min \Omega(x_i, x_{i-1}, x_{i-2}) = \left|\left|\sum_{m=0}^{3} (\alpha_m^2 x_{i-3+m})\right|\right|^2 + h^2 \left|\left|\sum_{m=0}^{3} (\alpha_m^1 x_{i-3+m})\right|\right|^2 + \tag{19}$$

$$+h^4 \left|\left|\sum_{m=0}^{3} (\alpha_m^0 x_{i-3+m})\right|\right|^2$$

under constraints of the equality type (10).

At $r = 1$ we have

2. Find

$$\min \Gamma(x_i, x_{i-1}, x_{i-2}) = \left|\left|\sum_{m=0}^{3} (\alpha_m^1 x_{i-3+m})\right|\right|^2 + h^2 \left|\left|\sum_{m=0}^{3} (\alpha_m^0 x_{i-3+m})\right|\right|^2 \tag{20}$$

under constraints of the equality type (10).

By analogy with (13), these problems are equivalent to finding the conditional minimum of the functions $\Omega(x_i, x_{i-1}, x_{i-2})$ and $\Gamma(x_i, x_{i-1}, x_{i-2})$, respectively.

Just as for the case $r = 3$, for the case $r = 2$ in formula (19) the summands containing h^4 or h^4 and h^2 can be discarded. And for $r = 1$ the summand containing h^2 can be discarded.

Then we obtain a family of algorithms: for $r = 2$ find

$$\min \Omega_1(x_i, x_{i-1}, x_{i-2}) = \left|\left|\sum_{m=0}^{3} (\alpha_m^2 x_{i-3+m})\right|\right|^2 + h^2 \left|\left|\sum_{m=0}^{3} (\alpha_m^1 x_{i-3+m})\right|\right|^2 \tag{21}$$

or

$$\min \Omega_2(x_i, x_{i-1}, x_{i-2}) = \left|\left|\sum_{m=0}^{3} (\alpha_m^2 x_{i-3+m})\right|\right|^2 \tag{22}$$

under constraints of the equality type (10).

For $r = 1$ we will have a family of methods: find

$$\min \Gamma_1(x_i, x_{i-1}, x_{i-2}, h) = \left|\left|\sum_{m=0}^{3} (\alpha_m^1 x_{i-3+m})\right|\right|^2 \tag{23}$$

under constraints of the equality type (10).

To solve problems (21)-(23) we can apply the method of Lagrange multipliers. The conditional minimum of the functions , ,$\Omega\Omega_1\Omega_2$ and ,$\Gamma\Gamma_1$ in this case is found exactly from the solution of the corresponding SLAEs.

Note that the study of stability and convergence rate of methods (19)-(23) is of separate interest. The properties of these algorithms will depend not only on the choice of quadrature formulas (see formula (6)), i.e., on the parameters$a$ and$b$, but also on the choice of approximation of the derivatives of the solution (see formula (12)), i.e., on the parameters$\alpha_m^l, 0 \leq l, m \leq 3$. Different variants of such approaches have been considered. Preliminary analysis of these algorithms showed that they have the property of stability.

4. NUMERICAL

In this section we present calculations of test cases using algorithm (17) with parameters$a = 1/3(1,4,1,0)^T$,$b = 1/4(3,0,9,0)^T$. The results are presented in the form of tables. The designation er= max $_{i=1,N}|x(t_i - x_i|$.

Example 1 (see [6], p. 149). Consider an IS

$$(r^2+1)\int_0^t \cos(t-\tau)x(\tau)d\tau = \sin(t) + r(\exp(rt) - \cos(t), t \in [0,1],$$

the exact solution of which $x(t) = \exp(rt)$. The results of calculations at , ,$r = 1 a = 1/3(1,4,1,0)^T b = 1/4(3,0,9,0)^T$ are presented in Table 1.

The calculation results of this example at the parameter values , ,$a = 1/3(0,7,-2,1)^T b = 1/4(0,9,0,3)^T r = 1$ are presented in Table 2.

Example 2 (see [6], p. 517). Consider an IS

$$\alpha\int_0^t \exp(\alpha(t-\tau)x(\tau)d\tau = (\exp(\alpha t) - \exp(-\alpha t))/2\sin(t) + r(\exp(rt) - \cos(t), t \in [0,1],$$

the exact solution of which$x(t) = \exp(-\alpha t)$. The results of calculations at parameter values , ,$\alpha = 3 a = 1/3(1,4,1,0)^T b = 1/4(3,0,9,0)^T$ are presented in Table 3.

The results of calculations of this example at the value of parameters , , $a = 1/3(0,7,-2,1)^T b = 1/4(0,9,0,3)^T \alpha = 3$ are presented in Table 4.

Numerical calculations of these examples agree with the statement. In addition to the above examples, numerous calculations of other test cases, which do not contain rigid components, have been performed with different parameter choices$a$ and$b$ using algorithm (17). These experiments also agree well with the statement.

5.

In this paper, a class of Volterra integral equations of the first kind has been identified, for the numerical solution of which collocation-variational methods of the third order have been proposed. These algorithms are reduced to the solution of a mathematical (quadratic) programming problem - the target function is quadratic (some analog of the square of the norm of the approximate solution) with equality-type constraints (collocation condition). Such a problem is equivalent to finding a solution to a nongenerated SLAU. Numerical calculations have shown that further development of this approach is promising. Further detailed investigation of collocation-variational methods (21)-(23), methods of higher order and for more general problems is planned, in particular, for Volterra integral equations having the degree of instability (see [3]) greater than one and for equations of the first kind with a kernel containing a weak singularity.

Table 1: Numerical calculations of example 1 at ,$r = 1 a = 1/3(1,4,1,0)^T$, $b = 1/4(3,0,9,0)^T$

h	0.1	0.05	0.025
er	0.0039	0.0006	0.00009

Table 2: Numerical calculations of example 1 at ,$r = 1a = 1/3(0,7,-2,1)^T$, $b = 1/4(0,9,0,3)^T$

h	0.1	0.05	0.025
er	0.0027	0.0004	0.00006

Table 3: Calculations for example 2 with parameter values , , $\alpha = 3a = 1/3(1,4,1,0)^T b = 1/4(3,0,9,0)^T$

h	0.1	0.05	0.025
er	0.085	0.012	0.0018

Table 4: Calculation results of example 2 with the parameters , , $\alpha = 3a = 1/3(0,7,-2,1)^T b = 1/4(0,9,0,3)^T$

h	0.1	0.05	0.025
er	0.02	0.0035	0.00052

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