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TWO STAGES IN THE FORMATION OF THE BRANCHING STRUCTURE OF A DECIDUOUS TREE

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Abstract. Fractal properties in the formation of the branching structure of deciduous trees have been studied by numerical Fourier analysis. It is shown that the lower levels of branching of adult trees are formed obeying the law of the logarithmic fractal in two-dimensional space, according to which the surface area of the lower branch is equal to the sum of the surface areas of the branches after its branching, i.e. the law of conservation of area when scaling is fulfilled. The structure of branches at the upper levels of branching obeys the law of the logarithmic fractal in three-dimensional space, i.e. the law of volume conservation during scaling, which is natural, since living tissue occupies completely an young branch, and not only its surface. A mathematical model is proposed that generalizes the concepts of a logarithmic fractal on the surface for adult branches and a logarithmic fractal in volume for young branches. Thus, an integral fractal concept of the growth and branching structure of deciduous trees is constructed.

Keywords: *Logarithmic fractal*, *the branching structure of deciduous trees*, *Fourier analysis*

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1. INTRODUCTION

The word "fractal" was introduced into scientific terminology in the second half of the twentieth century by Benoît B. Mandelbrot [1] and very quickly the concept of fractal description of objects was developed and applied in various fields of science and technology [2, 3]. The concept of "fractal" in its meaning is very close to the expression "self-similar fragmented system". Fractals are found everywhere, both in living and inanimate nature: mountain massifs, the coastline of the sea, riverbeds [3], moles [4], turbulence processes [5], vascular pattern in the lungs [6]. In itself, the study of fractals is of great interest, since the fractal concept can be used in various fields of science and technology, for example, in medicine when modeling various processes in humans and animals [6-9].

Biological systems in the course of their life often develop according to the laws of self-similarity and can be attributed to fractals [10-12]. One of the most striking manifestations of fractality in nature is trees. Leonardo da Vinci was the first to notice and formulate a pattern in the growth and branching of trees. He deduced the empirical law of tree branching, which refers us to the universal cyclicity of nature. Leonardo's law says: at each level of branching of a tree, the total cross—sectional area of all branches is the same and equal to the trunk cross-sectional area: $d_i^2 = kd_{i+1}^2$ where d — is the diameter of the branches, k is the number of branches after branching [13]. This formulation serves as the basis for describing structural properties in all known tree models [14-21] and is a real foundation in computer modeling of a tree of similar objects [22, 23]. And although the pattern noted by Leonardo is widely used in theoretical models and allometric studies [16-20], only a few works can be found that confirm the validity of this pattern with good statistics. Moreover, the authors of a broad and in-depth review [21], analyzing existing experimental studies [24-26], came to the conclusion that "Leonardo da Vinci's rule is not fulfilled in all cases." And indeed, as noted in numerous studies, experimental confirmation of this irregularity requires the experimenter to exert considerable physical effort and dexterity if he decides to measure the diameter of each branch of a tree without causing any harm to it. The fractal structure, as the dominant feature of coniferous trees, is considered in detail in [12, 27, 28] using the example of spruce.

In [29], it was proposed to use numerical Fourier analysis of tree images as a non-destructive method for studying the properties of self-similarity in their structure. It was shown [29] that the image photograph of the crown of some species of leafy trees, which is a two-dimensional projection of a tree onto a plane, belongs to a special class of fractals — logarithmic fractals in two-dimensional space. It can be concluded that the law of conservation of the lateral surface area at different branching levels is fulfilled for them, which is apparently explained by the concentration of conducting cells in the phloem (bark) and in the upper layers of xylem (wood). Experimental evidence in support of this hypothesis was obtained using the method of numerical Fourier analysis in the study of many images of various deciduous trees (such as oak, birch, linden, etc.), images of trees were taken in the cold season, after leaf fall. The study is carried out by numerical methods, the Fourier square of the object image is obtained — an isotropic Fourier intensity distribution, which, after azimuthal averaging of a two-dimensional map, gives the so-called scattering curve — the dependence of the Fourier intensity on the inverse coordinate. When examining images of trees, a crossover was switched to another scattering mode in the region of large momentum transfer, past the section of the scattering curve corresponding to the logarithmic fractal structure. In all images, a sharp slowdown in the decrease in Fourier intensity was detected at large momentum transfer (small scales in the real image).

In this article, we explore the nature of this crossover. An assumption has been made, which is confirmed by experiment, that the region of the large momentum transfer on the scattering curve corresponds to such areas of small scales in images of trees where young branches are depicted. Since young branches consist entirely of living tissue, life in them is distributed over the entire volume, unlike adult branches, in which living cells are concentrated only on the surface. The article shows that the part of the tree consisting of adult branches is a logarithmic fractal characterizing the surface of the tree, i.e. it obeys the law of a logarithmic fractal in two-dimensional space. At the same time, a part of the tree with young branches corresponds to a logarithmic fractal in three-dimensional space, which in the image of the tree (its projection onto the plane) turns into an object described as a massive two-dimensional fractal with dimension $D_f = 1.5-1.7$. A branching model is proposed that generalizes the concepts of a logarithmic fractal on the surface for adult branches and a logarithmic fractal in volume for young branches. Thus, a holistic concept of the growth and branching of deciduous trees has been built.

The work is organized as follows. In the section 2 the classification of fractal objects based on the methods of scattering of penetrating radiation (three-dimensional case) or light (two-dimensional case) is described. In the section 3 the study of images of trees of different ages by numerical Fourier analysis is presented. Images of mature trees and their individual sections with young branches are considered. In the section 4 a two-stage mathematical model is proposed that describes the law of formation of the branching structure of an adult tree. Section 5 presents remarkable consequences of the concept of self-similarity of branches. In particular, it is shown that experimental data for an adult tree, together with the hypothesis of self-similarity of branches, guarantee the fulfillment of Leonardo da Vinci's law on the structure of a tree. At the same time, Leonardo's law is not fulfilled for young branches. Section 6 presents the conclusions of the work.

2. CLASSIFICATION OF FRACTAL OBJECTS

The main characteristic of fractal objects is their fractal dimension D_f (Hausdorff-Bezikovich dimension). In contrast to the topological dimension of the object D_T, the Hausdorff-Bezikovich dimension D_f can be both an integer and a fractional one. And $D_T \leqslant D_f \leqslant D_E$, this means that the fractal dimension of the object, due to its fragmentation, exceeds its topological dimension, but is less than the dimension of Euclidean space.

To obtain information about the fractal dimensionality of an object in a three-dimensional space, the method of small-angle neutron scattering

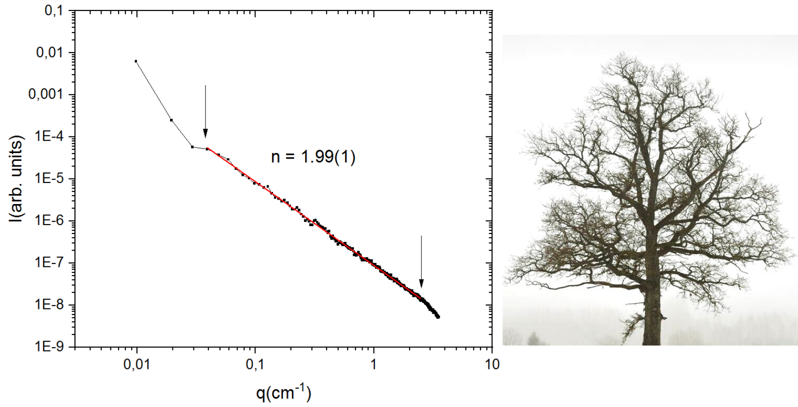

Fig. 1. The image of an old tree (an oak) is on the right. The scattering curve is on the left, as a result of Fourier analysis of the image: intensity as a function of momentum on a double logarithmic scale. In the momentum range from 10^{-2} to 5 cm^{-1}, the curve is described by the dependence $Q^{-\nu}$ with $\nu = 1.99 \pm 0.01$

