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SPINOR FIELD IN FLRW COSMOLOGY: SPHERICALLY SYMMETRIC CASE

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Abstract. Within the scope of a spherically symmetric FLRW cosmological model we have studied the role of nonlinear spinor field in evolution of the universe. It is found that if the FLRW model given by the spherical coordinates the energy-momentum tensor (EMT) of the spinor field possesses nontrivial non-diagonal components. These non-diagonal components of EMT neither depend on the spinor field nonlinearity nor on the value of parameter k defining the type of curvature of the FLRW model. The presence of such components imposes some restrictions on the spinor field. The problem is studied for open, flat and close geometries. In doing so we exploited the spinor description of sources such as perfect fluid and dark energies. Some qualitative numerical solutions are given.

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1. INTRODUCTION

Thanks to its ability to simulate different kinds of matter such as perfect fluid, dark energy etc. spinor field is being used by many authors not only to describe the late time acceleration of the expansion, but also to study the evolution of the Universe at different stages [1, 2, 3, 4, 5, 6, 7, 8].

It was found that the spinor field is very sensitive to spacetime geometry. Depending on the concrete type of metric the spinor field may possess different type of nontrivial non-diagonal components of the energy-momentum tensor. As a result the spinor field imposes various kinds of restrictions on both the spacetime geometry and the spinor field itself [9].

Recently spinor field is used in astrophysics to see whether its specific behavior can shed any new light in the study of objects like black hole and wormhole. Such studies were carried out within the scope of spherically symmetric [10, 11] and cylindrically symmetric spacetime [12, 13].

Since the present-day universe is surprisingly isotropic and the presence of nontrivial non-diagonal components of the spinor field leads to the severe restrictions on the spinor field, we have studied role of a spinor field in Friedmann–Lemaitre–Robertson–Walker (FLRW) model as well. But in those cases the space-time was given in Cartesian coordinates. In order to see influence of the coordinate transformations on spinor field some works were done by us earlier [14, 15]. In this paper we will further develop those studies and see how the spinor field behaves if the isotropic and homogeneous cosmological FLRW model given by spherical coordinates.

2. BASIC EQUATION

The action we choose in the form

$$S = \int \sqrt{-g}\left[\frac{R}{2\kappa} + L_{sp}\right] d\Omega, \tag{1}$$

where $\kappa = 8\pi G$ is Einstein's gravitational constant, R is the scalar curvature and L_{sp} is the spinor field Lagrangian given by [16]

$$L_{sp} = \frac{i}{2}[\bar\psi\gamma^\mu\nabla_\mu\psi - \nabla_\mu\bar\psi\gamma^\mu\psi] - m\bar\psi\psi - \lambda F(K). \tag{2}$$

To maintain the Lorentz invariance of the spinor field equations the nonlinear term $F(K)$ in (2) is constructed as some arbitrary functions of invariants generated from the real bilinear forms. On account of Fierz equality in (2) we set $K = K(I,J) = b_1 I + b_2 J$, where b_1 and b_2 takes the value 0 or 1 which leads to the following expressions for $K = \{I, J, I+J, I-J\}$. Here $I = S^2$ and $J = P^2$ are the invariants of bilinear spinor forms with $S = \bar\psi\psi$ and $P = i\bar\psi\bar\gamma^5\psi$ being the scalar and pseudo-scalar, respectively. In (2) λ is the self-coupling constant. Note that λ can be both positive and negative, while $\lambda = 0$ leads to linear case. Here m is the spinor mass.

The covariant derivatives of spinor field takes the form [16]

$$\nabla_\mu \psi = \partial_\mu \psi - \Omega_\mu \psi, \quad \nabla_\mu \bar\psi = \partial_\mu \bar\psi + \bar\psi \Omega_\mu, \quad (3)$$

where Ω_μ is the spinor affine connections, defined as [16]

$$\Omega_\mu = \frac{1}{4} g_{\rho\sigma}\left(\partial_\mu e_\tau^{(b)} e_{(b)}^\rho - \Gamma_{\mu\tau}^\rho\right)\gamma^\sigma\gamma^\tau. \quad (4)$$

In (4) $\Gamma_{\mu\alpha}^\beta$ is the Christoffel symbol and the Dirac matrices in curve space–time γ are connected to the flat space–time Dirac matrices $\bar\gamma$ in the following way

$$\gamma_\beta = e_\beta^{(b)} \bar\gamma_b, \quad \gamma^\alpha = e_{(a)}^\alpha \bar\gamma^a, \quad (5)$$

where $e_{(a)}^\alpha$ and $e_\beta^{(b)}$ are the tetrad vectors such that

$$g_{\mu\nu}(x) = e_\mu^a(x) e_\nu^b(x) \eta_{ab}, \quad (6)$$

and fulfil following relations

$$e_{(a)}^\alpha e_\beta^{(a)} = \delta_\beta^\alpha, \quad e_{(a)}^\alpha e_\alpha^{(b)} = \delta_a^b. \quad (7)$$

Here $\eta_{ab} = \mathrm{d}iag(1,-1,-1,-1)$ is the Minkowski spacetime. The γ matrices obey the following anti-commutation rules

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}, \quad \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}. \quad (8)$$

Varying the Lagrangian (2) with respect to $\bar\psi$ and ψ, respectively, we obtain the following spinor field equations

$$i\gamma^\mu \nabla_\mu \psi - m\psi - D\psi - iG\,\bar\gamma^5\psi = 0, \quad (9)$$

$$i\nabla_\mu \bar\psi \gamma^\mu + m\bar\psi + D\bar\psi + iG\,\bar\psi\bar\gamma^5 = 0, \quad (10)$$

where $D = 2\lambda F_K b_1 S$, $G = 2\lambda F_K b_2 P$.

