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LONG MEMORY EFFECTS IN THE DEVELOPMENT OF INSTABILITY IN A RANDOM MEDIUM

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Abstract. In studying the development of instabilities in a random medium, it is often assumed that memory in the medium is lost instantaneously at prescribed moments in time, while it is natural to assume that memory loss occurs gradually. This paper studies the effects arising from gradual memory loss. It turns out that long memory can increase the rate of instability development (increase the Lyapunov exponent). The connection between this effect and intermittency effects arising during the development of instabilities in a random medium is established. The study is conducted within the framework of a simple model proposed by Ya. B. Zeldovich for describing the development of instability arising under the action of curvature fluctuations during light propagation in a universe that is homogeneous and isotropic only on average.

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1. INTRODUCTION

Many problems concerning the development of instabilities in a random medium contain, as their integral part, the problem of multiplication of random matrices along a Lagrangian trajectory (for example, [1]). These include, for instance, the hydromagnetic dynamo problem, which studies the generation of magnetic field **H** in a non-relativistic random flow **v** of conducting fluid as a result of electromagnetic induction effect (for example, [2,3]). This process is described by the induction equation

$$\frac{\partial \mathrm{H}}{\partial t} + (\nabla \mathbf{v})\mathrm{H} = (\mathrm{H}\nabla)\mathbf{v} + \nu_m \Delta \mathrm{H}, \tag{1}$$

which, through transition to a Lagrangian reference frame and neglecting the small coefficient of magnetic diffusion ν_m reduces to a system of ordinary differential equations along the Lagrangian trajectory

$$\frac{\partial \mathrm{H}}{\partial t} = (\mathrm{H}\nabla)\mathbf{v}. \tag{2}$$

If we assume that the matrix random process

$$\hat{A} = \nabla \mathbf{v} = \left\| \frac{\partial v_i}{\partial x_j} \right\| \tag{3}$$

loses its memory at fixed moments of time, separated by equal time intervals τ (renewal intervals), then the solution of the initial value problem for equation (2) reduces to the multiplication of random matrices $\hat{B}$, describing the evolution of magnetic field in a random velocity field **v**. If, for example, matrix $\hat{A}$ is constant over the renewal interval, then $\hat{B} = \exp \tau(\hat{A})$, if matrix $\hat{A}$ changes with time, then instead of the exponent in the last relation, a Volterra multiplicative integral appears, or (in more physical terminology) t-exponent. Matrices B, corresponding to different renewal intervals, are obviously independent, so the complete solution of equation (2) reduces to the product of independent random matrices. If we consider a random medium that is stationary on average, these matrices are equally distributed. The determinants of these matrices, being scalars, can be factored out of the matrix product, therefore it is sufficient to consider products of unimodular matrices, and supplementing the obtained results with the product of their determinants poses no problems.

The properties of the product of a large number of independent unimodular random matrices are well studied in probability theory and constitute the content of the classical Furstenberg theory (its main features are described below). In particular, it turns out that this product grows exponentially with the increase in the number of factors, and methods are proposed for calculating the Lyapunov exponent that characterizes the rate of this growth. The phenomenon of intermittency is also discovered, which consists in the fact that the growth rate of statistical moments, normalized to the moment number, increases with the moment number (see, for example, [4, 5]).

Of course, the complete study of instability properties is not limited to this. In particular, it is necessary to somehow take into account the role of small diffusion ν_m, however, this is achieved by other methods .

The scheme of instability study described above contains a number of obvious idealizations. In particular, it assumes that the memory of the random flow completely disappears at the moments of renewal. A much more realistic idea is that the memory does not disappear completely, and the subsequent renewal interval retains a small memory of the previous renewal interval. The remnants of this memory, exponentially decreasing from one renewal moment to the next, in principle persist throughout the entire evolution of the magnetic field.

The study of effects associated with this long memory of flow within the described approach is practically absent in the literature. In particular, in the presence of such memory, the analytical methods of Furstenberg theory cease to work, however, the study of these effects through direct numerical modeling does not cause fundamental difficulties, since the system of ordinary differential equations (2) is much simpler than the original system (1). This study constitutes the content of this work.

It seems reasonable within such research to separate the study of memory effects from technical problems of calculating matrix exponentials, Lyapunov exponent, etc., which are important in themselves but play a secondary role in the context of this work. Therefore, we investigate memory effects using a simple model of instability in a random medium, which was proposed by Ya. B. Zeldovich back in 1964 [6]. This paper examines light propagation in a cosmological model that is homogeneous and isotropic only on average. It turns out that due to curvature fluctuations, light from distant sources propagates in such a way that it creates an impression that the spatial section curvature is negative, even if the average Universe density equals the critical value. This occurs due to a peculiar instability, resulting in exponential growth of distance between nearby geodesics along which light propagates, as the distance from the observer increases.