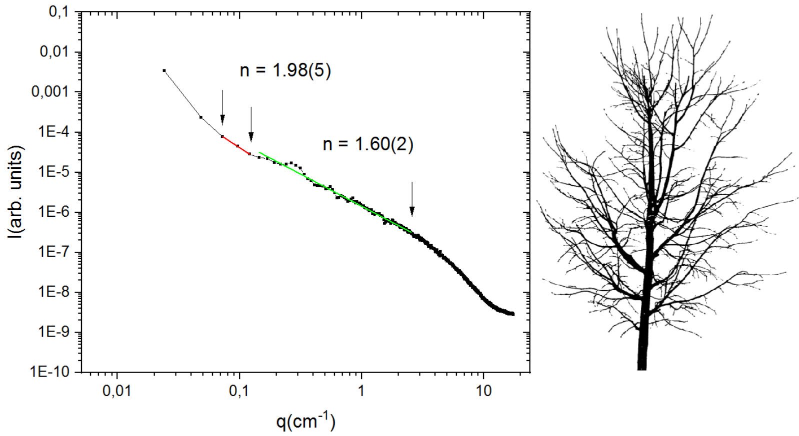

Fig. 2. The image of a young tree (an apple tree) is on the right. The scattering curve is on the left, as a result of Fourier analysis of the image: intensity as a function of momentum on a double logarithmic scale. In the momentum range from 2 • 10^{-2} to 2 cm^{-1}, the curve is described by the dependence of $Q^{-\nu}$ with $\nu = 1.60 \pm 0.02$

and X-ray radiation is widely used [30-32]. Using this method, fractals in three-dimensional space were divided into three classes: mass (fragmentation is distributed over the volume inside the object), surface (fragmentation is concentrated at the boundary of the object) and logarithmic, when the surface fractal extends deep into the substance, forming an intermediate object between the surface and volume fractals [33].

When describing fractal objects, the topological dimension of a single self-similar element of a fractal, the fractal dimension of the entire object and the Euclidean dimension of space should be distinguished and correlated. For example, flat objects in three—dimensional space ($D_E = 3$) and flat objects in two-dimensional space ($D_E = 2$) are physically different objects. Experimental measurement of the fractal dimension of “flat” fractals (in two—dimensional space) can be carried out by scattering light on them, registering the square of the Fourier image of the scattering object on the detector [34-38]. Such an experiment can be modeled using numerical Fourier analysis, examining images of various fractal and non-fractal objects in this way. Using this method, fractals in the two-dimensional case can be divided (by analogy with three-dimensional space) into mass, boundary and logarithmic [39, 40].

Logarithmic fractals are of particular interest. Unlike mass and surface (boundary) fractals, logarithmic fractals, although self-similar, are heterogeneous, and, as a result, are described by an additive scaling law and a hierarchical structure. At the same time, for logarithmic fractals, the law of equality of the amount of matter at each level of their hierarchy is fulfilled. An example of a tree constructed according to the principle of Leonardo da Vinci as a logarithmic fractal was proposed in work [39].

3. INVESTIGATION OF TREE IMAGES BY NUMERICAL FOURIER ANALYSIS METHODS

When studying images of fractal and non-fractal objects using Fourier analysis methods, the so-called scattering curve is obtained in the inverse, i.e. in Fourier space. This dependence of the scattering intensity on the momentum is a probability function and characterizes the amount of matter depending on the size. A detailed description of the numerical simulation of the scattering process on fractal objects using Fourier analysis in two-dimensional space is presented in [40-42].

As shown in [29] and can be seen from the example of Fig. 1 and Fig. 2, when studying photographs of trees using Fourier analysis methods, the scattering curves can be divided into three sections, each of which demonstrates its own character of decreasing intensity with increasing momentum. In the region of small momenta a slope of the scattering curve on a double logarithmic scale is close to 2, which corresponds to the structure of a logarithmic fractal. The next section is characterized by a slowdown in the decrease in intensity and the slope of the scattering curve on a double logarithmic scale turns out to be less than 2. And in the area of large momentum transfer, there is again a rapid decrease in intensity with a slope, close to 3. This area corresponds to the scattering on the minimal element of the image and does not represent much interest. The transition point from the second section to the third corresponds to the size of the minimal element of the image.

If we take into account that the first and second sections of the scattering curve in the inverse space correspond to two different ranges of linear sizes of scattering objects, then we can assume that the scattering curve at large sizes (the range of small momenta of the inverse space) is due to the presence of large, i.e. old, tree branches, and at small sizes — the presence of young branches. To experimentally test the hypothesis of the existence of two different types of branching structure of a deciduous tree, it is necessary to study two-dimensional images (photographs) old, mature and young trees. It is necessary to select images, develop a research methodology, obtain scattering curves by numerical Fourier analysis and, approximating them by a power law, find the parameters of scale invariance in accordance with the classification of fractal objects [40, 42]. For these studies, we used the fractal program [43].

The criteria for selecting images of trees suitable for this kind of research seem quite simple, but impose some restrictions on the tree under study. To achieve the best contrast in the image between the presence of a branch and its absence, it is necessary to take a picture of a tree without leaves against a light sky. Then the picture turns into black and white with maximum contrast, so that the background turns white and the tree turns black. The image is taken from a distance of several tens of meters, so that the characteristic distances inside the object would be much less than the distance from the object to the shooting location. This ensures that the proportions in the size of the individual branches of the tree in the picture are preserved. With such criteria, it is natural that only free-standing trees are involved in the study, and not trees growing in the middle of the forest.

Information about a three-dimensional object (tree) is obtained by direct photographing, i.e. an act during which a three-dimensional object is projected onto a two-dimensional plane. With this projection, most branches and the trunk of the tree are clearly distinguishable, i.e. the branches overlap only slightly with each other and with the trunk. Thus, when photographing/projecting a tree onto a plane, information about the structure of branches is not lost and the proportions between the branches (their size and number) are preserved when zooming in. The registered object is a black spot of a certain size (in pixels) on a white background.

Branches or parts of them growing along the line of mapping (along the projection axis) are not visible in the image, are not recorded, and therefore are not taken into account in the analysis. At the same time, their number obeys the general law of correspondence of the number of objects with a change in scale, so the photo correctly conveys the law of scaling, based on only one image (one projection). Note that by changing the angle of photography, it is possible to obtain and analyze information about another projection of the tree. Taking into account the fact that the tree has axial symmetry, the azimuthally averaged square of the Fourier image of photographs of the tree from different lateral angles gives the same scattering curves, and this was confirmed experimentally.

The research methodology of this work is aimed at differentiating old branches and young branches. First, we will select an image of an adult tree with a large number of old branches. We assume that it is the old branches that form a two-dimensional logarithmic fractal in the image, which corresponds to a section with a slope close to 2 on the scattering curve. It is expected that a large number of old branches makes this area the most pronounced. Secondly, we look at an adult tree in the stage of active growth, i.e. with a large number of both old branches and young branches, in order to understand how the scattering curve will change if young branches become more numerous. Thirdly, to make sure that it is the young branches of an adult tree that cause a deviation from the characteristic scattering law with an slope close to 2, we will highlight