The energy momentum tensor of the spinor field is defined in the following way [16]

$$T_\mu^{\ \rho} = \frac{i}{4} g^{\rho\nu}(\bar\psi\gamma_\mu\nabla_\nu\psi + \bar\psi\gamma_\nu\nabla_\mu\psi - \nabla_\mu\bar\psi\gamma_\nu\psi - \nabla_\nu\bar\psi\gamma_\mu\psi) - \delta_\mu^\rho L, \quad (11)$$

which in view of (3) we rewrite as

$$T_\mu^{\ \rho} = \frac{i}{4} g^{\rho\nu}(\bar\psi\gamma_\mu\partial_\nu\psi + \bar\psi\gamma_\nu\partial_\mu\psi - \partial_\mu\bar\psi\gamma_\nu\psi - \partial_\nu\bar\psi\gamma_\mu\psi) - \frac{i}{4} g^{\rho\nu}\bar\psi(\gamma_\mu\Omega_\nu + \Omega_\nu\gamma_\mu + \gamma_\nu\Omega_\mu + \Omega_\mu\gamma_\nu)\psi - \delta_\mu^\rho L. \quad (12)$$

Note that the non-diagonal components of the EMT arises thanks to the second term in (12). Moreover, let us emphasize that in view of the spinor field equations (9)–(10) the spinor field Lagrangian (2) can be expressed as

$$L = \lambda(2KF_K - F), \quad F_K = dF/dK. \quad (13)$$

We exploit this form of Lagrangian in solving Einstein equations, as they should be consistent with the Dirac one, as (13) is valid only when spinor fields obey Dirac equations (9)–(10). Let us also note that in case $F = \sqrt{K}$ the Lagrangian vanishes which is very much expected as in this case spinor field becomes linear. We are interested in nonlinear spinor field as only it can generate different kinds of source fields.

The isotropic and homogeneous cosmological model proposed by Friedmann, Lemaitre, Robertson and Walker independently is the most popular and thought to be realistic one among the cosmologists. Let us consider the FLRW model in spherical coordinates in its stanard form [17]:

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\phi^2\right], \quad (14)$$

with k taking the values +1, 0 and −1 which corresponds to a close, flat and open universe, respectively. Though the value of k defines the type of geometry of space-time, in reality it is defined by the contents that filled universe. As we see later, independ to the value of k the universe filled with dark energy is always open, whereas for perfect fluid the value of k really matters. In this case depending on the value of k we obtain close, flat or open universe.

In view of (6) the tetrad we will choose in the form

$$e_0^{(0)} = 1, \quad e_1^{(1)} = \frac{a}{\sqrt{1 - kr^2}},$$

$$e_2^{(2)} = ar, \quad e_3^{(3)} = ar\sin\vartheta.$$

Then from (5) we find the following γ matrices

$$\gamma^0 = \bar\gamma^0, \quad \gamma^1 = \frac{\sqrt{1 - kr^2}}{a}\bar\gamma^1,$$

$$\gamma^2 = \frac{\bar\gamma^2}{ar}, \quad \gamma^3 = \frac{\bar\gamma^3}{ar\sin\vartheta}.$$

Further from $\gamma_\mu = g_{\mu\nu}\gamma^\nu$ one finds the γ_μ as well.

The Christoffel symbols, Ricci tensor and scalar curvature and the Einstein tensor corresponding to the metric (14) are well known and can be found in [17].

Then from (4) we find the following expressions for spinor affine connection

$$\Omega_0 = 0, \tag{15}$$

$$\Omega_1 = \frac{1}{2\sqrt{1 - kr^2}}\dot a\bar\gamma^1\bar\gamma^0, \tag{16}$$

$$\Omega_2 = \frac{1}{2}r\dot a\bar\gamma^2\bar\gamma^0 + \frac{1}{2}\sqrt{1 - kr^2}\bar\gamma^2\bar\gamma^1, \tag{17}$$

$$\Omega_3 = \frac{1}{2}\dot ar\sin\vartheta\bar\gamma^3\bar\gamma^0 + \frac{1}{2}\sqrt{1 - kr^2}\sin\vartheta\bar\gamma^3\bar\gamma^1 + \frac{1}{2}\cos\vartheta\bar\gamma^3\bar\gamma^2. \tag{18}$$

Let us consider the case when the spinor field depends on t only, then in view of (15)–(18) the spinor field equations can be written as

$$\dot\psi + \frac{3}{2}\frac{\dot a}{a}\psi + \frac{\sqrt{1 - kr^2}}{ar}\bar\gamma^0\bar\gamma^1\psi + \frac{\cot\vartheta}{2ar}\bar\gamma^0\bar\gamma^2\psi + i(m + D)\bar\gamma^0\psi + G\bar\gamma^5\bar\gamma^0\psi = 0, \tag{19}$$

$$\dot{\bar\psi} + \frac{3}{2}\frac{\dot a}{a}\bar\psi - \frac{\sqrt{1 - kr^2}}{ar}\bar\psi\bar\gamma^0\bar\gamma^1 - \frac{\cot\vartheta}{2ar}\bar\psi\bar\gamma^0\bar\gamma^2 - i(m + D)\bar\psi\bar\gamma^0 + G\bar\psi\bar\gamma^5\bar\gamma^0 = 0, \tag{20}$$

Introducing $\varphi = a^{3/2}\psi$ we rewrite the equation (19)–(20)

$$\dot\varphi + \frac{\sqrt{1 - kr^2}}{ar}\bar\gamma^0\bar\gamma^1\varphi + \frac{\cot\vartheta}{2ar}\bar\gamma^0\bar\gamma^2\varphi + i(m + D)\bar\gamma^0\varphi + G\bar\gamma^5\bar\gamma^0\varphi = 0, \tag{21}$$

$$\dot{\bar\varphi} - \frac{\sqrt{1 - kr^2}}{ar}\bar\varphi\bar\gamma^0\bar\gamma^1 - \frac{\cot\vartheta}{2ar}\bar\varphi\bar\gamma^0\bar\gamma^2 - i(m + D)\bar\varphi\bar\gamma^0 + G\bar\varphi\bar\gamma^5\bar\gamma^0 = 0, \tag{22}$$