Zeldovich's paper is interesting from various perspectives and was well-received in cosmology, although the effect described in it is quantitatively small simply because, in natural units for its estimation, the Universe is still very young and the development of the considered instability has not yet progressed too far. For us now, only one aspect of this work is interesting – the formal description of the instability reduces to a very simple system of equations, which, on one hand, reproduces the main features of more complex problems, and on the other hand, is simple enough to avoid technically complex details not directly related to the issue under consideration. Therefore, in this work, we use the instability discovered by Zeldovich as a model problem to study memory effects in the development of instabilities in a random medium using its example.

The development of instabilities in a random medium contains many diverse and important effects that have been considered in widely known works. For example, paper [7] proposed a qualitative picture of the emergence of exponential divergence of trajectories in dynamical systems, while paper [8] clarified the dependence of growth rate on correlation time. In this work, we set ourselves a more localized but, as we believe, meaningful task – to determine how the development of instability, under the condition that the memory of the random medium is lost instantaneously at prescribed moments of time, differs from the development of instability in a medium where memory is lost gradually. Since the effect discovered in this process is small, we conduct our research in a way that separates it from other effects. Therefore, our research is not focused on determining the dependence of growth rates on other parameters of the problem. Such research is addressed, in particular, in works [9, 10].

2. BASIC EQUATIONS

In describing the instability of interest to us, we rely on work [11], in which the formal content of Zeldovich's original work is methodologically separated from the observational aspect. The original work was written when the study of instabilities in random media was still in its early stages, so it was unclear how the specific situation of this problem fits into a broader context.

Let us consider a curved Riemannian manifold (spatial section of a cosmological model), a point X and two nearby geodesics Γ_1 and Γ_2, emanating from this point. Let θ be a small angle between these geodesics at point X. One of the geodesics is considered basic and the angle θ is measured from it. Let us mark a segment of length x on each of the geodesics. The distance between the points obtained in this way is proportional to the small angle θ, and the resulting coefficient of proportionality y is called the geodesic deviation, or Jacobi field. An important observation by Zeldovich was that the described construction is essentially twodimensional, so one can abstract from the three-dimensionality of the spatial section of the cosmological model and consider the value y as a scalar.

It turns out that the Jacobi field satisfies a simple differential equation (geodesic deviation equation):

$$y'' + ky = 0, \tag{4}$$

where derivatives are taken with respect to distance x, which with an appropriate choice of unit system can be identified with time t, and k is the Gaussian curvature at the corresponding point on the base geodesic. If we wanted to consider more consistently the fourdimensional spacetime in which this construction is carried out, equation (4) would acquire tensor indices, the Gaussian curvature of two-dimensional space would transform into sectional curvature, we would have to account for general cosmological expansion, etc. A remarkable circumstance, fully utilized in the original work, is that all these complications can be introduced gradually, and within our work, we abstract from them.

By the meaning of Jacobi fields $y(0) = 0$, and the second initial condition, namely the value $y'(0)$, for the Jacobi equation can be considered as normalization for angle θ. For constant $k > 0$ the solution of the Jacobi equation oscillates as $\sin(\sqrt{kt})$, for $k < 0$ it grows exponentially as $\exp(\sqrt{|k|}t)$, and for $k = 0$ it grows linearly. Most cosmological tests are based on this difference; the construction itself was proposed by Gauss in his works on geodetic survey, and the idea of application to cosmology goes back to the works of N. I. Lobachevsky.

When studying the role of curvature fluctuations (in cosmological context – density) in light propagation, the value k should be considered not as a constant but as a random process. The Zeldovich effect consists in the fact that if the mean value of the random process k equals zero (the spatial section is flat on average, and the density equals critical on average), then the Jacobi field grows not linearly but exponentially, as in an open Universe.