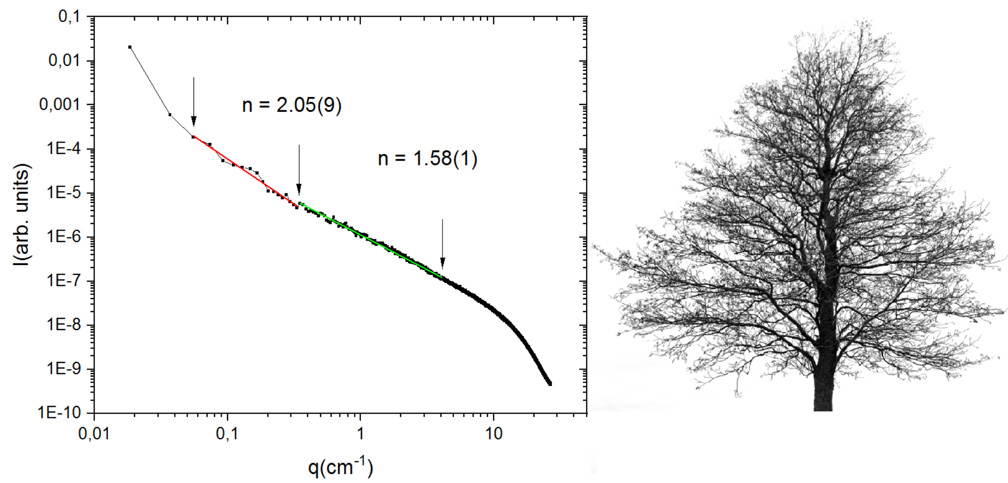

Fig. 3. The image of an adult (actively growing) tree (a linden) is on the right. The scattering curve is on the left, as a result of Fourier analysis of the image: intensity depending on the momentum in a double logarithmic scale. In the pulse range from $5 \cdot 10^{-2}$ to $3 \cdot 10^{-1}$ cm^{-1}, the curve is described by the $Q^{-\nu}$ dependence with $\nu = 2.05 \pm 0.09$, and in the range from $4 \cdot 10^{-1}$ to 4 cm^{-1} by the $Q^{-\nu}$ dependence with $\nu = 1.58 \pm 0.09$

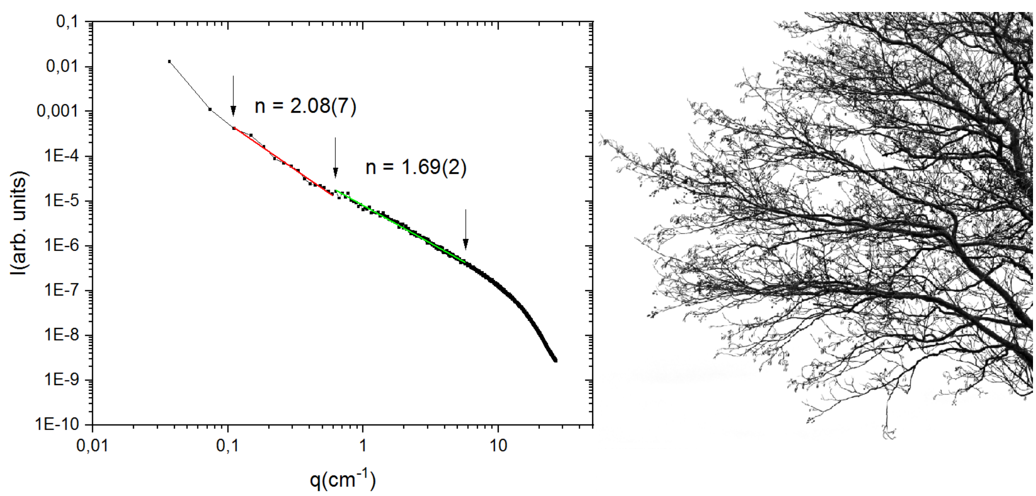

Fig. 4. A fragment of an image of an adult (actively growing) tree (a linden) containing a small number of «old» branches is on the right. The scattering curve is on the left: intensity depending on the momentum on a double logarithmic scale. In the momentum range from 10^{-1} to $6 \cdot 10^{-1}$ cm^{-1}, the curve is described by the $Q^{-\nu}$ dependence with $\nu = 2.08 \pm 0.07$, and in the range from $6 \cdot 10^{-1}$ to 6 cm^{-1} by the $Q^{-\nu}$ dependence with $\nu = 1.69 \pm 0.02$.

fragments of the image of an adult tree, which will include mainly young branches and only a few old branches, or not include old branches at all. Fourth, we will consider an image of a young tree in which old branches are missing, and consequently, there should be no area on the scattering curve with a slope close to 2.

Figure 1 shows an image of an old tree (oak) with a small current growth, in which the bulk of the branches are old. In this case of the old tree, the scattering curve is described by the dependence $Q^{-\nu}$ with $\nu = 1.99 \pm 0.01$ in the momentum range from $4 \cdot 10^{-2}$ to 3 cm^{-1}. Such a power dependence corresponds to a logarithmic fractal structure in a very large (about two orders) momentum range [29]. This range of momentum transfer can be converted into a range of average distances in direct space from 2 to 150 cm, which, taking into account the cylindrical shape of the branch, approximately corresponds to a range of diameters from 0.7 to 50 cm and a range of lengths from 7 to 500 cm. Here and further, we assume that the element of self-similarity in the image is a rectangle with sides d (thickness of the branch) and l (length of the branch). For certainty, we assume that the ratio $l/d = 9$, and the average linear size r of the rectangle is equal to the square root of its area: $r = \sqrt{S} = \sqrt{ld} = 3d = l/3$.

In the case of a very young tree (apple sapling), a section with a slope close to 2 is practically absent, and the section corresponding to the young branches occupies almost the entire scattering curve, which in the area from $2 \cdot 10^{-1}$ to 2 cm^{-1} has a slope $\nu = 1.60 \pm 0.02$ on a double logarithmic scale (Fig. 2). This range is converted into a range of average sizes from 3 to 30 cm, which, for example, in terms of branch lengths corresponds to sizes from 10 to 100 cm.

In the case of an adult (actively growing) tree with a large number of young branches (Fig. 3), the area corresponding to a two-dimensional logarithmic fractal occupies a significantly smaller part on the scattering curve with a slope $\nu = 2.05 \pm 0.09$ (one order) in the range from $5 \cdot 10^{-2}$ to $3 \cdot 10^{-1}$ cm^{-1}. The part of the scattering curve corresponding to the young branches has a slope in the double logarithmic scale $\nu = 1.58 \pm 0.09$ in the range from $4 \cdot 10^{-1}$ to 4 cm^{-1}. The inflection point in the q-dependence corresponds to an average distance of 15 cm, which in terms of branch thickness is converted to 5 cm, and in terms of branch length — to 50 cm.

The study of fragments of the image of a mature tree (a linden from Fig. 3) showed that the fewer branches of the tree remain in the fragment, the shorter becomes the length of the scattering curve section corresponding to a two-dimensional logarithmic fractal structure with a slope close to 2. In case the “old” branches of the tree are present in the image (Fig. 4, right), we observe a small section of the curve with an inclination $\nu = 2.08 \pm 0.07$, corresponding to a two-dimensional logarithmic fractal in the range from 10^{-1} to $6 \cdot 10^{-1}$ cm^{-1}.

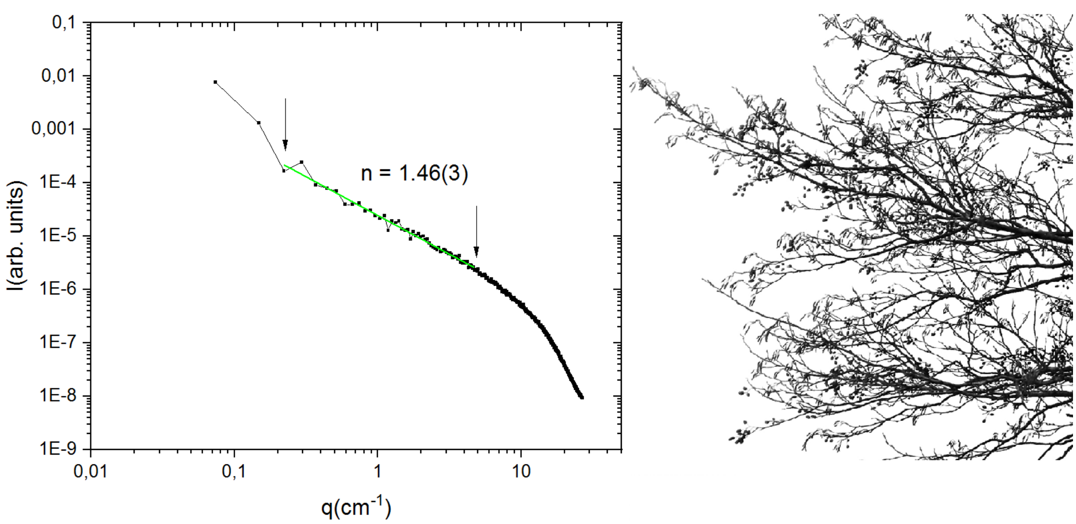

Fig. 5. A fragment of an image of an adult (actively growing) tree (linden) that does not contain «old» branches is on the right. The scattering curve is on the left: intensity depending on the pulse in a double logarithmic scale. In the pulse range from $2 \cdot 10^{-1}$ to 4 cm^{-1}, the curve is described by the dependence $Q^{-\nu}$ with $\nu = 1.46 \pm 0.03$