The equation (21) can be presented in the matrix form

$$\dot\varphi = A\varphi, \tag{23}$$

or

$$\begin{pmatrix}\dot\varphi_1\\ \dot\varphi_2\\ \dot\varphi_3\\ \dot\varphi_4\end{pmatrix} = \begin{pmatrix}-iD_1 & 0 & -G & B_1\\ 0 & -iD_1 & B_1^* & -G\\ G & B_1 & iD_1 & 0\\ B_1^* & G & 0 & iD_1\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\\ \varphi_4\end{pmatrix}, \tag{24}$$

where

$$D_1 = (m + D), \quad B_1 = -\frac{\sqrt{1 - kr^2}}{ar} + i\frac{\cot\vartheta}{2ar},$$

$$B_1^* = -\frac{\sqrt{1 - kr^2}}{ar} - i\frac{\cot\vartheta}{2ar}.$$

It can be shown that

$$\det A = \left(D_1^2 + G^2 - B_1B_1^*\right)^2.$$

We can choose the nonlinearity in such a way that the corresponding determinant is nontrivial. In that case the solution (23) can be formally written as [18]

$$\varphi(t) = T\exp\left(-\int_t^{t_1}A_1(\tau)d\tau\right), \tag{25}$$

where $T = \varphi(t_1)$ is the solution at $t = t_1$. Given the fact that the universe is expanding and the spinor field invariants are the inverse functions of scale factor, in case of a nonzero spinor mass one can assume

$$\varphi(t_1) = col\left(\varphi_1^0e^{-imt_1}, \varphi_2^0e^{-imt_1}, \varphi_3^0e^{imt_1}, \varphi_4^0e^{imt_1}\right),$$

whereas for a massless spinor field

$$\varphi(t_1) = col\left(\varphi_1^0, \varphi_2^0, \varphi_3^0, \varphi_4^0\right)$$

with φ_i^0 being constants.

The non-trivial components of the energy momentum tensor of the spinor field in this case read

$$T_0^0 = mS + \lambda F, \tag{26}$$

$$T_1^1 = T_2^2 = T_3^3 = -\lambda\left(2KF_K - F\right), \quad (27)$$

$$T_3^1 = \frac{a\cos\vartheta}{4\sqrt{1-kr^2}}A^0, \quad (28)$$

$$T_1^0 = \frac{\cot\vartheta}{4r\sqrt{1-kr^2}}A^3, \quad (29)$$

$$T_2^0 = -\frac{3}{4}\sqrt{1-kr^2}\,A^3, \quad (30)$$

$$T_3^0 = \frac{3}{4}\sqrt{1-kr^2}\sin\vartheta A^2 - \frac{1}{2}\cos\vartheta A^1. \quad (31)$$

From (28)–(31) we conclude that the energy-momentum tensor of the spinor field contains nontrivial non-diagonal components. The non-diagonal components

- do not depend on the spinor field nonlinearity;
- occur due to the spinor affine connections;
- appear depending on space-time geometry as well as the system of coordinates;
- impose restrictions on spinor field and/or space-time geometry;
- do not depend on the value of k which defines the type of curvature.

It should be emphasized that for a FLRW model given in Cartesian coordinate the EMT have only diagonal components with all the non-diagonal one being identically zero [19]. So in this case the non-diagonal components arise as a result of coordinate transformation. Note also that all cosmological spacetime defined by diagonal matrices of Bianchi type VI, VI_0, V, III, I, $LRS-BI$ and $FLRW$, possess same diagonal components of EMT, but has nontrivial non-diagonal elements that differ from each other in different cases [9]. Moreover, non-diagonal metrics such as Bianchi type II, $VIII$ and IX also have nontrivial non-diagonal components of EMT. Consequently, we see that the appearance of non-diagonal components of the energy-momentum tensor occurs either due to coordinate transformations or due to the geometry of space-time.

As one sees, the components of the EMT of the spinor field contains some spinor field invariants. To define those invariants let us write the system of equations for the invariants of the spinor field. It can be obtained from the spinor field equation (19)–(20):

$$\dot{S}_0 + 2GA_0^0 = 0, \quad (32)$$

$$\dot{P}_0 - 2\left(m + D\right)A_0^0 = 0, \quad (33)$$

$$\dot{A}_0^0 + 2GS_0 + 2\left(m + D\right)P_0 + 2\frac{\sqrt{1-kr^2}}{ar}A_0^1 + \frac{\cot\vartheta}{ar}A_0^2 = 0, \quad (34)$$

$$\dot{A}_0^1 + 2\frac{\sqrt{1-kr^2}}{ar}A_0^0 = 0, \quad (35)$$

$$\dot{A}_0^2 + \frac{\cot\vartheta}{ar}A_0^0 = 0, \quad (36)$$

that gives the following relation between the invariants:

$$P_0^2 - S_0^2 + \left(A_0^0\right)^2 - \left(A_0^1\right)^2 - \left(A_0^2\right)^2 = C_0, C_0 = \text{const.} \quad (37)$$

In (32)–(37) the quantities with a subscript "0" are related to the normal ones as follows: $X_0 = Xa^3$. From (37) we can conclude that since C_0 is an arbitrary constant, the each term of (37) should be constant as well.

In order to solve the Einstein equations we have to know how the components of the EMT are related to the metric functions. In order to know that let us find the invariant K in general. We consider the 4 cases separately.

