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Приложение. Оптимальная система подалгебр в случае $^{з}{}_{S} = 0, \tilde{N} = 0$

П1. Одномерные подалгебры

1.1. $X_{13} + \alpha\tilde{X}_{11} + \beta X_{12} + \gamma X_0$, 1.2. $X_{10} + \alpha\tilde{X}_{11} + \beta X_{12} + \gamma X_0$

1.3. $\tilde{X}_{11} + \beta X_{12} + \gamma X_0$, 1.4. $X_{12} + \alpha X_0$, 1.5. X_0

П2. Двухмерные подалгебры

2.1. $\left\{X_{10}, X_{13} + \alpha\tilde{X}_{11} + \beta X_{12} + \gamma X_0\right\}$

2.2. .$\left\{X_{10} + \beta X_{12} + \gamma X_0, \tilde{X}_{11} + \beta_1 X_{12} + \gamma_1 X_0\right\}$

2.3. $\left\{X_{10} + \alpha\tilde{X}_{11} + \beta X_0, X_{12} + \gamma X_0\right\}$, 2.4. $\left\{X_{10} + \alpha\tilde{X}_{11} + \beta X_{12}, X_0\right\}$

2.5. $\left\{X_{13} + \alpha X_{12} + \beta X_0, \tilde{X}_{11} + \alpha_1 X_{12} + \beta_1 X_0\right\}$

2.6. $\left\{X_{13} + \alpha\tilde{X}_{11} + \gamma X_0, X_{12} + \beta X_0\right\}$, 2.7. $\left\{X_{13} + \alpha\tilde{X}_4 + \beta\tilde{X}_{12}, X_0\right\}$

2.8. $\left\{\tilde{X}_{11} + \alpha X_0, X_{12} + \beta X_0\right\}$, 2.9. $\left\{X_{12}, X_0\right\}$, 2.10. $\left\{\tilde{X}_{11}, X_0\right\}$

П3. Трехмерные подалгебры

3.1. $\left\{X_{10}, X_{13} + \alpha X_{12} + \beta X_0, \tilde{X}_{11} + \alpha_1 X_{12} + \beta_1 X_0\right\}$

3.2. $\left\{X_{10}, X_{13} + \alpha\tilde{X}_{11} + \gamma X_0, X_{12} + \beta X_0\right\}$

3.3. $\left\{X_{10}, X_{13} + \alpha\tilde{X}_{11} + \beta X_{12}, X_0\right\}$, 3.4. $\left\{X_{10} + \delta X_0, \tilde{X}_{11} + \alpha X_0, X_{12} + \beta X_0\right\}$

3.5. $\left\{X_{10} + \delta X_{12}, \tilde{X}_{11}, X_0\right\}$, 3.6. $\left\{X_{10} + \delta\tilde{X}_{11}, X_{12}, X_0\right\}$

3.7. $\left\{X_{13} + \alpha X_0, \tilde{X}_{11} + \beta X_{10}, X_{12} + \gamma X_0\right\}$, 3.8. $\left\{X_{13} + \alpha X_{12}, \tilde{X}_{11}, X_0\right\}$

3.9. $\left\{X_{13} + \alpha\tilde{X}_{11}, X_{12}, X_0\right\}$, 3.10. $\left\{\tilde{X}_{11}, X_{12}, X_0\right\}$

П4. Четырехмерные подалгебры

4.1. $\left\{X_{10}, X_{13} + \alpha X_0, \tilde{X}_{11} + \beta X_0, X_{12} + \gamma X_0\right\}$

4.2. $\left\{X_{10}, \tilde{X}_{11}, X_{12}, X_0\right\}$, 4.3. $\left\{X_{13}, \tilde{X}_{11}, X_{12}, X_0\right\}$

Здесь α, β, γ – постоянные, $\delta = 0$ или 1. Для любой подалгебры произвольный оператор $X = B_0 X_{10} + N_0\tilde{X}_{11} + EX_{12} + BX_{13} + C_0 X_0$ алгебры L_5 равен линейной комбинации базисных операторов подалгебры с произвольными коэффициентами λ, μ, ν, σ. Постоянные B_1, Γ в равенстве (5.1) линейно выражаются через произвольные коэффициенты. Их нет в выражениях для оператора X. Они определяются уравнением состояния. Каждая подалгебра определяет с помощью соотношения (5.1) уравнения состояния, с которыми система (2.1) – (2.3) допускает эту подалгебру.