And at the same time, a long section with a slope $\nu = 1.69 \pm 0.02$ is observed on the curve on a double logarithmic scale, corresponding to young branches in the range from $6 \cdot 10^{-1}$ to 6 cm^{-1} (Fig. 4, left). If only young branches are present in the image (Fig. 5, right), then there is no section corresponding to the logarithmic fractal on the scattering curve, and a section with a slope $\nu = 1.46 \pm 0.03$ corresponding to the structure of young branches is visible in the range from $2 \cdot 10^{-1}$ to 4 cm^{-1} (Fig. 5).

Thus, we can conclude that it is the old branches in the image of the tree that are described by the law of the logarithmic fractal with Q^{-2} – dependence of the scattering intensity in the Fourier analysis of their image. At the same time, as shown in [29], their organization obeys the law of conservation of the lateral surface area, while young branches obey a different law, which, when analyzing two-dimensional images, gives a dependence of intensity on momentum in the form of $Q^{-\nu}$ with ν = 1.60-1.70. It is also important to note that the power dependence itself implies the realization of the hypothesis of self—similarity of scalable elements – in this case, branches of a tree. The question arises: is it possible to combine into a single model the two stages of tree growth that differ so clearly in the Fourier analysis conducted above, and what are the similarities and differences between the two different stages? We will formulate the answer to this question in section 4.

4. TWO-STAGE MODEL OF THE TREE BRANCHING STRUCTURE

In botany, trees are defined as “the life form of woody plants with a single, distinct, perennial, to varying degrees woody, persisting throughout life, branched main axis — the trunk” [44].

Conventionally, the trunk and branches of a tree can be radially divided into three main parts (layers): bast (floema, the living part of the bark), cambium, wood (xylem). The bast is the outermost part of the trunk, adjacent to the outer (deadened) bark and consists of living cells, the bast is involved in the transport of photosynthesis products from the leaves to all organs of the tree. Cambium is the main producing tissue of a tree, cambium consists of living cells, while the outer part of the cells becomes a bast, and the inner part becomes wood. It is due to cambium that the tree and its branches grow in thickness here. It is important to note that the bast and cambium are quite thin compared to the trunk of the tree and have a fixed thickness. Wood is formed from internal cambium cells and consists of stiffened cells that are not capable of action, it occupies up to 90% of the volume of the tree. The outer part of the wood (sapwood) is involved in the transport of water and minerals from the roots upwards, while the inner part (core) formed in many tree species is physiologically inactive and performs only a mechanical function. In addition to the trunk, the tree is characterized by multiple branching, the multiplicity of which increases every year. The mature branches have the same structure as the trunk in thickness and consist of wood, cambium and bast. Thus, the trunk of a tree, like any of its branches, can be divided into an inner, partially “deadened” part — wood and an outer layer consisting of living cells.

As shown in [29], the images of the branching structure of trees belong to the class of logarithmic fractals on the plane and the law of conservation of the area of the lateral surface as branching is performed for them. This conclusion fits well into the description of a tree whose life is concentrated on its surface, i.e. in the bast and cambium, as well as in the outer layer of the xylem. The law of conservation of the lateral surface area at different branching levels, being undoubtedly a simplified mathematical model of a tree, nevertheless connects the dimensional parameters of a branch of the i branching level and k branches of the $(i + 1)$ level emanating from it [29]:

$$d_i l_i = k d_{i+1} l_{i+1}. \quad (1)$$

Here d_i, l_i and d_{i+1}, l_{i+1} are the diameter of the cross section and the length of the i and $(i + 1)$ branches, respectively.

Developing this model, we take into account that the “living” layer of the branch has a finite thickness x, the same (in the first approximation) at different levels of the tree branching. Then not only the area is preserved, but also the volume of the surface layer, and the conservation law is written as

$$x d_i l_i = k x d_{i+1} l_{i+1}. \quad (2)$$

Such a transformation of the “conservation law” during scaling does not change anything from the point of view of the mathematical model, but it is of great importance for the essence of the

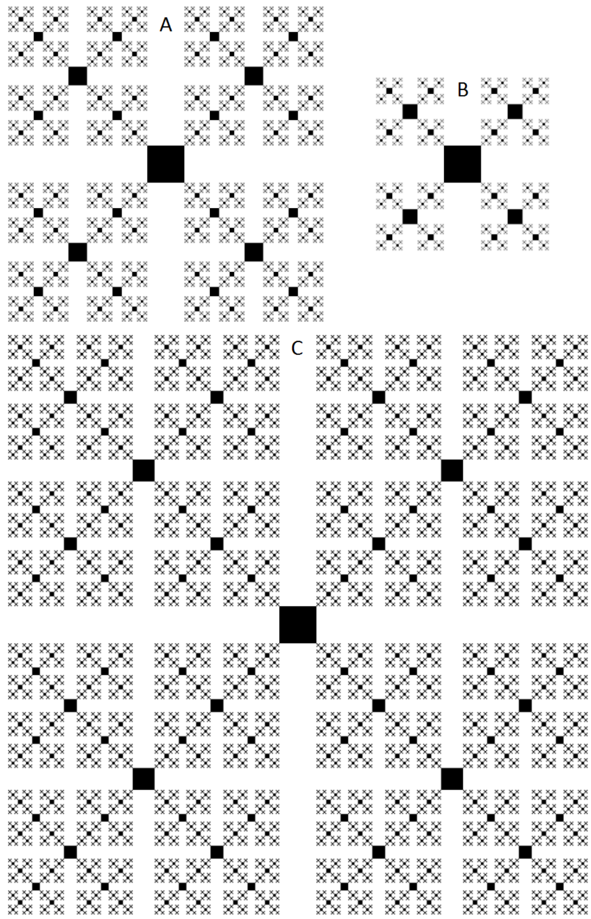

Fig. 6. The model of branching a Leonardo da Vinci tree into four branches with the condition of preserving the area (A). A model of branching a tree into four branches with the condition of reducing the area (B). A model of branching a tree into four branches with the condition of increasing the area (C)

matter, since it changes the very concept in which the volume of living matter is now preserved at each level of the branching tree.

Taking into account the thickness of the surface layer makes it possible to introduce into a mathematical model and describe the structural transition from old branches to young branches. To do this, note that the concept of a surface "living" layer of a branch works as long as the condition is met that the thickness of this layer is less than the radius of the cross-section of the branch ($x < d_{i+1}/2$). This condition is violated when the thickness of the "living" layer becomes equal to the radius of the cross section of the branch, i.e. when the entire branch consists only of living cells. This corresponds to the definition of a single branch, inside which a spring has not yet appeared. Thus, we can consider, as two separate cases, the structure of the old branches, for which $x < d/2$, and the structure of the young branches, for which $x_i = d_i/2$.

In the first case, x in the right and left parts (2) are reduced, we obtain (1) and, therefore, the law of conservation of area is fulfilled. In the second case, expression (2) takes on a new form and acquires a new meaning:

$$x_i d_i l_i = k \cdot x_{i+1} d_{i+1} l_{i+1},$$

or

$$d_i^2 l_i = k d_{i+1}^2 l_{i+1}, \quad (3)$$

that is, the volume of branches is preserved, and not their surface area. The expression (3), provided that the shape of the branch is self-similar at different branching levels, is a mathematical definition for a logarithmic fractal in three-dimensional space.