In case of $K = I$, $G = 0$. In this case from (32) we find

$$S = \frac{C_s}{a^3}, \quad \Rightarrow K = \frac{C_s^2}{a^6}. \quad (38)$$

If $K = J$, then in case of a massless spinor field from (33) we find

$$P = \frac{C_p}{a^3}, \quad \Rightarrow K = \frac{C_p^2}{a^6}. \quad (39)$$

Let us consider the case when $K = I + J$. In this case $b_1 = b_2 = 1$. Then on account of expression for D and G from (32) and (33) for the massless spinor field we find

$$\dot{S}_0 + 4\lambda a^3 F_K P A^0 = 0, \quad (40)$$

$$\dot{P}_0 - 4\lambda a^3 F_K S A^0 = 0, \quad (41)$$

which yields

$$K = I + J = S^2 + P^2 = \frac{C_1^2}{a^6}. \quad (42)$$

Finally in case when $K = I - J$, i.e. $b_1 = -b_2 = 1$ from (32) and (33) for the massless spinor field we find

$$\dot{S}_0 + 4\lambda a^3 F_K P A^0 = 0, \quad (43)$$

$$\dot{P}_0 + 4\lambda a^3 F_K S A^0 = 0, \quad (44)$$

which yields

$$K = I - J = S^2 - P^2 = \frac{C_2^2}{a^6}. \quad (45)$$

Thus we see that the invariant K is a function of metric function a, namely, $K = const.a^{-6}$ and it is what we need to solve the Einstein equation. In what follows we solve the Einstein equation.

Let us recall that the Einstein tensor G_μ^ν corresponding to the metric (14) possesses only nontrivial diagonal components. Hence the general Einstein system of equations

$$G_\mu^\nu = -8\pi G\, T_\mu^\nu, \quad (46)$$

leads to the following non-diagonal expressions

$$0 = T_\mu^\nu, \quad \mu \neq \nu. \quad (47)$$

In view of (28)–(31) from (47) one dully finds that

$$A^0 = 0, \quad A^3 = 0, \quad A^1 = (3/2)\sqrt{1 - kr^2}\,\vartheta A^2. \quad (48)$$

Note that since the FLRW model given by the Cartesian coordinate the non-diagonal components of EMT are identically zero, hence relation such as (48) does not exist.

In view of $A^0 = 0$, $A^3 = 0$ from the system (32)–(36) we find

$$S_0 = C_S, \quad P_0 = C_P, \quad A_0^1 = C_0^1, \quad A_0^2 = C_0^2, \quad (49)$$

with C_S, C_P, C_0^1 and C_0^2 being some arbitrary constants. Thus we see that $K = const.a^{-6}$. Note that the equation (34) in this case in redundant and (48) gives relations between the constants C_0^1 and C_0^2.

We are now ready to consider the diagonal components of the Einstein system of equations which for the metric (14) takes the form

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) = 8\pi G T_1^1, \quad (50)$$

$$3\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) = 8\pi G T_0^0. \quad (51)$$

On account of (51) we rewrite (50) in the form

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(T_0^0 - 3T_1^1\right) = -\frac{4\pi G}{3}(\varepsilon + 3p), \quad (52)$$

where ε and p are the the energy density and and pressure, respectively:

$$\varepsilon = T_0^0 = mS + \lambda F, \quad (53)$$

$$p = -T_1^1 = \lambda\left(2KF_K - F\right). \quad (54)$$

On account of (26) and (27) from (52) we find

$$\ddot{a} = -\frac{4\pi G}{3}\left(mS - 2\lambda F + 6\lambda K F_K\right)a. \quad (55)$$

Note that the equations (52) or (55) do not contain k that defines the type of space-time curvature. In order to take this very important quantity into account we have to exploit (51) as the initial condition for $\dot{a}$. The equation (51) we rewrite in the form

$$\dot{a} = \pm\sqrt{(8\pi/3)G\,\varepsilon a^2 - k} = \pm\sqrt{(8\pi/3)G\,(mS + \lambda F)a^2 - k}, \quad (56)$$

Now we can solve (55) with the initial condition given by (56). It comes out that these equations are consistent when one takes the negative sign in (56). Alternatively, one can solve (56), but for the system to be consistent he has to check whether the result satisfies (55).

As we have already established, S, K, hence $F(K)$ are the functions of a. Consequently, given the spinor field nonlinearity the foregoing equation can be solved either analytically or numerically.

The equation (55) can be solved analytically. The first integral of (55) takes the form

$$\dot{a} = \sqrt{\int f(a)da + C_c}, \quad (57)$$

where we define

$$f(a) = -\frac{8\pi G}{3}\left(mS - 2\lambda F + 6\lambda K F_K\right)a$$

and C_c is a constant which should be defined from (56). The solution to the equation (57) can be given in quadrature

$$\int \frac{da}{\sqrt{\int f(a)da + C_c}} = t. \quad (58)$$

1. In what follows we solve the system (50)–(51) numerically. In doing so we rewrite it in the following way:

$$\dot{a} = Ha, \quad (59)$$

$$\dot{H} = -\frac{3}{2}H^2 - \frac{1}{2}\frac{k}{a^2} - 4\pi G\lambda(2KF_K - F), \quad (60)$$

$$H^2 = \frac{8\pi G}{3}(mS + \lambda F) - \frac{k}{a^2}, \quad (61)$$

where H is the Hubble constant.

As one sees, in the foregoing system the first two are differential equations, whereas the third one is a constraint, which we use as the initial condition for H:

$$H = \pm\sqrt{8\pi G\,(mS + \lambda F)/3 - k/a^2}. \quad (62)$$

Since the expression under the square-root must be non-negative, it imposes some restrictions on the choice of initial value of a as well. Note that initial value of H depends on spinor mass m, coupling parameter λ and the value of k.