Подалгебра 1.1 $X = \lambda\left(X_{13} + \alpha\tilde{X}_{11} + \beta X_{12} + \gamma X_0\right) \Rightarrow B_0 = \lambda\gamma, N_0 = \lambda\alpha, E = \lambda\beta,$

$B = \lambda, C_0 = \lambda\gamma, B_1 = \lambda b_1, \Gamma = \lambda b_0$. Равенство (5.1) после сокращения на λ определяет уравнение для функции, задающей уравнение состояния, чтобы подалгебра допускалась уравнениями газовой динамики (2.1)–(2.3),

$$te_t + \beta V e_V + \gamma\eta\left(S\right)e_S = 2e\left(\alpha - 1\right) - Vb_1 - b_0$$

Аналогичные вычисления для других подалгебр из оптимальной системы определяют уравнения для функций, задающих уравнения состояния.

Подалгебра 1.2. $\Rightarrow B_0 = \lambda, N_0 = \lambda\alpha, E = \lambda\beta, B = 0, C_0 = \lambda\gamma$

$$e_t + \beta V e_V + \gamma\eta\left(S\right)e_S = 2\alpha e - Vb_1 - b_0$$

Подалгебра 1.3. $\Rightarrow B_0 = 0, N_0 = \lambda, E = \lambda\beta, B = 0, C_0 = \lambda\gamma$

$$\beta Ve_V + \gamma r(S)e_S = 2e - Vb_1 - b_0$$

Подалгебра 1.4. $\Rightarrow B_0 = N_0 = B = 0, E = \lambda, C_0 = \lambda\alpha$

$$Ve_V + \alpha r(S)e_S = -Vb_1 - b_0$$

Подалгебра 1.5. $\Rightarrow B_0 = N_0 = E = B = 0, C_0 = \lambda$

$$e = g(t,V) - (Vb_1 + b_0)(S \text{ или } \ln|S|)$$

Подалгебра 2.1. $X = \lambda X_{10} + \mu\left(X_{13} + \alpha\tilde{X}_{11} + \beta X_{12} + \gamma X_0\right) \Rightarrow B_0 = \lambda, N_0 = \mu\alpha,$ $E = \mu\beta, B = \mu, C_0 = \mu\gamma, B_1 = \lambda b_1 + \mu b_2, \Gamma = \lambda b_{01} + \mu b_{02}$.

Равенство (5.1) после расщепляется по λ и μ дает 2 уравнения для функции e

$$e_t = -b_1V - b_{01} \Rightarrow e = -(b_1V + b_{01})t + e_1(V,S)$$

$$te_t + \beta Ve_V + \gamma r(S)e_S = 2(\alpha - 1)e - Vb_2 - b_{02} \Rightarrow (2\alpha - 3)b_{01} = 0$$

$$(2\alpha - \beta - 3)b_1 = 0, \beta Ve_{1V} + \gamma r(S)e_{1S} = 2(\alpha - 1)e_1 - Vb_2 - b_{02}$$

Подалгебра 2.2. $\Rightarrow B_0 = \lambda, N_0 = \mu, E = \lambda\beta + \mu\beta_1, B = 0, C_0 = \lambda\gamma + \mu\gamma_1$

$(5.1) \Rightarrow e_t + \beta Ve_V + \gamma r(S)e_S = -b_1V - b_{01}$

$$\beta_1 Ve_V + \gamma_1 r(S)e_S = 2e - b_2V - b_{02} \Rightarrow b_{02} \sim 0$$

Если $\beta \neq 0$, то $e = -\dfrac{b_1}{\beta}V - \dfrac{b_{01}}{\beta}\ln V + e_1(V_1,S_1), V_1 = Ve^{-\beta t}$

$$S_1 = -\gamma t + (S \text{ или } \ln|S|); b_{01} = 0, b_2 = (\beta_1 - 2)b_1\beta^{-1}$$

$$\beta_1 V_1 e_{1V_1} + \gamma_1 e_{1S_1} = 2e$$

Если $\beta = 0$, то $e = -t(b_1V + b_{01}) + e_1(V,S_1);\ b_{01} = 0$

$$b_1(\beta_1 - 2) = 0, \beta_1 Ve_{1V} + \gamma_1 e_{1S_1} = 2e_1 - b_2V$$

Подалгебра 2.3. $\Rightarrow B_0 = \lambda, N_0 = \alpha\lambda, E = \mu, B = 0, C_0 = \lambda\beta + \mu\gamma$