Thus, the system of young branches of the tree forms a three-dimensional logarithmic fractal, since the volume of branches at each branching level is preserved, and the system of adult branches is a two-dimensional logarithmic fractal, since their surface area is stored. This can explain the presence of a transition point from the first to the second section on the scattering curve, as well as the fact that the length of the section of the scattering curve corresponding to the two-dimensional logarithmic fractal varies depending on the age of the tree and the number of mature branches in the image. The mature branches, in which the main part is occupied by wood, form a logarithmic fractal structure on the photo of the tree or on its surface, and the young branches form a logarithmic fractal structure in volume. The transition point itself is determined by the condition when the thickness of the living layer becomes equal to the radius of the branch $x_i = d_i/2$. It is interesting to note that such a two-stage model of the branching structure of a tree nevertheless obeys a single law of conservation of the volume of living material at each level of branching, regardless of whether we consider old branches or young branches. In other words, the number of living cells at each level of branching remains constant, although on old branches it is distributed over the surface of the branch, and on young branches — over their entire volume.

The model proposed above finds experimental confirmation in terms of mature and old branches [29], however, the hypothesis of a three-dimensional logarithmic fractal to describe the branching structure of young branches still requires its confirmation. The experimental data presented above demonstrate a power dependence with the index $v = 1.60$-1.70, which implies the very similarity of branches at different branching levels. Based on the hypothesis of self-similarity (see (4)) and the law of volume storage at scaling (3), it can be shown at least on a qualitative level that the projection of young tree branches onto a two-dimensional plane is a fractal on a plane with dimension D*f* should be in lower case ~1.70.

5. THE HYPOTHESIS OF SELF-SIMILARITY OF BRANCHES

The hypothesis of self-similarity of the shape of branches during scaling can be formulated as follows. A single branch from the point of its branching from the mother branch to the point of branching to the daughter is well described by a cylinder of length l and diameter d. The self-similarity of a single element (cylinder) within the entire structure of the tree means that two conditions are met:

$$d_i = \alpha d_{i+1}, \quad l_i = \alpha l_{i+1}, \tag{4}$$

that is, the length and width of the branch at the next level of branching decreases by the same number of times α.

If the hypothesis of self-similarity is valid for old branches and the law of conservation of the lateral area of branches is fulfilled when scaling (1), then it is easy to show that from (1) and (4) it follows that $a^2 = k$, where k is the number of child branches. From here we get

$$d_i^2 = kd_{i+1}^2, \quad l_i^2 = kl_{i+1}^2. \tag{5}$$

The first expression in (5) is a formulation of Leonardo da Vinci's law to describe the structure of a tree [13, 39]. The second expression is a formula for constructing a Pythagorean tree, for the case $k = 2$ [29, 45]. As shown in [29, 39], both constructions are logarithmic fractals in two-dimensional space, as is the mixed construction expressed by (1). Note that the old (excluding young branches) part of the tree in three-dimensional space is not a logarithmic fractal, since the volume of the branch of the lower level is not equal to the sum of the volumes of the branches of the upper level. On the contrary,

$$d_i^2 l_i = k^{3/2} d_{i+1}^2 l_{i+1} \to V_i / kV_{i+1} = k^{1/2}, \tag{6}$$

i.e. the volume of the lower-level branch is $k^{1/2}$ times larger than than the sum of the volumes of the upper-level branches.

If the hypothesis of self-similarity is valid for young branches and the law of conservation of the volume of branches is fulfilled when scaling (3), then it is easy to see that equality $a^3 = k$ follows from (3) and (4).

It is interesting to note that the primary law of tree growth is branching, embedded in the gene, while the proportions of branches are determined by the factors of branch formation during its growth and are most likely due to external conditions and the distribution of the resource base. So, by setting the number $k = 2$ (which is true for most branches), we can confidently predict that the length and thickness of the child branch will be $\sqrt[3]{2} \approx 1.26$ times shorter and thinner than the mother branch.

If the genetics of the tree forces the branch to be divided into 3 daughter branches, then their length and thickness will be $\sqrt[3]{3} \approx 1.44$ times less than the maternal. It would be interesting to find out experimentally which of the rules "works".

However, in photographs of trees we see their projection onto a plane, i.e. the area of the branches, not their volume. In order to obtain the ratio of the areas of the i generation of S_i and the $(i + 1)$th generation of S_{i+i}, it is necessary to multiply the first and second equations from the system (4):

$$d_i l_i = \alpha^2 d_{i+1} l_{i+1} \Leftrightarrow S_i = \sqrt[3]{k^2} S_{i+1}, \qquad (7)$$

$$S_{i+1} = \frac{S_i}{\sqrt[3]{k^2}}.$$

That is, if the total volume of branches of each generation is preserved, then the surface area of the $(i + 1)$ (child) generation should be $\sqrt[3]{k^2}$ times less than the surface area of the branch of the i (mother) generation. Then the surface area of all k branches of the $(i + 1)$ generation will be $\sqrt[3]{k}$ times larger than the surface area of the branch of the i generation:

$$kS_{i+1} = \sqrt[3]{k} S_i. \qquad (8)$$

That is, at each branching level, the total surface area of the branches will increase by $\sqrt[3]{k}$ times. Let's model this situation using the example of the logarithmic fractal of the Leonardo da Vinci in two-dimensional space.

An object in which the total area of the elements is the same at each level of the hierarchy, when followed by the Fourier analysis method, gives an intensity curve that decreases according to the q^{-2} law [29]. To illustrate how the intensity curve changes if, instead of the law of conservation of area, we use the «law of reduction» or «by increasing» the area, we generated three objects — three analogues of the Leonardo da Vinci tree. (Fig. 6). The first (Fig. 6 *A*) is a model of branching into four branches with the condition of preserving the area, which is a logarithmic fractal. The process of its construction is as follows: we take a square and add four squares to its corners from the outside, the length of the sides of which is two times less than the side of the original square. Accordingly, the area of each such square is four times smaller than the area of the original square, and the total area of all four such squares is equal to the area of the original square. In the next step, we repeat this process by selecting the squares added in the previous step as the starting ones. Such an object obeys the law of equal area when scaled. A total of 8 such iterations were made. The second object (Fig. 6 *B*) is a model of branching a tree into four branches with the condition of reducing the area, it differs in that the side of the squares decreases not by 2 times, but by 2.5 times, and the total area of these squares is less than the total area of the squares of the previous generation. When generating a third object (Fig. 6 *C*) — a model of branching a tree into four branches with the condition of increasing the area, squares are added at each next step, the sides of which are only 1.7 times smaller than the hundred squares added in the previous step, and the total area of these squares is greater than the total area of the squares of the previous generation.

The results of the Fourier analysis of the objects constructed in Fig. 6 are presented in Fig. 7. For convenience of comparison, a product of the the Fourier intensity and the square of the coordinate, $I(q)q^2$, are shown, depending on the coordinate q – the so—called Kratky representation. This representation highlights the dependence of q^{-2}, additionally highlighting the features of the curve. Thus, in particular, the production of $I(q)q^2$ in the range q of interest to us should be a constant, which is very convenient for detecting a logarithmic fractal in the object under study. However, the product $I(q)q^2$ shows an oscillating character with the growth of the momentum transfer q. The oscillations are caused by the regular structure of the objects of study. They have a quasi-periodic character on a logarithmic scale, which indicates the fractal properties of the object under study.

The exponent (the slope of the curves on a double logarithmic scale) turns out to be different for different objects (Fig. 7). If the area of the pre--supplied generators does not change (Fig. 6 *A*), the slope of the scattering curve is $\nu = 1.99 \pm 0.04$. If the total area of the added generation decreases with decreasing scale with a factor (4/6.25) per generation (Fig. 6 B), then the slope of the scattering curve is $\nu = 2.24 \pm 0.03$ (greater than 2), and if the area of the added generations increases with a factor (4/2.89) per generation (Fig. 6 C), then the slope of the scattering curve is $\nu = 1.68 \pm 0.06$ (less than 2).