3. NUMERICAL SOLUTIONS

In what follows we solve the equations (59) and (60), numerically. The third equation of the system (61) we exploit as initial condition for $H(t)$ in the form (62). We do it for both massive and massless spinor field. Beside this, we consider close, flat and open universe choosing different values for k. As it was mentioned earlier, the coupling constant λ can be positive or negative. Let us recall that

$$K = \frac{K_0}{a^6}, \quad K_0 = \text{const.} \quad (63)$$

The foregoing relation holds for $K = \{I, J, I \pm J\}$ for a massless spinor field, whereas for $K = I = S^2$ it is true for both massive and massless spinor field. Hence we assume that $K = I = S^2$. We consider different kind of spinor field nonlinearities $F(K)$ (equivalently, $F(S)$), that describes various types of sources from perfect fluid to dark energy.

3.1 Barotropic equation of state

Let us consider the case when the Universe is filled with perfect fluid or dark energy given by quintessence, Λ-term or phantom matter. It can be implemented by the barotropic equation of state (EoS), which gives a linear dependence between the pressure and energy density and was exploited by many authors [20, 21, 22, 23]. The corresponding EoS takes the form

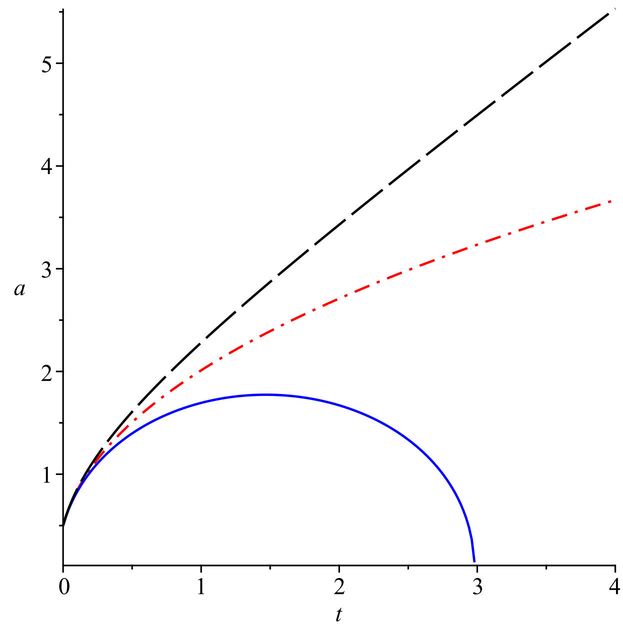

Fig. 1. Evolution of the FLRW Universe (scale factor $a(t)$) in presence of a radiation given by a massless spinor field. The blue solid, red dash-dot and black long dash lines stand for close, flat and open ($k = +1, 0, -1$) universe, respectively

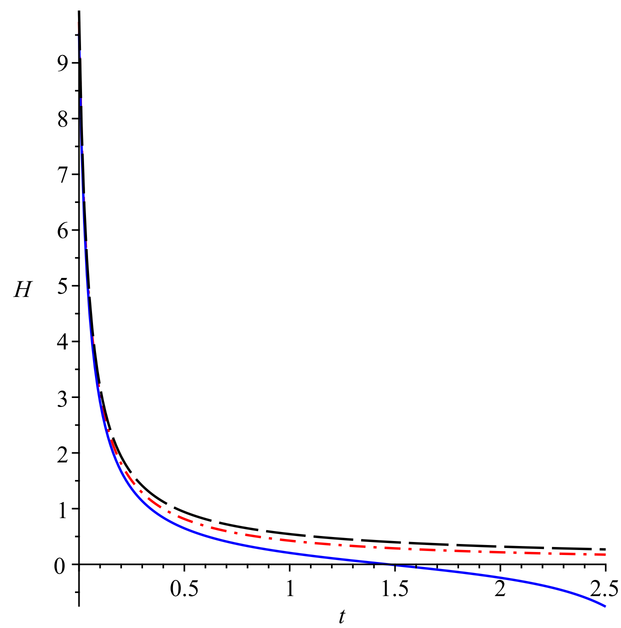

Fig. 2. Evolution of the corresponding Hubble parameter $H(t)$ and corrsponds to differnt values of k as in Fig. 1

$$p = W\varepsilon, \quad (64)$$

where the EoS parameter W is a constant. Depending on the value of W, the Eq. (64) can give rise to both perfect fluid, such as dust, radiation etc. and dark

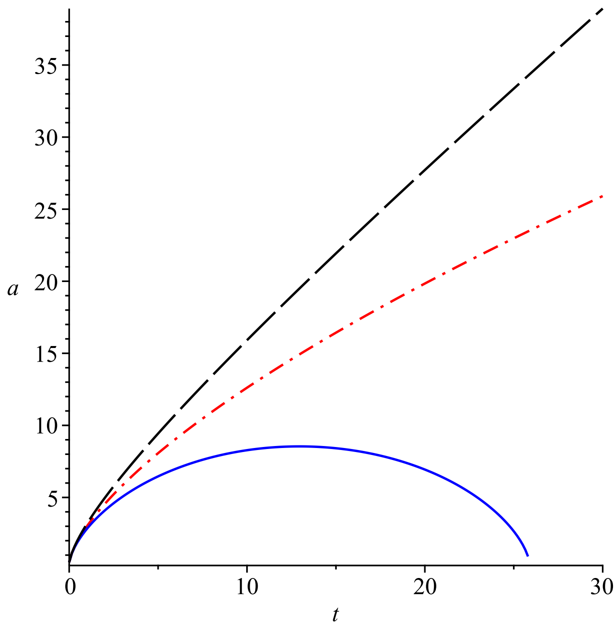

Fig. 3. Evolution of the FRW Universe (scale factor $a(t)$) in presence of a radiation given by a massive spinor field. The blue solid, red dash-dot and black long dash lines stand for $k = +1, 0, -1$, respectively