$(5.1) \Rightarrow e_t + \beta r(S)e_S = -b_1V - b_{01} \Rightarrow e = -t(b_1V + b_{01}) + e_1(V,S_1)$

$$S_1 = -\beta t + (S \text{ или } \ln|S|); Ve_V + \gamma r(S)e_S = -b_2V - b_{02} \Rightarrow b_1 = 0,\ b_2 \sim 0$$

$$Ve_{1V} + \gamma e_{1S_1} = -b_{02}$$

Подалгебра 2.4. $\Rightarrow B_0 = \lambda, N_0 = \lambda\alpha, E = \lambda\beta, B = 0, C_0 = \mu$

$(5.1) \Rightarrow \eta(S)e_S = -b_2V - b_{02} \Rightarrow e = e_1(t,V) - (b_2V + b_{02})(S \text{ или } \ln|S|)$

$$e_t + \beta Ve_V = 2\alpha e - b_1V - b_{01} \Rightarrow \alpha b_{02} = 0, (\beta - 2\alpha)b_2 = 0$$

$$e_{1t} + \beta Ve_{1V} = 2\alpha e_1 - b_1V - b_{01}$$

Подалгебра 2.5. $\Rightarrow B_0 = 0, N_0 = \mu, E = \lambda\alpha + \mu\alpha_1, B = \lambda, C_0 = \lambda\beta + \mu\beta_1$

$(5.1) \Rightarrow te_t + \alpha V e_V + \beta\eta(S)e_S = -2e - b_1 V - b_{01} \Rightarrow b_1 \sim 0; b_{01} \sim 0$

$$e = t^{-2} g(V_1, S_1), V_1 = Vt^{-\alpha}, S_1 = -\beta\ln|t| + (S \text{ или } \ln|S|)$$

$$\alpha_1 V e_V + \beta_1 \eta(S) e_S = 2e - b_2 V - b_{02} \Rightarrow b_{02} = 0$$

$$\alpha_1 V_1 g_{V_1} + \beta_1 g_{S_1} = 2g - b_2 V_1, b_2 = 0 \text{ при } \alpha \neq -2 \text{ или } \alpha_1 \neq 2$$

Подалгебра 2.6. $\Rightarrow B_0 = 0, N_0 = \lambda\alpha, E = \mu, B = \lambda, C_0 = \lambda\gamma + \mu\beta$

$(5.1) \Rightarrow te_t + \gamma\eta(S)e_S = 2(\alpha - 1)e - b_1 V - b_{01}, Ve_V + \beta\eta(S)e_S = -b_2 V - b_{02}$

Если $\alpha \neq 1$, то $b_1 \sim 0$, $b_{01} \sim 0$, $e = 2(\alpha - 1)\ln|t| + e_1(V, S_1)$

$$S_1 = t^{-\gamma}(e^S \text{ или } S), e_{1V} + \beta S_1 e_{1S_1} = -b_{02}, b_2 \sim 0$$

Если $\alpha = 1$, то $e = -b_{01}\ln|t| + e_1(V, S_1), Ve_{1V} + \beta S_1 e_{1S_1} = -b_{02}$

$$b_1 = 0, b_2 \sim 0$$

Подалгебра 2.7. $\Rightarrow B_0 = 0, N_0 = \lambda\alpha, E = \lambda\beta, C_0 = \mu, B = \lambda$

$(5.1) \Rightarrow \eta(S)e_S = -b_2 V - b_{02} \Rightarrow e = e_1(t, V) - (b_2 V + b_{02})(S \text{ или } \ln|S|)$

$$te_t + \beta V e_V = 2(\alpha - 1) - Vb_1 - b_{01} \Rightarrow (\alpha - 1)b_{02} = 0, (\beta - 2\alpha - 2)b_2 = 0$$

$$te_{1t} + \beta V e_{1V} = 2(\alpha - 1)e_1 - Vb_1 - b_{01}$$

Подалгебра 2.8. $\Rightarrow B_0 = 0, N_0 = \lambda, E = \mu, B = 0, C_0 = \lambda\alpha + \mu\beta$

$(5.1) \Rightarrow \alpha\eta(S)e_S = 2e - b_1 V - b_{01} \Rightarrow b_1 \sim 0, b_{01} \sim 0, \alpha \neq 0$

$$e = e_1(t, V)(e^S \text{ или } S)^{\frac{2}{\alpha}}$$

$$Ve_V + \beta\eta(S)e_S = -b_2 V - b_{02} \Rightarrow b_2 = b_{02} = 0, e_1 = V^{-\frac{2\beta}{\alpha}} g(t)$$