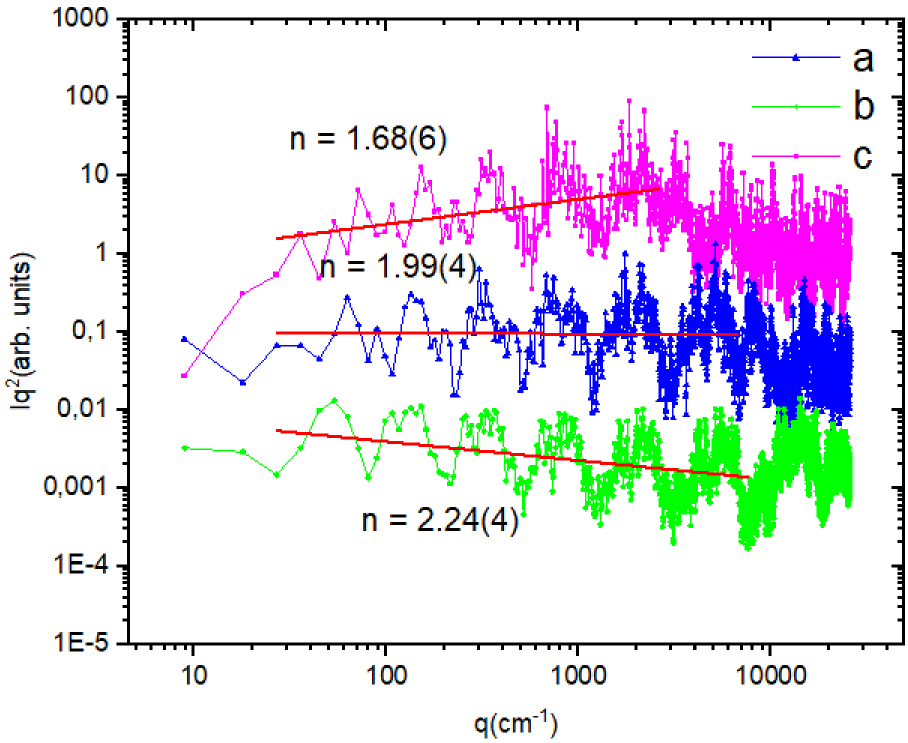

Fig. 7. Investigation of constructed objects by numerical Fourier analysis

Thus, it is shown that if the total area of the child branches in the tree image is larger by $\sqrt[3]{k}$, which is a consequence of conservation the total volume of branches at different levels of branching, then the exponent of the degree of the scattering curve is less than 2.

That is, the tree branching model with volume preservation and with the preservation of the shape of branches during branching gives results similar to the results of studying photographs of young trees and photographs of young branches when studying them using method of numerical Fourier analysis.

It is interesting to discuss the question of converting portions of a logarithmic fractal in three-dimensional space for young branches into the proportion of a two-dimensional logarithmic fractal for adult branches. At a certain stage of branch growth, such a transformation process should be essential precisely for branches (or part of a branch) with characteristic dimensions corresponding to the inflection point at q_c in the dependence $I(q)$ in Fig. 8-15 (see Application). This inflection point q_c characterizes a branches with a diameter from 3 to 6 cm, depending on the type of tree and, possibly, the conditions of its growth. Obviously, as the branch passes from one growth mode to another, it is the proportions of the branch that change. It can be argued that the inner part of the branches or trunk should obey the law of the logarithmic fractal in three-dimensional space, but it is much smaller than the outer part of the trunk, which obeys the law of the logarithmic fractal in two-dimensional space. Taking into account the logarithmic scale laid down along the q axis (the axis of dimensions), we confidently see the point of inflection or transition from one mode to another, although on a linear scale this transition would look more stretched. Therefore, experimental curves on a double logarithmic scale demonstrate a bifractal scattering pattern

6. CONCLUSIONS

Fractal properties in the formation of the branching structure of deciduous trees are studied by the method of numerical Fourier analysis. It has been shown that the intensity curve, depending on the Fourier space coordinate $I(q) \sim q^{-\nu}$, has two sections with different exponents of degree ν and an inflection point between them. For old (large) branches with small q, the exponent is 2, which corresponds to the law of the logarithmic fractal