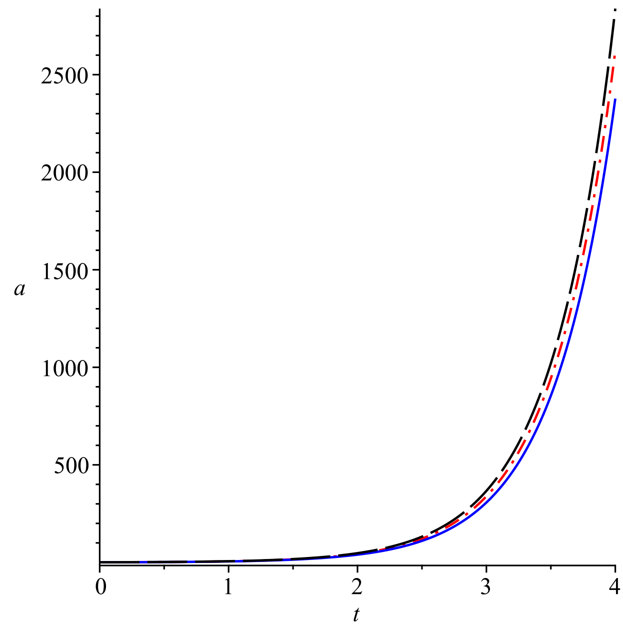

Fig. 5. Evolution of the FRW Universe (scale factor $a(t)$) in presence of a modified Chaplygin gas given by a massless spinor field. As one sees, independent to the value of k in this case the universe expand rapidly

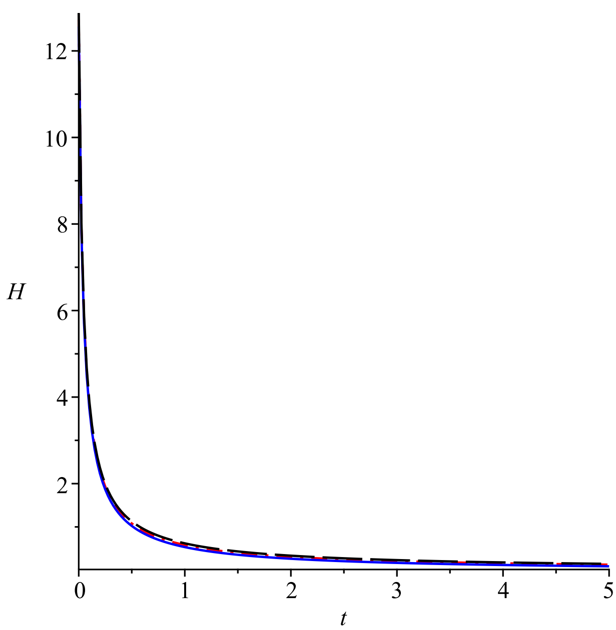

Fig. 4. Evolution of the corresponding Hubble parameter $H(t)$

energy such as quintessence, cosmological term, phantom matter etc. For $W \in [0,1]$, it describes a perfect fluid. The value $W = -1$ represents a typical cosmological constant (Λ-term) [24, 25, 26], whereas $W \in [-1, -1/3]$ gives rise to a quintessence, while for $W < -1$ it ascribes a phantom matter.

It was shown in [9, 27] that inserting (26)–(27) into (64) the matter or energy corresponding to Eq. (64) can be simulated by the nonlinear term given by

$$F(S) = \lambda S^{1+W} - mS, \quad \lambda = \text{const.}, \qquad (65)$$

in the spinor field Lagrangian (2).

Let us now solve (59)–(61) numerically for the nonlinear term given by (65). We consider both massive and massless spinor field. The values of W are taken to be $1/2$, $-1/2$ and -1 describing the radiation, quintessence and cosmological constant, respectively. For simplicity we set $S_0 = 1$, $G = 1$, $\lambda = 0.5$ here and in the cases to follow. We also set $m = 0$ for a massless and $m = 1$ for a massive spinor field.

In Fig. 1 we have illustrated the evolution of the Universe filled with radiation, given by a massless spinor field, while Fig. 2 shows the evolution of the Hubble parameter corresponding to the case in question. Figs. 3 and 4 describes the evolution of the Universe filled with radiation and the corresponding Hubble parameter in case of a massive spinor field. In the figures blue solid line stands for a closed universe given by $k = 1$, red dash-dot line stands for a flat universe with $k = 0$ and black long dash line stands for an open universe with $k = -1$.

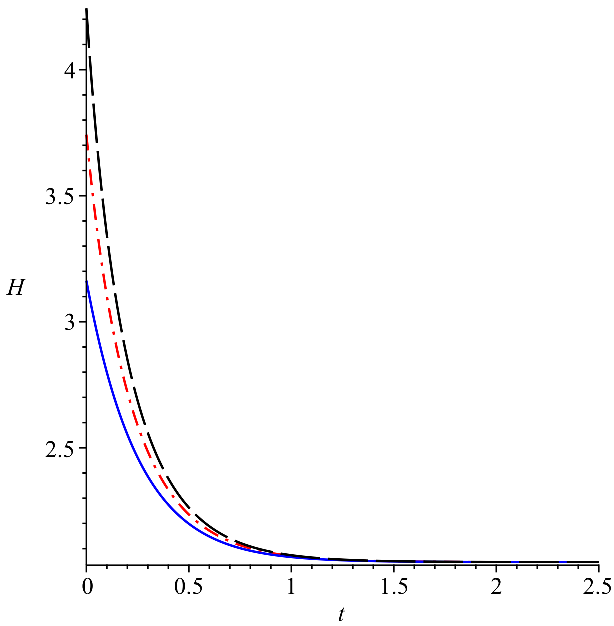

Fig. 6. Evolution of the corresponding Hubble parameter $H(t)$

We have also considered the case with the spinor field nonlinearity describing a quintessence $(W = -1/2)$ and cosmological constant $(W = -1)$. Both massive and massless spinor fields are taken into account. Since in both cases the energy density is less than the critical density, independent to the value of k we have only open type of universe. The behavior of the evolution is qualitatively same as that of in case of a modified Chapligin gas. The corresponding figures will be similar to those in Figs. 5 and 6, only the rate of expansion being much slower.