Подалгебра 2.9. $\Rightarrow B_0 = B = N_0 = 0, E = \lambda, C_0 = \mu$

$(5.1) \Rightarrow Ve_V = -b_1 V - b_{01} \Rightarrow b_1 \sim 0; b_{01} \neq 0, e = -b_{01}\ln V + e_1(t, S)$

$$\eta(S)e_S = -b_2 V - b_{02} \Rightarrow b_2 = 0; b_{02} \neq 0, e_1 = g(t) - b_{02}(S \text{ или } \ln|S|)$$

Подалгебра 3.1. $\Rightarrow B_0 = \lambda, N_0 = \nu, E = \alpha\mu + \alpha_1\nu, B = \mu, C_0 = \beta\mu + \beta_1\nu$

$(5.1) \Rightarrow e_t = -b_1 V - b_{01} \Rightarrow e = -(b_1 V + b_{01})t + e_1(V, S)$

$$te_t + \alpha V e_V + \beta\eta(S)e_S = -2e - b_2 V - b_{02} \Rightarrow b_{02} \sim 0; b_{01} = 0, b_1 \neq 0$$

$$\alpha = -3, b_2 \sim 0$$

$$\alpha_1 V e_V + \beta_1 r(S) e_S = 2e - b_3 V - b_{03} \Rightarrow \alpha_1 = 2, b_3 = b_{03} = 0; \beta_1 \neq 0.$$

Изучая совместность последних двух уравнений раздельно при $\beta \neq 0$ и при $\beta = 0$, получим единую формулу

$$e = -b_1 tV + GV^{\frac{2}{3}+\frac{\beta}{9\beta_1}}\left(e^S \text{ или } |S|\right)^{\frac{1}{3\beta_1}}$$

Подалгебра 3.2. $\Rightarrow B_0 = \lambda, N_0 = \alpha\mu, E = \nu, B = \mu, C_0 = \gamma\mu + \beta\nu$

$(5.1) \Rightarrow e_t = -b_1 V - b_{01} \Rightarrow e = -t\left(b_1 V + b_{01}\right) + e_1\left(V, S\right)$

$$te_t + \gamma\eta\left(S\right)e_S = 2\left(\alpha - 1\right)e - b_2 V - b_{02} \Rightarrow \alpha = \frac{3}{2}, \gamma \neq 0, e_1 = g\left(V\right)\left(e^S \text{ или } |S|\right)^{\frac{1}{\gamma}}$$

$$Ve_V + \beta\eta\left(S\right)e_S = -b_3 V - b_{03} \Rightarrow b_3 = b_{03} = 0, b_1 = 0, b_{01} \neq 0, g = GV^{\frac{\beta}{\gamma}}$$

Уравнение состояния имеет вид

$$e = -b_{01}t + GV^{\frac{\beta}{\gamma}}(e^S \text{ или } |S|)^{\frac{1}{\gamma}} \text{ при } \alpha = \frac{3}{2}, \gamma \neq 0$$

Подалгебра 3.3. $\Rightarrow B_0 = \lambda, N_0 = \mu\alpha, E = \mu\beta, B = \mu, C_0 = \nu$

$(5.1) \Rightarrow e_t = -b_1 V - b_{01} \Rightarrow e = -t\left(b_1 V + b_{01}\right) + e_1\left(V, S\right)$

$$te_t + \beta Ve_V = \left(\alpha - 1\right)e - b_2 V - b_{02} \Rightarrow b_{01}\left(\alpha - 2\right) = 0, b_1\left(\beta - \alpha + 2\right) = 0$$

$$\eta e_S = -Vb_3 - b_{03} \Rightarrow e_1 = g\left(V\right) - \left(Vb_3 + b_{03}\right)\left(S \text{ или } \ln|S|\right)$$

$$b_{03}\left(\alpha - 1\right) = 0, b_3\left(\beta - \alpha + 1\right) = 0, \beta Vg' = \left(\alpha - 1\right)g - Vb_2 - b_{02}$$

Если $b_1 \neq 0$, $b_{01} \neq 0$, то $\alpha = 2$, $\beta = 0$, $b_{03} = 0$, $b_3 = 0 \Rightarrow e_S = 0$ противоречие.