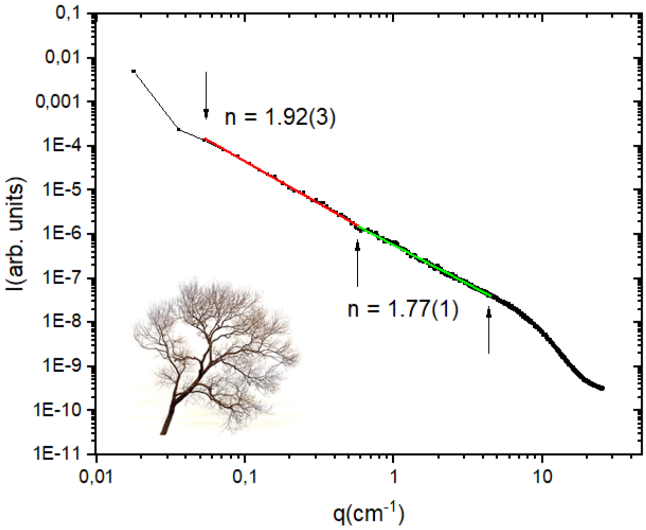

Fig. 8. Study of the willow image by numerical Fourier analysis

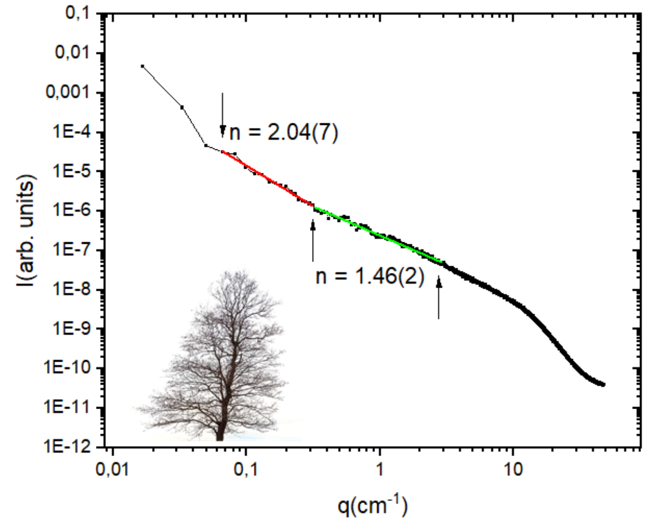

Fig. 9. Study of the image of a lime tree by numerical Fourier analysis

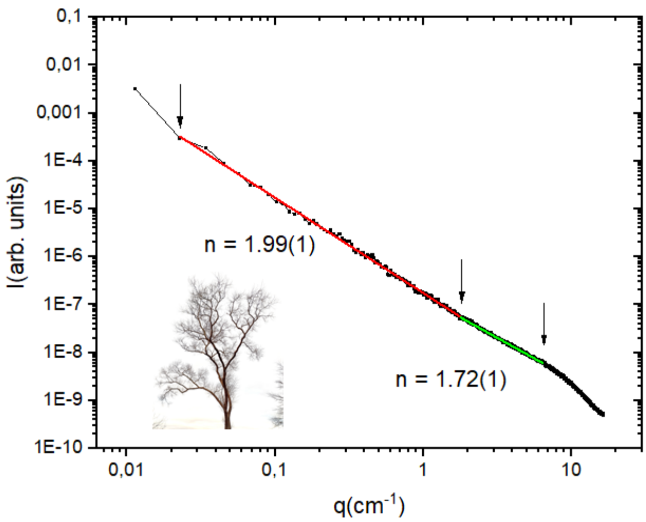

Fig. 10. Study of the image of an apple tree by numerical Fourier analysis

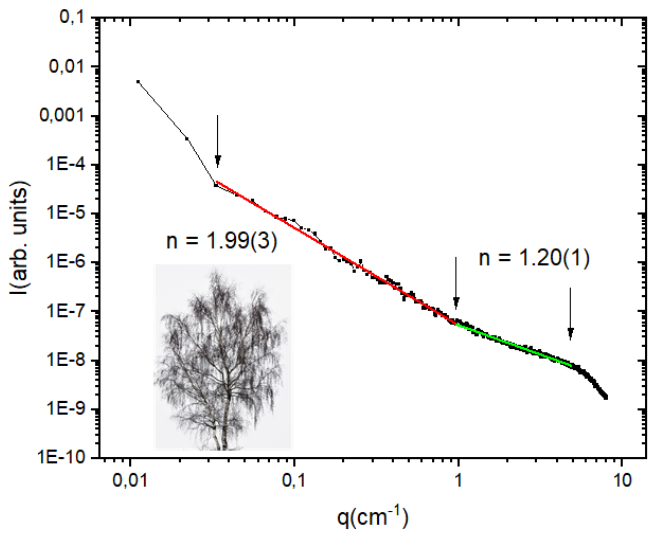

Fig. 11. Study of the birch image by numerical Fourier analysis

in two-dimensional space: the surface area of the lower branch is equal to the sum of the surface areas of the branches after its branching, i.e. the law of conservation of area when scaling is fulfilled. It can be concluded that the living tissue of old (large) branches of adult trees is formed only on their surface. The exponent of the Fourier curve in the region of large q is $v = 1.6$-1.7 and is due to the law of the structure of young branches, which consist entirely of living biological cells involved in the growth of the tree. Unlike the old branches, life in them is concentrated in the entire volume of the branch, not just on the surface. It is assumed that the young branches retain not the surface area, but their volume at different levels of branching. That is, the structure of branches at the upper levels of branching obeys the law of conservation of volume when scaling, which corresponds to the model of a logarithmic fractal in three-dimensional space. A mathematical model is proposed that generalizes the concepts of a logarithmic fractal on the surface for adult branches and a logarithmic fractal in volume for young branches, which is equivalent to the statement that the number of living cells at each level of branching remains constant, although on old branches it is distributed along the surface of the branch, and on young branches — over their entire volume. Numerical models of two-dimensional images of a tree obeying the law of conservation of volume in three-dimensional space (the law of growth of a young tree) are constructed. Numerical Fourier analysis of the models demonstrates a coincidence within the error bars with the results of Fourier analysis obtained for real trees. Thus, a generalized fractal concept of the growth and branching structure of free-standing deciduous trees has been built.

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APPENDIX

To demonstrate the generality of the law of scaling the structure of trees, we present a series of images of various trees and Fourier analyses of these images. All of them have the same kind of curves with an inflection, characterizing the two-stage structure of the formation of branches of a deciduous tree. For large scales (adult branches), the exponent for all the above images is close to or within the error bars of $\nu_1 = 2.00$, and for small scales (young branches), the exponent turned out to be less than 2 and approximates $\nu_2 \approx 1.6$-1.8. In all cases, the inflection point $q_c \approx 0.3$ cm^{-1}, which corresponds to the linear average size $r_c = 20$ cm. In this case, the average linear size of the rectangular element $r_c = \sqrt{l_c d_c} = 20$ cm corresponds to the thickness of the branch $d_c =$ 6.6 cm and its length $l_c = 60$ cm. That is, branches with sizes smaller than d_c and l_c are described by one scaling law, and branches with sizes larger than d_c and l_c are described by another. The structure of large branches is described by the law of the logarithmic fractal in the two—dimensional space of the tree image – the law of conservation of area during scaling. The structure of small branches is described by the law of a logarithmic fractal in three-dimensional space, which, when projected onto a two-dimensional plane, forms a dependence for a classical fractal with $\nu_2 = 1.6$-1.8.

Figure 8 shows an image of willow and a Fourier intensity curve depending on the momentum on a double logarithmic scale with degree values for large branches $\nu_1 = 1.92(3)$ in the range of coordinates of the inverse space from $6 \cdot 10^{-2}$ to $5 \cdot 10^{-1}$ cm^{-1} and for small branches $\nu_2 = 1.77(1)$ in the range from $5 \cdot 10^{-1}$ to 4 cm^{-1}. The inflection point $q_c = 5 \cdot 10^{-1}$ cm^{-1} corresponds to an average distance of 12 cm, which in terms of branch thickness is 4 cm, and in terms of branch length is 36 cm. Thus, the structure of branches with a length from 36 to 300 cm (thickness from 4 to 30 cm) is described by the structure of a logarithmic fractal in two-dimensional space. And the structure of branches from 4.5 to 36 cm long (0.5 to 4 cm thick) is described by the structure of a logarithmic fractal in three-dimensional space.

Fig. 9 shows an image of a lime tree and a Fourier intensity curve depending on the momentum on a double logarithmic scale with degree values for large branches $\nu_1 = 2.04 \pm 0.07$ in the range of coordinates of the inverse space from $6 \cdot 10^{-2}$ to $3 \cdot 10^{-1}$ cm^{-1} and for small branches $\nu_2 = 1.74(1)$ in the range from $3 \cdot 10^{-1}$ to 3 cm^{-1}. The inflection point $q_c = 3 \cdot 10^{-1}$ cm^{-1} corresponds to an average distance of 20 cm, which in terms of branch thickness is converted to 6.6 cm, and in terms of branch length – to 60 cm. Thus, the structure of branches from 60 to 300 cm long (6.6 to 30 cm thick) is described by the structure of a logarithmic fractal in two-dimensional space. And the structure of branches from 6 to 60 cm long (0.6 to 6 cm thick) is described by the structure of a logarithmic fractal in three-dimensional space.

Figure 10 shows an image of an apple tree and a Fourier intensity curve depending on the momentum in a double logarithmic scale with exponents for large branches $\nu_1 = 1.99 \pm 0.01$ in the range of coordinates of the inverse space from $2 \cdot 10^{-2}$ to $1.5 \cdot 10^{-1}$ cm^{-1} and for small branches $\nu_2 = 1.72 \pm 0.01$ in the range from 1.5 to 6 cm^{-1}. The inflection point $q_c = 1.5$ cm^{-1} corresponds to an average distance of 4 cm, which in terms of branch thickness is converted to 1.3 cm, and in terms of branch length – to 12 cm. Thus, the structure of a network with a length from 12 to 1000 cm (thickness from 1.3 to 100 cm) is described by the structure of a logarithmic fractal in two-dimensional space. And the structure of branches from 3 to 12 cm long (0.3 to 1.3 cm thick) is described by the structure of a logarithmic fractal in three-dimensional space.

Figure 11 shows an image of a birch tree and a Fourier intensity curve depending on the momentum in a double logarithmic scale with exponents for large branches $\nu_1 = 1.99 \pm 0.03$ in the range of coordinates of the inverse spaces from $3 \cdot 10^{-2}$ to 1 cm^{-1} and for small branches $\nu_2 = 1.20 \pm 0.01$ in the range from 1 to 6 cm^{-1}. The inflection point $q_c = 1$ cm^{-1} corresponds to an average distance of 6 cm, which in terms of branch thickness is converted to 2 cm, and in terms of branch length — to 18 cm. Thus, the structure of branches with a length from 18 to 600 cm (thickness from 2 to 60 cm) is described by the structure of a logarithmic fractal in two-dimensional space. And the structure of branches from 3 to 18 cm long (0.3 to 2 cm thick) is described by the structure of a logarithmic fractal in three-dimensional space.