3. GROWTH OF THE JACOBI FIELD WITHOUT MEMORY

For the coherence of presentation, let us briefly recall how the solutions of the Jacobi equation are studied without considering memory effects. It is convenient to reduce equation (4) to a system of first-order linear equations:

$$\frac{d\mathbf{z}}{dx} = \mathbf{z}\hat{A}, \tag{5}$$

where the two-dimensional row vector $\mathbf{z}$ has components $y, y'\tau$, and the traceless matrix

$$\hat{A} = \begin{Vmatrix} 0 & -K\Delta \\ 1/\Delta & 0 \end{Vmatrix} \tag{6}$$

is constant on each update interval. The initial condition for this system has the form $\mathbf{z}_0 = (1,0)$. The solution of system (5) on the n-th update interval has the form

$$\mathbf{z}_n = \mathbf{z}_{n-1}\hat{B}_n = \mathbf{z}_0\hat{B}_1 \ldots \hat{B}_n, \tag{7}$$

where $\mathbf{z}_n$ is the value of vector $\mathbf{z}$ at the end of the n-th update interval, and, $\hat{B}_n = \exp \hat{A}_n\tau$ ($\hat{A}_n$ is the value of matrix $\hat{A}$ on the n-th update interval). Note that matrices $\hat{B}_n$ are uniformly distributed, independent, and unimodular. Thus, the problem reduces to studying the behavior of the product of a large number of independent uniformly distributed unimodular random matrices.

Using the representation of the Jacobi equation solution in the form (7), various growth characteristics of the solution are calculated. The simplest is to find the growth rate of the mean value of the solution modulus, i.e.,

$$\gamma_1 = \lim_{n\to\infty} \frac{\ln\langle|z(n\Delta)|\rangle}{n\Delta}. \quad (8)$$

For this, it is sufficient to find the mean value of matrices $\hat{B}_n$, which, due to the uniform distribution of these matrices, does not depend on the number n and is also a matrix $\langle\hat{B}\rangle$, and then find the largest eigenvalue of the resulting matrix, which determines the growth rate of the first moment. A somewhat more complex construction [9] gives a matrix whose largest eigenvalue determines the growth rate of the second moment:

$$\gamma_2 = \lim_{n\to\infty} \frac{\ln\langle|z(n\Delta)|^2\rangle}{2n\Delta}. \quad (9)$$

The growth rate of the solution itself (the logarithm of the solution modulus is averaged, not the logarithm of the mean value of the modulus powers!), i.e., the Lyapunov exponent

$$\gamma = \lim_{n\to\infty} \frac{\langle\ln|z(n\Delta)|\rangle}{n\Delta}, \quad (10)$$

can be found as a solution to some integral equation [10]. Remarkably, these growth characteristics do not coincide with each other, which manifests the phenomenon of intermittency [4]. In principle, the growth rates of the solution are calculated similarly. The model problem proposed by Zeldovich is important because in it the structure of matrix $\hat{A}$ is simple enough to be easily parameterized using a random number generator, and then explicitly or approximately find specific growth rates.

Note that, in principle, these growth rates can also be found by direct numerical simulation of solutions to the original equation with random curvature, although detecting the intermittency effect would require a very large (up to half a million) set of realizations.

The central role in this analytical theory is played by the assumption that at the moment of renewal, the memory of the random process is completely lost $K(x)$. The purpose of this work is to clarify how important this assumption is and how a small memory, passing from one renewal interval to another, changes these results.

4. MODEL OF INSTABILITY

4.1. Memory Loss Model

Now we need to describe how exactly memory loss occurs in the considered model. We will start from a construction where curvature values k at different renewal intervals, i.e., random variables k_n, are independent in aggregate at different n. Let's replace in equation (4) the random process $k(x)$ with another random process

$$q_1(t_n) = \frac{k(t_n) + k(t_{n-1})a}{\sqrt{1+a^2}}, \quad (11)$$

where the value $a \in [0,1]$ shows how many times memory is lost during transition from $(n-1)$-th renewal interval to n-th interval, and the denominator is introduced so that random processes k and q have equal variances. The random process q_1, like the random process k, is piecewise constant: both take constant values at each renewal interval. In the standard Gaussian case, their distribution functions also coincide. However, the values of process q_1 at different renewal intervals are not independent. Memory of the value q_1 at $(n-1)$-th renewal interval is preserved at n-th interval, but is already lost at $(n+1)$-th interval, although at this interval the process still remembers its value at n-th renewal interval. We can say that the process q_1 has a memory depth equal to τ.

Similarly, we can construct a process with memory depth 2τ:

$$q_2(t_i) = \frac{k(t_i) + k(t_{i-1})a + k(t_{i-2})a^2}{\sqrt{1+a^2+a^4}}, \quad (12)$$

and if necessary, construct models with arbitrarily long but fixed memory depth.

Let us emphasize that the considered memory loss model is not the only possible one, however, it is already complex enough to make it impossible to use the analytical methods described in the previous section. For specificity, in this work, we do not engage in developing analytical methods and limit ourselves to studying the selective growth of the Jacobi field for our model using direct numerical simulation methods.