3.2 Chaplygin gas

In order to combine two different physical concepts such as dark matter and dark energy, and thus reduce the two physical parameters in one, a rather exotic equation of state was proposed in [28] which was further generalized in the works [29, 30]. Generalized Chaplygin gas model is given by the EoS

$$p_{ch} = -A/\varepsilon_{ch}^{\alpha}, \tag{66}$$

where A is a positive constant and $0 < \alpha \leq 1$.

It was shown that such kind of dark energy can be modeled by the massless spinor field with the nonlinearity [9] inserting (26)–(27) into (66)

$$F(S) = \left(A + \lambda S^{1+\alpha}\right)^{1/(1+\alpha)}. \tag{67}$$

We have solved (59)–(61) numerically for the nonlinear term given by (67). We consider only massless spinor field setting $m = 0$. The parameters S_0, G and λ were taken as in previous case. We have also set $A = 1/2$ and $\alpha = 1/3$.

As in case of quintessence and cosmological constant, the evolution of the universe filled with Chaplygin gas and corresponding behavior of the Hubble parameter are qualitatively same as in case of a modified Chaplygin gas which are illustrated in Figs. 5 and 6. The expansion rate in this case is higher than the previous case but slower than in the case to follow.

3.3 Modified Chaplygin gas

Though the dark energy and the dark matter act in a completely different way, many researchers suppose that they are different manifestations of a single entity. Following such an idea a modified Chaplygin gas was introduced in [31] and was further developed in [32]. Corresponding EoS takes the form

$$p = W\varepsilon - A/\varepsilon^{\alpha}, \tag{68}$$

with W being a constant, $A > 0$ and $0 \leq \alpha \leq 1$.

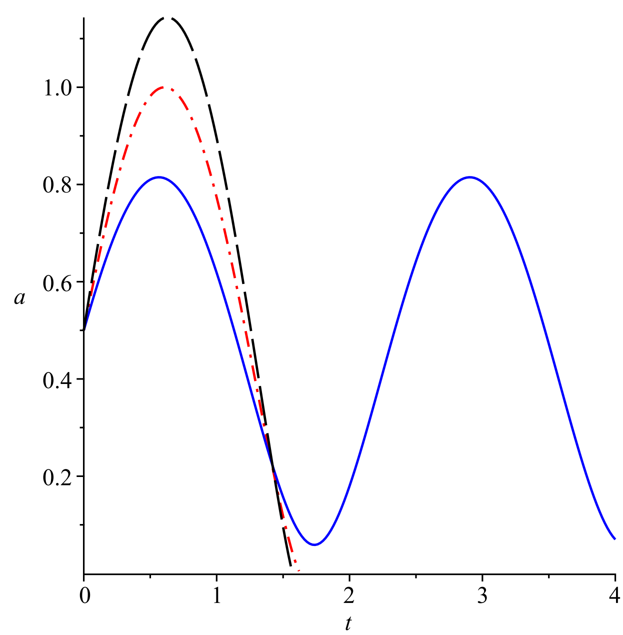

Fig. 7. Evolution of the FRW Universe (scale factor $a(t)$ in presence of a modified quintessence given by a massless spinor field. In case of $k = +1$ there occurs a periodic solution, whereas for $k = 0$ or $k = -1$, we have Big Crunch like solutions