Если $b_1 \neq 0$, $b_{01} = 0$, то $\beta = \alpha - 2$, $b_3 = 0$, $b_{03} \neq 0$, $\alpha = 1$, $\beta = -1$

$$g \sim b_{02}\ln V, e = -b_1 Vt + b_{02}\ln V - b_{03}\left(S \text{ или } \ln|S|\right)$$

Если $b_1 = 0, b_{01} \neq 0$, то $\alpha = 2, b_{03} = 0, \beta = \alpha - 1 = 1, g \sim -b_2 V\ln V$

$$e = -b_{01}t - b_2 V\ln V - b_3\left(S \text{ или } \ln|S|\right)$$

Подалгебра 3.4 $\Rightarrow B_0 = \lambda, N_0 = \mu, E = \nu, B = 0, C_0 = \lambda\delta + \mu\alpha + \nu\beta$

$(5.1) \Rightarrow \alpha\eta\left(S\right)e_S = 2e - b_2 V - b_{02} \Rightarrow b_2 \sim 0, b_{02} \sim 0, \alpha \neq 0$

$$e = e_1\left(t, V\right)(e^S \text{ или } |S|)^{\frac{2}{\alpha}}$$

$$e_t + \delta\eta e_S = -b_1 V - b_{01} \Rightarrow b_1 = b_{01} = 0, e_1 = g\left(V\right)e^{-\frac{2\delta}{\alpha}t}$$

$$Ve_V + \beta\eta e_S = -b_3 V - b_{03} \Rightarrow b_3 = b_{03} = 0, g = GV^{-\frac{2\beta}{\alpha}}$$

Подалгебра 3.6. $\Rightarrow B_0 = \lambda, N_0 = \mu, E = \lambda\delta, B = 0, C_0 = \nu$

$(5.1) \Rightarrow Ve_V = -b_2 V - b_{02} \Rightarrow b_2 \sim 0, e = -b_{02}\ln V + e_1\left(t, S\right)$

$$\eta e_S = -b_3 V - b_{03} \Rightarrow e_1 = g\left(t\right) - \left(b_3 V + b_{03}\right)\left(S \text{ или } \ln|S|\right)$$

$$e_t = 2\delta e - b_1 V - b_{01} \Rightarrow \delta = 0, b_1 = 0, g = -b_{01}t$$

Подалгебра 3.7. $\Rightarrow B_0 = 0, N_0 = \mu, E = \nu, B = \lambda, C_0 = \lambda\alpha + \mu\beta + \nu\gamma$

(5.1) $\Rightarrow \beta\eta e_S = 2e - b_2 V - b_{02} \Rightarrow b_2 \sim 0, b_{02} \sim 0, \beta \neq 0, e = e_1(t,V) e^{\frac{2}{\beta}S}$

$$\gamma\eta e_S + V e_V = -b_3 V - b_{03} \Rightarrow b_3 = b_{03} = 0, e_1 = g(t) V^{-\frac{2\gamma}{\beta}}$$

$$t e_t + \alpha\eta e_S = -2e - b_1 V - b_{01} \Rightarrow b_1 = b_{01} = 0, g = G t^{-2\left(1+\frac{\alpha}{\beta}\right)}$$

Подалгебра 3.9. $\Rightarrow B_0 = 0, N_0 = \lambda\alpha, E = \mu, B = \lambda, C_0 = \nu$

(5.1) $\Rightarrow V e_V = -b_2 V - b_{02} \Rightarrow b_2 \sim 0, e = -b_{02} \ln V + e_1(t,S)$

$$\eta e_S = -b_3 V - b_{03} \Rightarrow e_1 = -(b_3 V + b_{03})(S \text{ или } \ln|S|) + g(t)$$

$$t e_t = 2(\alpha - 1)e - b_1 V - b_{01} \Rightarrow \alpha = 1, b_1 = 0, g = -b_{01} \ln|t|$$

Замечание. Подалгебры 2.10, 3.5, 3.8, 3.10, 4.1, 4.2, 4.3 не производят уравнения состояния с условиями $e_{VV} \neq 0, e_S \neq 0, e_t \neq 0$.

Итак, в случае 4° для коэффициентов уравнения типа (3.2) проведена групповая классификация уравнений газовой динамики методом построения оптимальной системы подалгебр, расширяющих ядро.

Methods of Group Classification for Relaxing Gasdynamics

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Group classification is the basic problem of the group analysis of differential equations with an arbitrary element. For the equations of the ideal gas dynamics with a state equation invariable on time the problem was solved by enumerating simplifications of the determining relations using equivalence transformations. For a state equation depending on time the exhaustive search is vast and it can be used optimal systems of subalgebras for the subalgebra extending the kernel of admitted algebras. Combination of the both methods solves the problem of the group classification for the relaxing gas dynamics.

Keywords: gas dynamics, relaxing state equation, equivalence transformation, determining relation, group classification, optimal system of subalgebras

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