4.2. Calculation of the Lyapunov exponent

The theoretical definition of the Lyapunov exponent assumes a transition to the limit as the number of renewal intervals tends to infinity. Under these assumptions, the existence of a positive Lyapunov exponent is proved in Furstenberg's mathematical theory (see the exposition of this theory oriented towards physicists in work [12]). Naturally, in the strict sense, this property cannot be verified within the framework of direct numerical simulation, which by its nature implies calculations on a finite interval of argument variation. Therefore, we will (without specifically mentioning it each time) speak about an approximate estimation of the solution growth rate. Proving or disproving whether these estimates converge to a certain limit with an infinite increase in the number of renewal intervals in our model represents an important mathematical problem, but is of limited interest for physics, since practically interesting instabilities develop only for a finite, albeit large time (in the case of the original Zeldovich problem, it is limited by the age of the Universe). Therefore, we solve the Jacobi equation on a finite interval, typically comprising one hundred renewal times. The experience of numerical modeling of the problem shows that without taking into account memory effects, one hundred renewal intervals are quite sufficient for calculating the Jacobi exponent with reasonable accuracy. Since the Lyapunov exponent, calculated reasonably in normalized units, is comparable to unity, the length of this integration interval seems more than sufficient in the context of physics.

The desired estimate of the Lyapunov exponent will be determined from a graph, where the logarithm of the solution is plotted along the vertical axis, and the integration interval length along the horizontal axis. The Lyapunov exponent represents the slope coefficient of the line approximating this relationship. The direct motivation of the problem suggests that $\ln|y|$ is plotted along the vertical axis. However, y although growing exponentially overall, in rare cases at points that practically never coincide with renewal points, turns to zero, so that narrow and deep minima (often called spikes) form on such a graph. They are interesting in themselves. Within the framework of the original cosmological problem, they are interpreted as gravitational lenses formed not by individual celestial bodies but by the Universe as a whole. However, for our task, these spikes are not important and only complicate the study. Therefore, we calculate the logarithm of the vector length $\mathbf{z}$, equal to $\sqrt{y^2 + (\tau y')^2}$. In work [13], it was verified that this method provides a correct estimation of the Lyapunov exponent in the problem without accounting for memory.

Since the Jacobi equation is explicitly solved at each renewal interval in the form of some matrix acting on the solution from the previous renewal interval, we obtain the solution to our equation over several renewal intervals by simply multiplying the corresponding matrices.

4.3. Statistical Properties *k*

To obtain estimates of the Lyapunov exponent, we must specify the probability distribution for k. According to the original meaning of the problem, it is natural to set the probability density k, symmetric with respect to $k = 0$. In this case, k can be considered Gaussian distributed. However, we find it reasonable to also consider cases of fluctuating but remaining positive (and in the opposite case – negative) curvature, and compare them with results obtained for alternating curvature. The fact is that the instability of interest to us can be of three types [10]:

- the solution grows exponentially even without curvature fluctuations, but they are responsible for fluctuations in the growth rate ($k < 0$);
- the solution grows in some renewal intervals and oscillates in others, but, as it turns out, growth overcomes oscillations (k is alternating);
- in all renewal intervals, the solution oscillates but may ultimately grow due to fluctuations. This phenomenon is similar to resonance ($k > 0$). Such

behavior of the solution resembles the behavior of solutions of linear differential equations of the form $dX/dt = \hat{A}X$ with a constant matrix in the right part, where the highest eigenvalue is complex, and the real part of the highest eigenvalue is positive.

Historically, Zeldovich was interested, of course, in the second of these cases, but he noted the necessity of investigating the third possibility. In work [10], it is shown that the third situation can indeed be realized in this model problem, although in other problems, such phenomenon was known earlier [14]. We will consider how the presence of memory affects these cases as well. Since a Gaussian random variable is always, albeit with small probability, alternating in sign, to maintain sign-definiteness we consider models with uniform distribution k.

5. RESULTS

5.1. Uniform distribution of curvature, symmetric relative to its zero value

Let us first consider how memory affects the solution growth with a uniform and symmetric distribution of curvature relative to the value $K = 0$ (Fig. 1). The figure shows that the solution grows exponentially in all three cases, the growth rates (see table) are close to each other, however, overall the graph for models with memory lies even slightly higher than the similar graph without memory effects. It is also remarkable that the solution with memory depth 2 lies higher than the solution with memory depth 1, although the difference between these curves is less than when transitioning from the first curve to the second. Naturally, the results somewhat depend on the realization of the random sequence of curvatures. Figure 1 shows results for two different realizations.