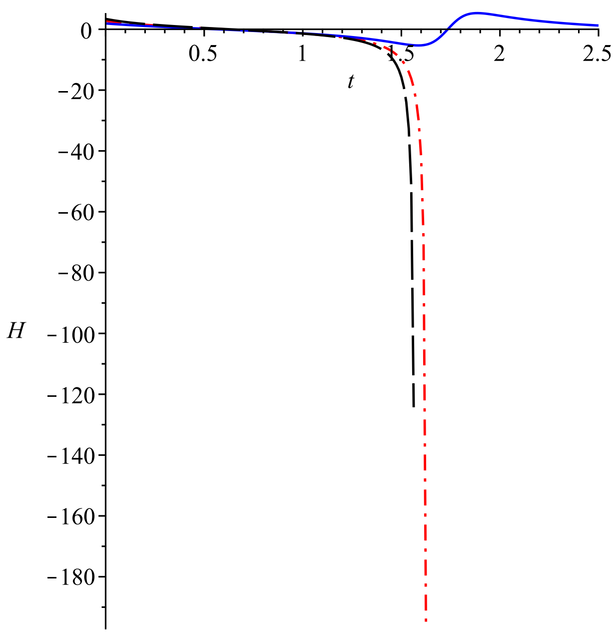

Fig. 8. Evolution of the corresponding Hubble parameter $H(t)$

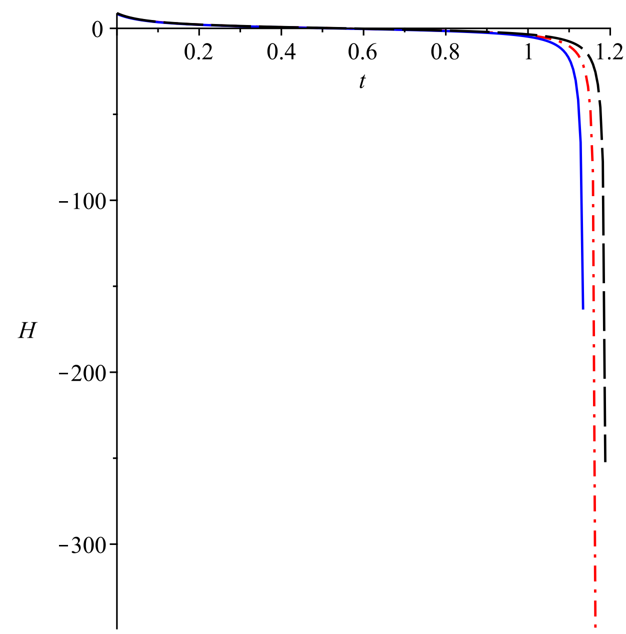

Fig. 10. Evolution of the corresponding Hubble parameter $H(t)$

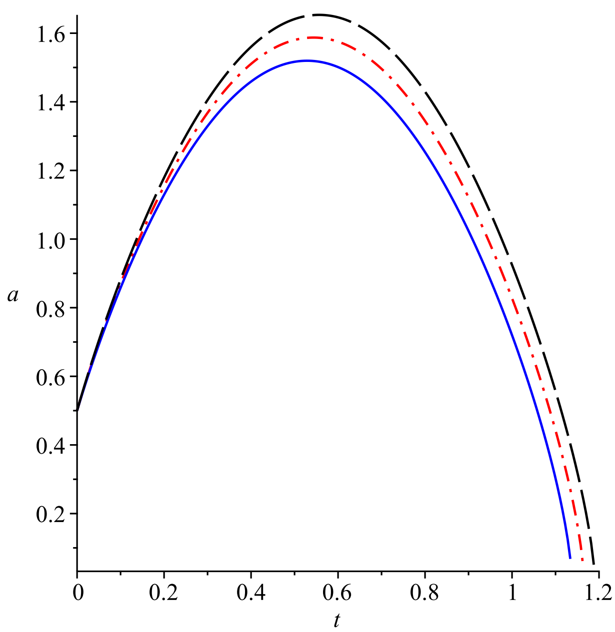

Fig. 9. Evolution of the FRW Universe (scale factor $q(t)$) in presence of a modified quintessence given by a massive spinor field. Unlike massless spinor field, in this case there is no periodic solutions for the given value of problem parameters

Inserting (26)–(27) into (68) the modified Chaplygin gas can be generated by a massless spinor field with the nonlinearity given by [9]

$$F(S) = \left[\frac{A}{1+W} + \lambda S^{(1+\alpha)(1+W)}\right]^{1/(1+\alpha)}. \quad (69)$$

In fact, mathematically it is a combination of quintessence and Chaplygin gas. We have solved (59)–(61) numerically for the nonlinear term given by (69). Since we consider only massless spinor field, we set $m = 0$. For simplicity we set S_0, G, λ, A, and α as in previous cases. Beside that we set $W = -1/2$.

In Figs. 5 and **6** we have illustrated the evolution of the universe and corresponding Hubble parameter when the Universe is filled with nonlinear spinor field simulating a modified Chaplygin gas.

3.4 Modified quintessence

A modified Quintessence was proposed in order to avoid eternal acceleration of the universe. In some cases it gives cyclic universe that pops up from a Big Bang singularity, expands to some maximum value and then decreases and finally ends in Big Crunch. In some cases it might be periodic without singularity. A spinor description of a modified quintessence was proposed in [23]

$$p = W(\varepsilon - \varepsilon_{cr}), \quad W \in (-1, 0), \quad (70)$$

with ε_{cr} being some critical energy density. The model gives rise to cyclic or oscillatory universe. Setting $\varepsilon_{cr} = 0$ one obtains ordinary quintessence. As one sees from (70), the pressure is negative as long as $\varepsilon > \varepsilon_{cr}$. Since with the expansion of the universe the energy density decreases, at some moment of time ε becomes less than ε_{cr}, i.e., $\varepsilon < \varepsilon_{cr}$. This leads to the positive pressure and the contraction of the universe. It can be shown that a modified quintessence can be modeled by a spinor field nonlinearity inserting (26)–(27) into (70)

$$F(S) = \lambda S^{1+W} + \frac{W}{1+W}\varepsilon_{cr}. \qquad (71)$$

We solve the system (59)–(61) for the values of parameters as in case of quintessence. For critical density we set $\varepsilon_{cr} = 1$.

In Figs. 7 and 8 we have illustrated the evolution of the universe and corresponding Hubble parameter when the universe is filled with nonlinear massless spinor field simulating a modified quintessence. The corresponding cases with massive spinor field are illustrated in Figs. 9 and 10

In the figures, evolution of Hubble parameter H is drawn for a much smaller time interval than the scale factor a. It is just for technical reason. For example, if in Figs. 3 and 4 we use interval 30 for both a and H, as we see from Fig. 4 Hubble parameter after crossing mark 5 it becomes almost zero, thus giving rise to a visually ugly picture. Whereas, setting interval 5 for both, we have a on rising phase for all three values of k [cf. Fig. 3]. These two figures correspond to the same values of problem parameter, only for good visual pictures we have drawn them for different intervals. The same can be told for all other cases.

4. CONCLUSION

Within the scope of a spherically symmetric FLRW model we have studied the role of a nonlinear spinor field in the evolution of the universe. It is found that in this case the spinor field possesses nontrivial non-diagonal components of the EMT. Since the Einstein tensor in this case is diagonal, this fact imposes some restrictions on the components of spinor field: $A^0 = 0$, $A^3 = 0$ and $A^1 \propto A^2$. Corresponding equations are solved. It is shown that if the spinor field nonlinearity repesents ordinay matter such as radiation, the factor k plays decisive role giving rise to close, flat or open universe depending on its positive, trivial or negative values. It is also shown that in this case spinor mass influences the result quantatively. If the spinor feild nonlinearity generates a dark energy we have only rapidly expanding universe independent to the value of k. Finally in case of a modified quintessence the model gives rise to as oscillating universe. Depending on the value of k and spinor mass m there might be periodic solutions or the one that ends in Big Crunch.

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