Figure 12 shows an image of an oak tree and a Fourier intensity curve depending on the momentum in a double logarithmic scale with exponents for large branches $\nu_1 = 1.99 \pm 0.04$ in the range of coordinates of the inverse space from $2 \cdot 10^{-2}$ to $3 \cdot 10^{-1}$ cm^{-1} and for small branches $\nu_2 = 1.72 \pm 0.03$ in The range is from $3 \cdot 10^{-1}$ to 1 cm^{-1}. The inflection point $q_c = 3 \cdot 10^{-1}$ cm^{-1} corresponds to an average distance of 20 cm, which in terms of branch thickness is converted to 6.6 cm, and in terms of branch length — to 60 cm. Thus, the structure of branches from 60 to 900 cm long (6.6 to 100 cm thick) is described by the structure of a logarithmic fractal in two-dimensional space. And the structure

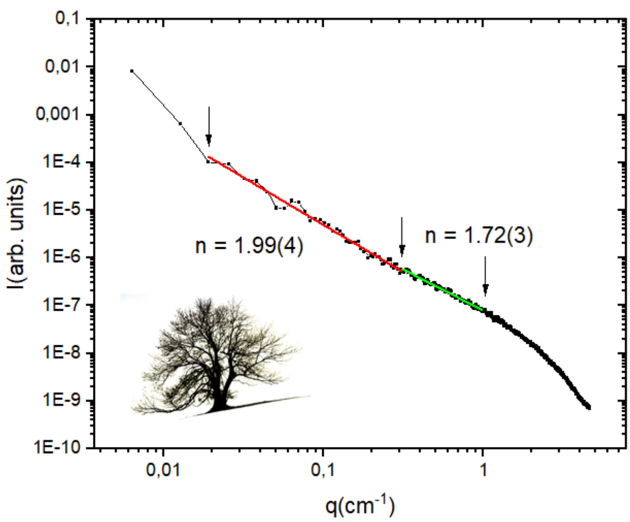

Fig. 12. Study of the image of an oak tree by numerical Fourier analysis

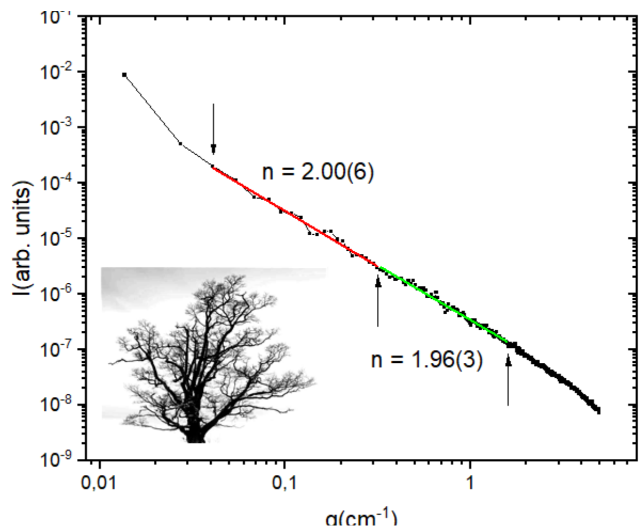

Fig. 13. Study of the image of another oak tree by the method of numerical Fourier analysis

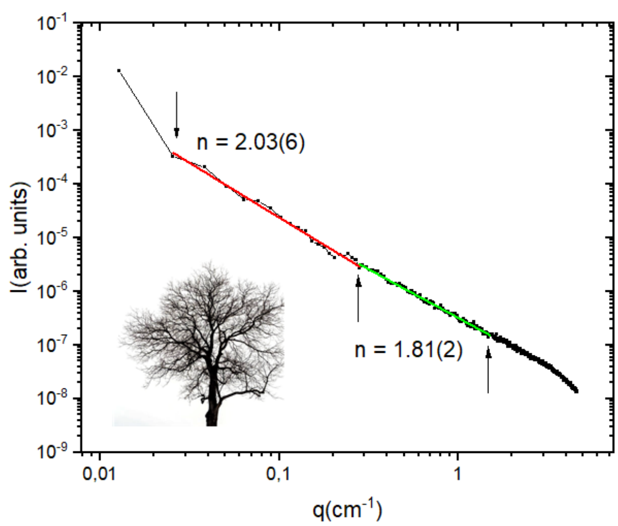

Fig. 14. Study of the image of an unidentified tree by numerical Fourier analysis

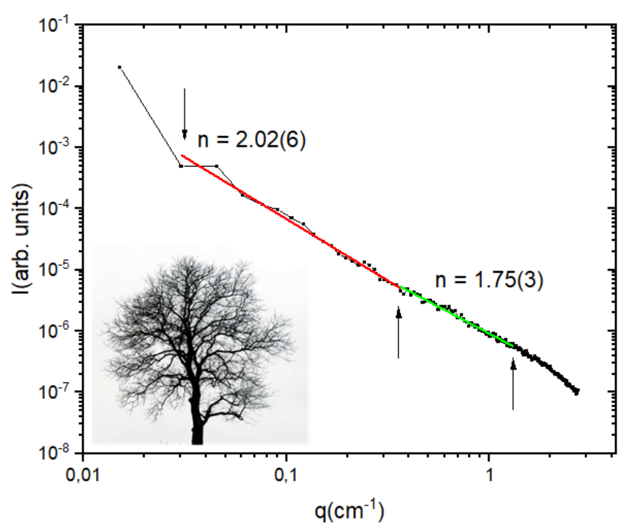

Fig. 15. Study of the image of a tree of the same unidentified species by numerical Fourier analysis

of branches with a length from 18 to 60 cm (thickness from 2 to 6.6 cm) is described by the structure of a logarithmic fractal in three-dimensional space.

Figure 13 shows an image of another oak tree and a Fourier intensity curve depending on the momentum in a double logarithmic scale with exponents for large branches $v_1 = 2.00 \pm 0.06$ in the range of coordinates of the inverse space from $4 \cdot 10^{-2}$ to $3 \cdot 10^{-1}$ cm^{-1} and for small branches $v_2 = 1.96 \pm 0.03$ in the range from $3 \cdot 10^{-1}$ to 1.5 cm^{-1}. The inflection point $q_c = 3 \cdot 10^{-1}$ cm^{-1} corresponds to an average distance of 20 cm, which in terms of branch thickness is converted to 6.6 cm, and in terms of branch length — to 60 cm. Thus, the structure of branches from 60 to 600 cm long (6.6 to 60 cm thick) is described by the structure of a logarithmic fractal in two-dimensional space. And the structure of branches from 12 to 60 cm long (1.3 to 6.6 cm thick) is described by the structure of a logarithmic fractal in three-dimensional space.

Figure 14 shows an image of an unidentified tree and a Fourier intensity curve depending on the momentum on a double logarithmic scale with exponents for large branches $v_1 = 2.03 \pm 0.06$ in the range of coordinates of the inverse space from $2 \cdot 10^{-2}$ to $3 \cdot 10^{-1}$ cm^{-1} and for small branches $v_2 = 1.81 \pm 0.02$ in The range is from $3 \cdot 10^{-1}$ to 1.5 cm^{-1}. The inflection point $q_c = 3 \cdot 10^{-1}$ cm^{-1} corresponds to an average distance of 20 cm, which in terms of branch thickness is converted to 6.6 cm, and in terms of branch length – to 60 cm. Thus, the structure of branches from 60 to 600 cm long (6.6 to 60 cm thick) is described by the structure of a logarithmic fractal in two-dimensional space. And the structure of branches from 12 to 60 cm long (1.3 to 6.6 cm thick) is described by the structure of a logarithmic fractal in three-dimensional space.

Figure 15 shows an image of a tree of the same unidentified species, a Fourier intensity curve depending on the momentum on a double logarithmic scale with exponents for large branches $v_1 = 2.02 \pm 0.06$ in the range of coordinates of the inverse space from $3 \cdot 10^{-2}$ to 340^{-1} cm^{-1} and for small branches $v_2 = 1.75 \pm 0.03$ in the range from $3 \cdot 10^{-1}$ to 1.5 cm^{-1}. The inflection point $q_c = 3 \cdot 10^{-1}$ cm^{-1} corresponds to an average distance of 20 cm, which in terms of branch thickness is converted to 6.6 cm, and in terms of branch length — to 60 cm. Thus, the structure of branches from 60 to 900 cm long (6.6 to 90 cm thick) is described by the structure of a logarithmic fractal in two-dimensional space. And the structure of branches from 12 to 60 cm long (1.3 to 6.6 cm thick) is described by the structure of a logarithmic fractal in three-dimensional space.

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