Before moving on to a more detailed comparison of results for the model with memory with results obtained without accounting for memory effects, let us note that this comparison can be made in two ways, and both methods seem to be of independent interest. First, when constructing a sequence of random curvatures for different update intervals, one can use the same set of random numbers in both models. Second, independent sets of random numbers can be used in these models. In the first case, we identify, as much as possible, memory

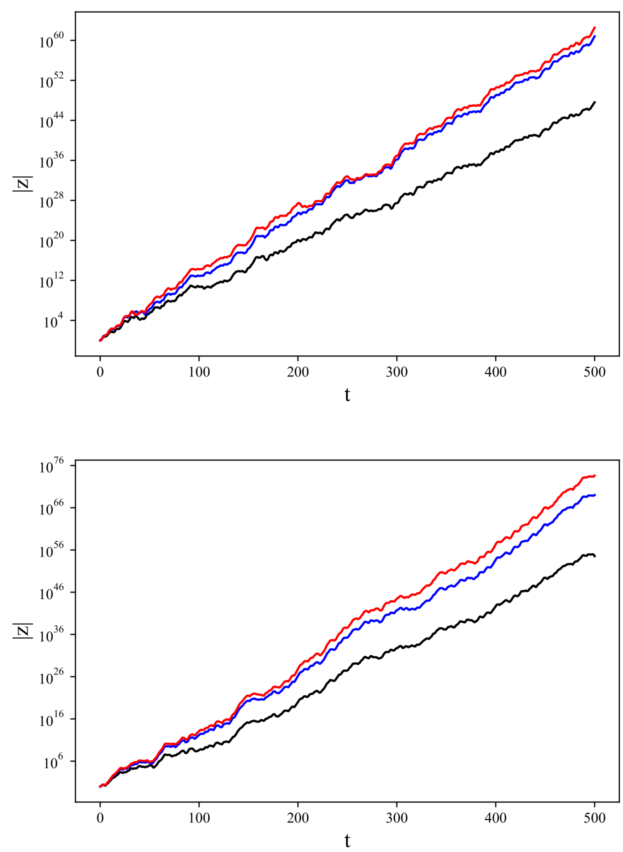

Fig. 1. Dependencies $|\mathbf{z}|$ on t at $a = 0.5$, black curves – model without memory, blue – model with memory depth 1, red – model with memory depth 2. Upper and lower panels – results for two independent realizations K

effects, abstracting from all other randomness effects, and in the second case, we determine how significant these effects are compared to other effects caused by randomness. Since both seem important, in Figures 2, 3 we provide both types of comparison.

It turns out that further increase in memory depth does not lead to significant changes in solution growth (Fig. 2).

Let's now increase the degree of curvature dependence, i.e., coefficient a. Now the second and third curves are much further apart from each other than in the previous figures. However, perhaps a more significant difference is that on the curve with the greatest memory depth, there is a clearly visible section where growth practically stops, forming a distinctive plateau.

Quantitative estimates of growth rates for several implementations in several curvature distribution

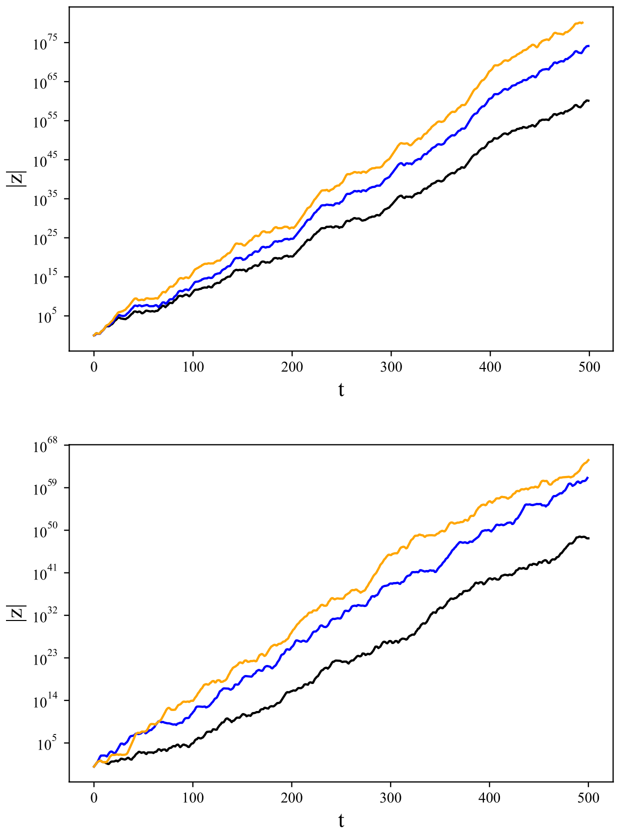

Fig. 2. Dependencies |**z**| on t at $a = 0.5$, black curves – model without memory consideration, blue – model with memory depth 1, yellow – model with memory depth 10. Upper panel – comparison of results for models with and without memory for the same set of random numbers used to generate the curvature sequence. Lower panel – comparison of results for models where curvature sequences are generated by different sets of random numbers

models are shown in the table. These data allow us to judge the degree of statistical stability of the obtained results, which we consider consistent with the conclusions made.

5.2. Dependence of the Lyapunov exponent on the lengths of update time intervals

Our approach is oriented towards identifying long-term memory effects rather than exploring the parametric space of models. However, as an example, we investigated the dependence of the Lyapunov exponent on memory time. Figure 4 shows estimates for Lyapunov exponents averaged over 100 implementations, with statistical error estimates of the means indicated. By the nature of the task, we used independent sets of random numbers each time

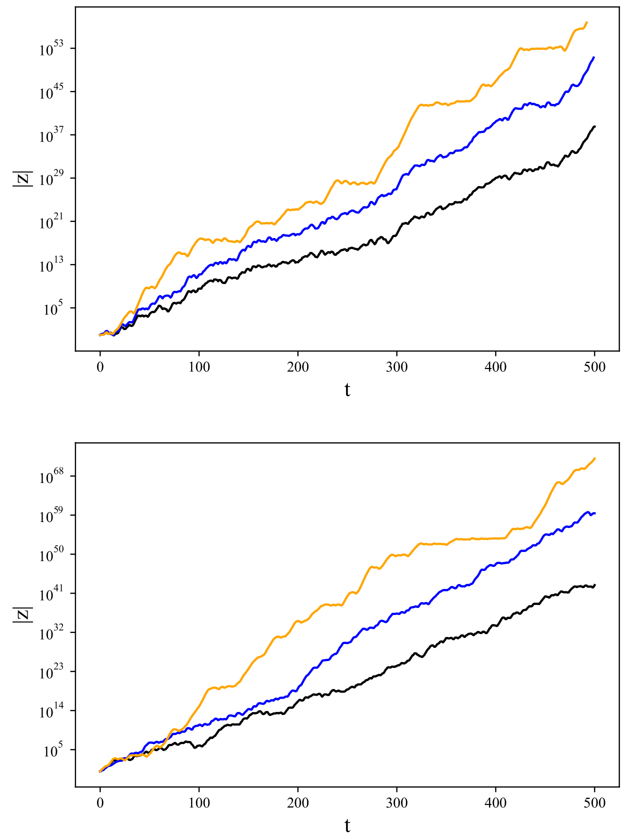

Fig. 3. Dependencies |**z**| on t at $a = 1$, black curves –model without memory consideration, blue – model with memory depth 1, yellow – model with memory depth 10. Upper and lower panels are constructed similarly to Fig. 2

to construct the sequence of random curvatures, i.e., we used the second comparison method.

The effect we are interested in is greater for small memory times Δ, which does not seem to contradict common sense.

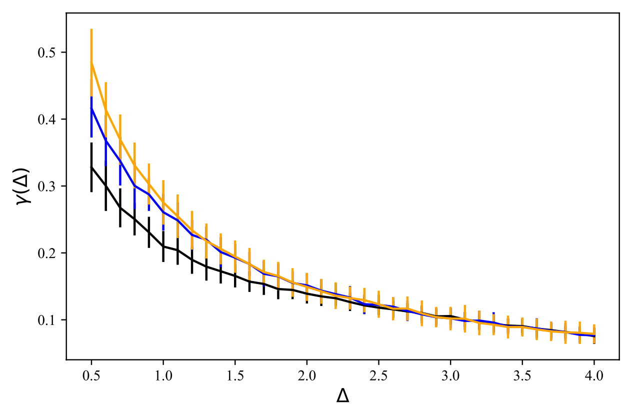

Fig. 4. Dependencies $\gamma(\Delta)$, black curve – model without memory, blue – model with memory depth 1, yellow – model with memory depth 10

Table. Lyapunov coefficient estimates for 10 implementations in several curvature distribution models and average values of these estimates

	Angular coefficients					
	Uniform on (−1, 1)			**Gaussian** *N* (0, 1)		
	Memory depth					
n	0	1	10	0	1	10
	0.197	0.257	0.283	0.37	0.43	0.46
	0.21	0.256	0.283	0.273	0.31	0.312
	0.196	0.249	0.337	0.333	0.37	0.35
	0.21	0.29	0.233	0.313	0.347	0.352
	0.233	0.308	0.343	0.303	0.363	0.372
	0.22	0.283	0.369	0.34	0.37	0.40
	0.24	0.288	0.322	0.32	0.378	0.364
	0.195	0.237	0.286	0.363	0.388	0.432
	0.208	0.264	0.298	0.298	0.368	0.401
	0.208	0.281	0.332	0.28	0.328	0.305
$\bar{y}$	0.211	0.272	0.309	0.319	0.365	0.375

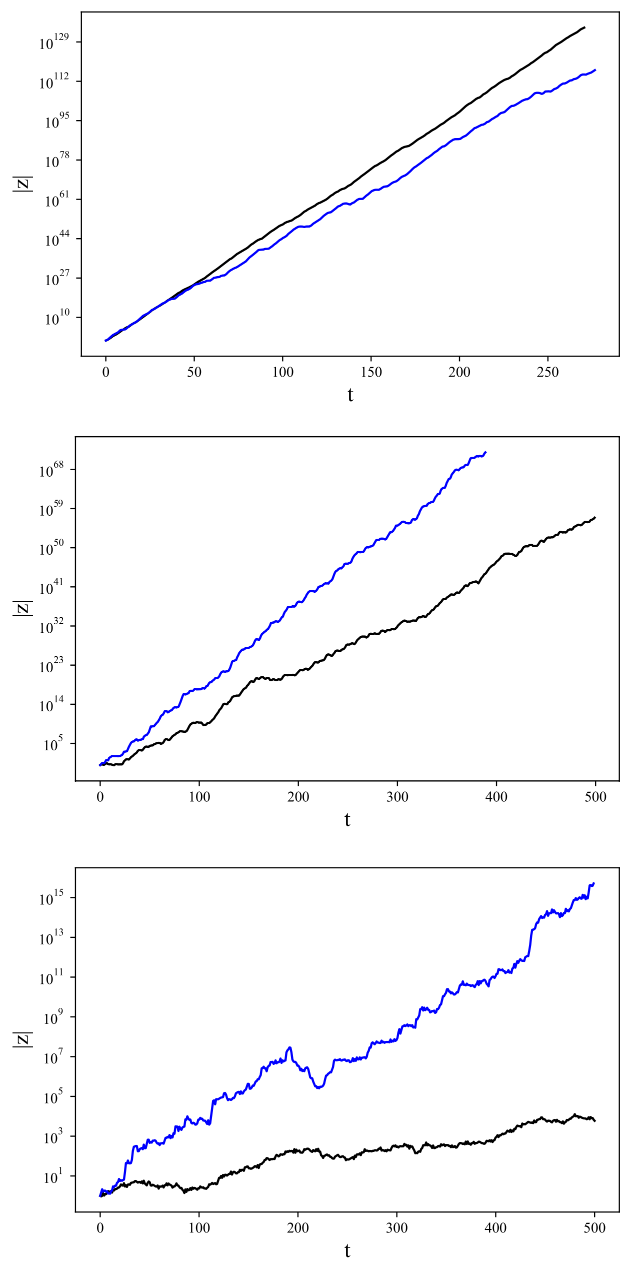

Fig. 5. Dependencies |**z**| on *t* at $a = 0.5$, blue curves represent the model using Gaussian distribution with memory depth 1, black curves represent the model using equiprobable distribution with memory depth 1. For the first graph (top to bottom), the mean equals -1, for the second – 0, for the third – 1

6. MODEL WITH GAUSSIAN CURVATURE DISTRIBUTION

Since we study the effects of long memory by comparing the behavior of the Jacobi field for cases where curvature fluctuates, being negative, positive and alternating, it is inconvenient for us to consider Gaussian curvature distribution, which is always alternating, as a typical example. However, outside our comparison, studying the case where curvature is Gaussiandistributed is also of obvious interest. Consideration of this case does not cause technical difficulties. To make the results comparable with those of uniform distribution, we choose the parameters of Gaussian distribution so that the mean coincides with the mean in uniform distribution, i.e., equals zero, and the root mean square value equals half the range of uniform distribution. The corresponding results are shown in Fig. 5 and in the table. It can be seen that the estimates of Lyapunov exponents when switching to Gaussian distribution typically increase somewhat, which, as we believe, fits into our interpretation of the results.

7. CONCLUSIONS AND DISCUSSION

Summarizing the obtained results, the following conclusions can be noted. Taking into account memory effects, at least within the framework of the considered model, does not lead to a radical revision of the results obtained for the renewal model. The solution of the studied system still grows exponentially, the obtained growth rates (Lyapunov exponents) are close to those obtained for the renewal model, in any case, these are values of the same order. Naturally, one wants to extrapolate these results, obtained for a very simple model, to much more complex problems – primarily to the study of small-scale dynamo, i.e., to the study of magnetic field self-excitation in turbulent or convective flow of conducting medium, not directly related to the effects of flow mirror asymmetry. In this context, the results look optimistic and remove natural concerns arising from neglecting memory effects in analytical theory based on the Kazantsev equation. As for the results of direct numerical modeling, their capabilities are also not unlimited, and interpretation of the obtained results causes complex problems even in cases where all problems of obtaining solutions to three-dimensional equations governing the evolution of magnetic field and flow have been solved (see, for example, [15, 16]). In any case, obtaining growth rate estimates similar to those obtained above would require years of specialized research by a scientific group without material constraints in their work. In principle, studying these issues by experimental methods is also possible, however, even for those instabilities whose documentation is of direct economic interest, finding a sample of experimental data with fixed statistical properties and volume sufficient for such studies causes serious difficulties [17]. Therefore, one can hope that from a pragmatic point of view, the role of memory effects in the considered range of problems has been clarified sufficiently, so that memory effects when extrapolating existing information, for example, about small-scale dynamo to realistic situations, do not cause concerns, at least at the current level of knowledge.

On the other hand, there are elements in the obtained results that seem unexpected and counterintuitive. First of all, it seems unexpected that accounting for memory leads not to a decrease but to some increase in the growth rate of the studied field. It would be natural to think that accounting for memory makes the medium less random, therefore the effect closely related to randomness should decrease.

One would like to understand more deeply the cause leading to such an enhancement of instability. This becomes possible if we somewhat modify the memory model. Let's assume that at each update moment, it is randomly determined whether the value will K remain the same as in the previous evolution stage (this happens with a small probability μ), or it will take a new value independent of the previous one (this happens with probability $1-\mu$). This probabilistic model also, we believe, presents scientific interest, but its full study goes beyond the scope of this article. However, it is important for us now that this model allows for analytical investigation. Within this investigation, the following important effect is discovered.

The nature of the considered instability is associated with the vector character of the evolving quantity. Therefore, in the solution (7) of the evolutionary equation, matrices appear which, generally speaking, are non-commutative. If we were to consider the evolution of a scalar quantity described by the equation

$$(d/dt)y = a(t)y, \tag{13}$$

where $a(t)$ is a renewal random process with zero mean and root-mean-square value σ, constant over renewal intervals, then its solution would grow as

$$y = \exp(\sqrt{t\Delta}\sigma\zeta), \tag{14}$$

where ζ is a Gaussian random variable with zero mean and unit variance, i.e., subexponentially (for definiteness, we assumed that a is Gaussian distributed at each moment of time). If we now consider the same equation but assume that the random process a has memory described by our second model, then when calculating the Lyapunov exponent, we will have to separately consider different variants of how memory is lost in this realization of the random process, and among these variants there will be one in which the value a takes a positive value $a_0 > 0$ and will maintain it during all subsequent renewals. Naturally, the statistical weight of this variant decreases as $\mu^n \propto \exp(t/\Delta \ln \mu)$, however, the decrease in statistical weight is compensated by the exponential growth of the contribution of this realization $\propto \exp(a_0 t)$. As a result, the solution can grow exponentially, which is impossible without taking into account memory effects.

In other words, at least part of the discovered memory effect is related to the fact that memory provides an opportunity for such realizations K, that are particularly favorable for solution growth to work longer than usual. It is now clear why there are also segments of the solution where growth decreases or stops altogether – memory sometimes gives the opportunity for unfavorable realizations to work longer as well K.

The question of how realistic is the memory model, which in principle allows (albeit with exponentially decreasing probability) for realizations where the environment effectively ceases to be random, goes beyond the capabilities of probability theory or theoretical physics. In many cases, such low-probability events can be freely neglected, but the presence of intermittency phenomena in the development of instabilities in random media shows that this must be done with certain caution. In any case, such an explanatory model is unusual in the range of questions considered in physics. However, we note with some surprise that humanities easily employ similar models when explaining how an undisputed leader emerges from a group of more or less similar competition participants (see, for example, [18]).

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